WORKING PAPERS

2025-21

05 Juin 2025

Contract Terms Monotonicity in Matching Markets

Cyril Rouault

CEPS
Centre for Economics at Paris-Saclay

Contract Terms Monotonicity in Matching Markets*

Cyril Rouault†

Abstract

We study a doctor-hospital matching market with contracts, where hospitals can offer a range of contract terms to doctors. Surprisingly, expanding the set of available terms can reduce doctors' welfare without improving the allocation for others. In contrast, our results suggest that limiting the options available to doctors can lead to a Pareto improvement. We then examine the necessary conditions on agents' preferences to prevent reductions in doctors' welfare. We demonstrate that only *agent-lexicographic* preferences for all agents guarantee that no doctor experiences a decrease in welfare when available terms are added.

JEL Classification: C78, D47, D71, D86.

Keywords: Matching with contracts; Stability; Adding terms; Welfare; Agent-lexicographic preferences.

1 Introduction

Contract terms play a crucial role in various matching problems. In labor markets, workers are assigned shift schedules and receive salaries; similarly, doctors are assigned to specific specialties within hospitals, and cadets are allocated to branches

*I am grateful to Olivier Bos, David Delacrétaz, Michael Greinecker, Guillaume Haeringer, Vincent Iehlé, Olivier Tercieux, and Bumin Yenmez for their valuable comments.

†ENS Paris-Saclay, Université Paris-Saclay, CEPS. *E-mail address:* cyril.rouault@universite-paris-saclay.fr

in the military. These terms are typically determined by firms, hospitals, or the military, often with minimal involvement from regulators. Despite this limited oversight, contract terms profoundly influence the welfare of market participants. This raises a critical question: does offering a broader range of terms to agents improve their welfare? This paper addresses this question and demonstrates that the answer is negative in most markets. Furthermore, we show that narrowing the range of available terms can lead to improved welfare for agents.

For concreteness, we use the terminology of doctors and hospitals to describe market participants, but our framework is applicable to a variety of contexts, such as school choice or housing markets.[1] Typically, a centralized mechanism allocates doctors to hospitals under specified contract terms, ensuring a stable allocation (Roth, 1982). For example, the *National Resident Matching Program* (NRMP) in the United States is a centralized system that matches medical students with residency training programs.[2] Hospitals offer terms and rank doctors for different specialties, while doctors express their preferences over the terms available at each hospital. A mechanism then determines the resulting allocation.

We adopt the many-to-one matching with contracts model (Hatfield and Milgrom, 2005) to analyze the effects of adding possible contract terms. Unlike the standard model, our framework allows each hospital to offer a set of terms to doctors. If a specific term t is available at a hospital h, then for every doctor, a contract involving the doctor, hospital h, and term t is available. Conversely, if a term t' is not available at hospital h, then no contract involving hospital h and term t' is available to any doctor. This framework aligns with the models used by Sönmez (2013) and Sönmez and Switzer (2013) in the context of cadet-branch matching.

Intuitively, one might expect that offering more options would benefit doctors, as suggested by the matching model proposed by Gale and Shapley (1962). In the marriage problem, when an additional woman enters the market, each man weakly prefers the new man-optimal stable allocation, and some men strictly prefer

[1] In school choice, contract terms can represent tuition fees or the academic pathways chosen by students, while in housing markets, terms may reflect factors such as family size or income.

[2] The NRMP processes over 48,000 applications for 38,000 positions annually, spanning more than 60 subspecialties. In this paper, we treat subspecialties as potential contract terms. https://www.nrmp.org/.

it if the new woman is not single.[3] However, using examples in which contract terms represent salaries (Echenique 2012, Schlegel 2015) we show that offering more terms can reduce doctors' well-being—even when one of the new terms is utilized.[4] The intuition behind these examples is that additional terms intensify competition among doctors, compelling some to accept less desirable contracts to secure employment. This welfare reduction can occur even when the added terms lie on the Pareto frontier (Echenique, 2012).

These findings challenge the conventional intuition that expanding options always benefits agents, prompting us to explore solutions that mitigate the negative impact of adding contract terms. In this regard, we propose an approach that results in an allocation that either Pareto improves the allocation of doctors or maintains the same allocation. This approach consists in withdrawing terms that are not used. However, we identify two limitations to this approach. First, the Pareto improvement is not always possible. Second, it is not possible to (strictly) improve the allocation of all doctors. This result also implies that adding terms cannot reduce the well-being of all doctors. More generally, our results highlight the importance of the design of contract terms in the well-being of agents in the market.

We then analyze conditions on agents' preferences that prevent welfare reductions when new terms are introduced, aiming to identify which markets are susceptible to this phenomenon. Through illustrative examples, we demonstrate that traditional approaches based on *acyclic* preference structures (Ergin, 2002; Pakzad-Hurson, 2023) fail to prevent welfare reductions. Instead, our results indicate that only *agent-lexicographic* preferences across all agents can guarantee no welfare loss. Agent-lexicographic preferences prioritize the agent over the contract term, treating the latter as secondary. Consequently, if even one agent in the market, whether a doctor or a hospital, does not have agent-lexicographic preferences, a reduction in doctors' welfare can occur. This finding highlights the vulnerability of many markets, as agent-lexicographic preferences impose a stringent condition.

[3]This result is introduced by Proposition 2 of Gale and Sotomayor (1985) and is further explored by Roth and Sotomayor (1992), who present these results as Theorems 2.25 and 2.26 (pp. 44-45).

[4]The key difference in this paper is that we are adding terms rather than agents.

Building on these theoretical insights, we explore markets where agents' preferences are *common* and *polarized*. This setting is particularly relevant for markets with salaries, in which doctors prefer higher salaries for a given hospital, while hospitals prefer lower salaries for a given doctor. We identify three necessary conditions to prevent welfare reductions. First, at least one of the added terms must be utilized. Second, the hospital offering the new terms must employ the same doctors. Third, the added terms must be preferred by doctors over the existing ones. Under these conditions, however, the hospital offering the new terms is worse off, raising questions about its incentives to introduce them.

In markets without contracts, it is well-known that hospitals can cooperate to improve their allocations (Roth, 1985). We show that cooperation is unnecessary when it comes to offering terms. Our findings suggest that if the mechanism used in the market leads to a hospital-optimal stable allocation, each hospital always has an incentive to expand its set of terms, regardless of the terms offered by other hospitals. This further highlights the importance of regulating the terms made available by hospitals.

Related Literature

This paper connects to several strands of the literature. First, it builds on the centralized matching literature initiated by Gale and Shapley (1962) and extended to include contracts by Crawford and Knoer (1981), Kelso and Crawford (1982), Fleiner (2003), and Hatfield and Milgrom (2005). Within this framework, many applications have been studied using contract terms as representations of agents' opportunities. For example, Sönmez and Switzer (2013) and Sönmez (2013) model the options available to cadets regarding the duration of their enlistment in the U.S. Army. Similarly, Hatfield and Kominers (2015) explore interns' choices across hospital departments, and Kominers and Sönmez (2016) analyze airline upgrade systems. Markets involving monetary transfers can also be modeled within this framework, with contract terms representing salaries. Echenique (2012) and Schlegel (2015) demonstrate that when firms' preferences satisfy the *substitutes* condition, the matching with contracts framework effectively captures labor markets with salaries. Our results apply to all markets analyzed in these contributions. While these studies provide significant insights into agents' welfare, none have addressed the design of contract terms. Hatfield and Kominers (2017) contribute to the literature by emphasizing the importance of contract design in many-to-many matching with contracts. Unlike their work, which focuses on restricting the set of contracts, we examine how limiting the set of available terms can improve doctors' welfare within stable allocations. This enables us to analyze the welfare impact of adding terms, a dimension absent from their contribution.

Our findings also contribute to the literature studying the role of preference structures in agents' welfare. Ergin (2002) shows that stable and efficient matchings exist when preferences are acyclic. Pakzad-Hurson (2023) extends this analysis to markets with contracts by introducing agent-lexicographic preferences as an additional condition alongside acyclicity to ensure stability and efficiency. Additionally, common and polarized preferences have been widely studied in job-matching contexts, especially concerning salaries. Kelso and Crawford (1982) and Roth (1984) examine these preferences in markets without contracts, while Echenique (2012), Kominers (2012), and Schlegel (2015) study their implications in the presence of contracts. To the best of our knowledge, our paper is the first to identify conditions on agents' preferences that prevent reductions in doctors' welfare when terms are added.

Finally, this paper contributes to the literature on incentives and welfare in matching markets. Much of the existing work in the literature focuses on choice functions within the matching with contracts framework. For example, Chambers and Yenmez (2017) and Yenmez (2018) analyze how choice functions influence stable allocations, with an emphasis on improving doctors' allocations. In contrast, we examine how the availability of contract terms affects doctors' welfare, while maintaining hospitals' incentives to choose their preferred contracts. Afacan (2017) investigate the cumulative offer process when doctors expand their set of acceptable contracts and its impact on welfare. Our approach diverges by exploring how expanding the set of terms available to doctors influences welfare, regardless of their acceptability. Furthermore, we provide insights into hospitals' incentives to offer terms to doctors.

The structure of the paper is as follows: Section 2 introduces the model, presents two motivating examples, and outlines our approach. In Section 3, we analyze the impact of adding terms on the set of stable allocations and welfare. Section 4 examines the conditions on preferences that prevent reductions in welfare. Section 5 studies hospital incentives. Section 6 concludes. Additional results on the structure of stable allocations are provided in Appendix A. All proofs are collected in Appendix B.

2 Model

2.1 Allocation Problem

There are finite sets $D = \{d_1, d_2, ..., d_n\}$ and $H = \{h_1, h_2, ..., h_m\}$ of *doctors* and *hospitals*, and a finite set $\mathcal{T} = \{t_1, t_2, ..., t_\ell\}$ of *terms*. There is a set $\mathcal{X}$ of *contracts* specifying relationships between doctor-hospital pairs, represented as $\mathcal{X} = D \times H \times \mathcal{T}$. Each contract $x \in \mathcal{X}$ is associated with a doctor $x_d \in D$, a hospital $x_h \in H$, and a term of their match $x_t \in \mathcal{T}$. Each doctor can sign at most one contract. The null contract, which indicates that the doctor or hospital has no contract, is denoted by $\varnothing$. For a set of contracts $X \subseteq \mathcal{X}$, we define $X_d \equiv \{x \in X : x_d = d\}$ as the set of contracts in X associated with doctor $d \in D$. Similarly, we define $X_h \equiv \{x \in X : x_h = h\}$ as the set of contracts in X associated with hospital $h \in H$. A set of contracts $X \subseteq \mathcal{X}$ is an *allocation* if each doctor $d \in D$ is involved in at most one contract, formally, $|X_d| \leq 1$.

Each hospital h offers a set of *available terms* $T_h \subseteq \mathcal{T}$. Let $T \equiv (T_h)_{h \in H}$ be the *vector of terms*. For relevance, we assume that $T_h \neq \emptyset$ for each hospital h. For each doctor $d \in D$, $\succ_d$ represents a *strict preference* relation over $\mathcal{X}_d \cup \{\varnothing\}$. Let $\succ_D \equiv (\succ_d)_{d \in D}$ denote the *preference profile of doctors*. A contract is *acceptable* if it is strictly preferred to the null contract and *unacceptable* if it is strictly dispreferred to the null contract. For each $d \in D$ and $X \subseteq \mathcal{X}$, we define the *chosen set* $C_d(X)$ as $C_d(X) \equiv \max_{\succ_d}[\{x \in X_d : x_t \in T_{x_h}\} \cup \{\varnothing\}]$.[5] In words, doctors choose their preferred contracts among those whose terms are available

[5] We use the notation $\max_{\succ_d}$ to indicate maximization with respect to the preferences of doctor d.

at the associated hospital, or the null contract if all contracts are unacceptable. Let $C_D(X) \equiv \bigcup_{d \in D} C_d(X)$ be the set of contracts chosen from X by doctors.

Hospitals can sign multiple contracts. Each hospital $h \in H$ has a strict preference relation $\succ_h$ on $2^{\mathcal{X}_h} \cup \{\varnothing\}$. Let $\succ_H \equiv (\succ_h)_{h \in H}$ denote the *preference profile of hospitals*. For each $h \in H$ and $X \subseteq \mathcal{X}$, we define the chosen set $C_h(X)$ as $C_h(X) \equiv \max_{\succ_h}[\{X' \subseteq X_h :$ for each $d \in D, |X'_d| \leq 1$, and for each $x \in X', x_t \in T_{x_h}\} \cup \{\varnothing\}]$.[6] In words, each hospital selects its preferred subset of contracts from X, ensuring that for the chosen contracts, the associated term is offered and that each doctor is involved in at most one selected contract. Let $C_H(X) \equiv \bigcup_{h \in H} C_h(X)$ be the set of contracts chosen from X by hospitals.

A *problem* is a tuple $(D, H, T, \succ_D, \succ_H)$. Let Π be the set of all problems. We fix $D, H, \succ_D$ and $\succ_H$ throughout the paper and denote a problem by $\pi(T)$. An allocation X is *feasible* if for each $x \in X$ we have $x_t \in T_{x_h}$. Let $\mathcal{F}(\pi(T))$ denote the *set of feasible allocation* for problem $\pi(T)$.

A doctor d's preferences $\succ_d$ over contracts implicitly define a preference relation $\succeq_d$ over allocations as follows: $X_d \succeq_d Y_d$ if and only if $X_d \succ_d Y_d$ or $X_d = Y_d$.[7] Similarly, hospital h's preferences $\succ_h$ over sets of contracts implicitly define a preference relation $\succeq_h$ over allocations as follows: $X_h \succeq_h Y_h$ if and only if $X_h \succ_h Y_h$ or $X_h = Y_h$.

We can now introduce the notion of *stability* for allocations.

Definition 1. An allocation $X \subseteq \mathcal{X}$ is *stable* if

(i) $C_D(X) = C_H(X) = X$,

(ii) there exists no hospital h, and set of contracts $Y \neq C_h(X)$ such that $Y = C_h(X \cup Y) \subseteq C_D(X \cup Y)$.

When (ii) is violated by some Y, we say that Y *blocks* X. We denote by $S(\pi(T))$ the set of stable allocations for problem $\pi(T)$.

A *mechanism* φ maps any problem to a feasible allocation, formally, $\varphi : \pi(T) \in \Pi \to \varphi(\pi(T)) \in \mathcal{F}(\pi(T))$. An allocation $X \subseteq \mathcal{X}$ *Pareto dominates* an allocation $Y \subseteq \mathcal{X}$ if for each doctor $d \in D, X_d \succeq_d Y_d$ and for at least one doctor $d' \in$

[6] Hospitals can only choose allocations, meaning that each doctor is concerned by at most one contract.

[7] In an allocation, each doctor is involved in at most one contract, ensuring that preferences over contracts determine preferences over allocations.

$D, X_{d'} \succ_{d'} Y_{d'}$. An allocation is *Pareto efficient* if it is not Pareto dominated by any other allocation. We utilize the notion of *lattice structure* to focus on optimal stable allocations. Given a problem $\pi(T)$, we say that an allocation $\overline{X} \in S(\pi(T))$ is the *doctor-optimal stable allocation* if every doctor weakly prefers $\overline{X}$ to all other stable allocation. Similarly, an allocation $\underline{X} \in S(\pi(T))$ is the *hospital-optimal stable allocation* if every hospital weakly prefers $\underline{X}$ to all other stable allocations.

2.2 Substitutes Condition

In this article, we assume that contracts are *substitutes* for hospitals.

Definition 2 (Hatfield and Milgrom, 2005)**.** Contracts are *substitutes* for h if there do not exist contracts $x, x' \in \mathcal{X}$ and a set of contracts $X \subseteq \mathcal{X}$ such that $x' \notin C_h(X \cup \{x'\})$ and $x' \in C_h(X \cup \{x, x'\})$.

The substitute condition ensures the monotonicity of the doctor-proposing *cumulative offer process* (COP hereafter),[8] thereby making renegotiation unnecessary.

Theorem 0. (Theorem 3 of Hatfield and Milgrom (2005)) Suppose contracts are substitutes for hospitals. Then, for a given vector of terms T, $S(\pi(T))$ is nonempty and forms a lattice. In addition, the COP generates the doctor-optimal stable allocation.

Note that since we consider strict preferences, the *irrelevance of rejected contracts* condition is not required (see Aygün and Sönmez, 2013).

2.3 Motivating Examples with Salaries

A common application of contract terms is salaries. Echenique (2012) demonstrates that matching with contracts is equivalent to matching with salaries if hospital preferences satisfy the substitute condition.[9] In the case of salaries, the Pareto frontier of contracts is one-dimensional, meaning that if a term benefits

[8]The COP was first introduced by Hatfield and Milgrom (2005). We introduce it formally in Appendix B.3.

[9]This result is introduced in the Theorem 1 of Echenique (2012).

the doctor, it necessarily becomes less favorable for the hospital. For simplicity, we omit the mention of doctors and hospitals from the contract notation when their identities are clear from the context (e.g., we write $(d,t) \succ_h (d',t')$ instead of $(d,h,t) \succ_h (d',h,t')$).

Example 1. Consider a problem where $D = \{d_1, d_2\}, H = \{h_1, h_2\}$, and terms are salaries with $\mathcal{T} = \{90, 100, 110\}$. The available salaries for each hospital are $T' = (T'_{h_1} = \{90, 100\}, T'_{h_2} = \{100\})$ and $T_{h_1} = \{90, 100, 110\}$. Contracts are substitutes for hospitals, and preferences are given as follows:

Doctor			
d_1	$(h_1, 110)$	$(h_1, 100)$	$(h_1, 90)$
d_2	$(h_1, 110)$	$(h_2, 100)$	

Table 1: Preference profile of doctors

Hospital				
h_1	$(d_1, 90)$	$(d_2, 110)$	$(d_1, 100)$	$(d_1, 110)$
h_2	$(d_2, 100)$			

Table 2: Preference profile of hospitals

In the preference profile, we only note the acceptable contracts for both doctors and hospitals. Thus, in Table 1, the contracts $(d_2, h_1, 90)$ and $(d_2, h_1, 100)$ are not acceptable to doctor d_2, and the contract $(d_1, h_2, 100)$ is not acceptable to doctor d_1.

Considering the problem $\pi(T')$, where the salary of 110 is unavailable at h_1. The doctor-optimal stable allocation is:

$$\overline{X}' = \{(d_1, h_1, 100), (d_2, h_2, 100)\}.$$

Now suppose that hospital h_1 offers an additional salary of 110. Consider problem $\pi(T_{h_1}, T'_{-h_1})$, where the new doctor-optimal stable allocation is:

$$\overline{X} = \{(d_1, h_1, 90), (d_2, h_2, 100)\}.$$

Although hospital h_1 offers a higher salary, d_1's wage decreases to 90. This example illustrates the significance of available terms and salaries in determining the doctor-optimal stable allocation. Notably, the salary of 110 is on the Pareto frontier. The interpretation is that the introduction of the higher salary creates competition between the doctors, both of whom aim for the higher salary, which ultimately forces d_1 to accept a lower salary in order to secure a position.[10]

Example 1 carries important policy implications. By offering a salary of 110, hospitals h_1 can emphasize that it provides a higher salary compared to other hospitals. This could strengthen the hospital's image as an employer of choice. However, the allocation is preferred by h_1 without changing the doctor assigned to it. Example 2 complements Example 1 by showing that even when the added term is utilized, a reduction in well-being can still occur.

Example 2. Consider a problem where $D = \{d_1, d_2, d_3\}$, $H = \{h_1, h_2, h_3\}$, and terms are salaries with $\mathcal{T} = \{100, 110\}$. The available salaries for each hospital are $T' = (T'_{h_1} = \{100\}, T'_{h_2} = \{100\}, T'_{h_3} = \{100\})$ and $T_{h_1} = \{100, 110\}$. Contracts are substitutes for hospitals, and preferences are given as follows:

Doctor			
d_1	$(h_1, 110)$	$(h_1, 100)$	$(h_2, 100)$
d_2	$(h_2, 100)$	$(h_1, 110)$	$(h_1, 100)$
d_3	$(h_1, 110)$	$(h_3, 100)$	$(h_1, 100)$

Table 3: Preference profile of doctors

Hospital						
h_1	$(d_2, 100)$	$(d_3, 100)$	$(d_2, 110)$	$(d_3, 110)$	$(d_1, 100)$	$(d_1, 110)$
h_2	$(d_1, 100)$	$(d_2, 100)$				
h_3	$(d_3, 100)$					

Table 4: Preference profile of hospitals

The doctor-optimal stable allocation of $\pi(T')$ is:

$$\overline{X}' = \{(d_1, h_1, 100), (d_2, h_2, 100), (d_3, h_3, 100)\}.$$

[10]This interpretation relates to the concept of the *Reserve army of labour* introduced by Engels and theorized by Marx (1867).

Now, if hospital h_1 offers an additional salary of 110 in problem $\pi(T_{h_1}, T'_{-h_1})$, the new doctor-optimal stable allocation is:

$$\overline{X} = \{(d_2, h_1, 110), (d_1, h_2, 100), (d_3, h_3, 100)\}.$$

Doctor d_2 accepts the new term but is worse off, while hospitals h_1 and h_2 benefit.

While Example 1 shows that the introduction of a higher salary results in a reduction in the salary of one doctor, Example 2 demonstrates that introducing a higher salary can reduce the welfare of doctors and attract a doctor preferred by the hospital. Even though the salary offered by h_1 is higher, it prefers d_2 over d_1 for every available salary.

These examples highlight the two forces at play in such scenarios: reducing the salary of doctors and attracting preferred doctors. While offering a higher wage may help hospitals attract their preferred doctors, as shown in Example 2, doctors may also be employed by hospitals not because of the higher salary, but because they have lost their previous job.

2.4 Sub-Problems and Preliminary Result

In this section, we introduce our approach and define the *sub-problems*. A problem $\pi(T)$ is defined in Section 2.1. A sub-problem of $\pi(T)$ is a problem that mirrors $\pi(T)$, except that the set of available terms for at least one hospital is reduced. We denote a sub-problem of $\pi(T)$ with an apostrophe on the vector of terms. Formally, given a problem $\pi(T) \in \Pi$, $\pi(T') \in \Pi$ is a sub-problem of $\pi(T)$ if $\pi(T) \equiv (D, H, T, \succ_D, \succ_H)$ and $\pi(T') \equiv (D, H, T', \succ_D, \succ_H)$ with $T' \subset T$.[11] Let $\tilde{\Pi}(\pi(T))$ denote the *set of sub-problems* of $\pi(T)$. We can study the effects of adding or withdrawing one or more terms for hospitals in a given problem. Given a sub-problem $\pi(T') \in \tilde{\Pi}(\pi(T))$, we denote by $\overline{X}'$ the doctor-optimal stable allocation and $\underline{X}'$ the hospital-optimal stable allocation.

When the set of available terms for a hospital h is reduced while all other hospitals retain the same set of terms, we represent this sub-problem as $\pi(T'_h, T_{-h}) \in$

[11] As stated in Section 2, we assume that T'_h is non-empty for each $h \in H$.

$\tilde{\Pi}(\pi(T))$. We can also consider the case where the set of terms available to hospital h is expanded. To do so, we define the sub-problem $\pi(T')$ by expanding the set of terms for hospital h only, leading to the problem $\pi(T) \equiv \pi(T_h, T'_{-h})$, where $T'_h \subset T_h$.

Proposition 1 formalizes the observations presented in Example 1 and Example 2 for any given vector of terms. It establishes that the introduction of additional terms can (i) diminish the welfare of certain doctors and (ii) do so even in cases where one of the added terms is actively utilized. Additionally, it states the impact of a reduction in doctors' welfare on hospital allocation.

Proposition 1. There exist problems $\pi(T) \in \Pi$ and a sub-problem $\pi(T') \in \tilde{\Pi}(\pi(T))$ such that:

(i) There exists a non-empty subset of doctor $D' \subset D$ such that for each $d \in D', \overline{X}'_d \succ_d \overline{X}_d$. If $D' \neq \emptyset$, then there exists a non-empty subset of hospital $H' \subseteq H$ such that for each $h \in H', \overline{X}_h \succ_h \overline{X}'_h$.

(ii) There exists a hospital $h \in H$ and a term $t \in T_h \setminus T'_h$ such that there exists $x \in \overline{X}_h, x_t = t$, and $\overline{X}'_{x_d} \succ_{x_d} \overline{X}_{x_d}$.

Proposition 1 thus states that when additional terms are introduced, if at least one doctor is worse off, then at least one hospital prefers the new doctor-optimal stable allocation. Note that the doctors in D' are involved in contracts with the hospitals in H' under the allocation $\overline{X}'$. Consequently, when the allocation deteriorates for doctors, the hospitals to which these doctors are allocated under $\overline{X}'$ prefer their new allocation under $\overline{X}$.[12]

In Examples 1 and 2, terms are added for only one hospital, but there may be policies that add terms available for specific sectors, thus affecting several hospitals (or firms in the job market). These policies could include salary increases or the possibility of longer working hours, for instance. A decrease in doctors' welfare is possible when the set of available terms is extended for a hospital, making it evident that when new terms become available for several hospitals, a decrease in welfare is also possible.

[12] Otherwise, they would prefer to rematch, which would contradict the stability property.

3 Stability and Welfare

To investigate the impact of the addition of terms on doctors' well-being, we begin by examining the effect on the set of stable allocations.

3.1 Set of Stable Allocation

From Example 1, it is straightforward that by offering the term 110, an allocation blocks the doctor-optimal stable allocation:

- $C_{d_2}(\{(d_2, h_1, 110), \overline{X}'_{d_2}\}) = (d_2, h_1, 110)$, and
- $C_{h_1}(\{(d_2, h_1, 110), \overline{X}'_{h_1}\}) = (d_2, h_1, 110)$.

A similar reasoning applies in Example 2. Following this observation, if a stable allocation in a given problem becomes unstable when the set of available terms for a hospital is expanded, then some of the newly added terms are used in contracts forming a blocking allocation.[13] The set of stable allocations can thus be reduced by extending the set of available terms for a hospital. It is also clear that new stable allocations can be obtained by adding new terms. Our approach is, therefore, to consider sub-problems by reducing the set of available terms. Our first main result states that any allocation that is stable in the problem $\pi(T)$ remains stable in each sub-problem of $\pi(T)$ where the allocation is feasible.[14]

Theorem 1. For any problem $\pi(T) \in \Pi$,

(i) Suppose that $X \in S(\pi(T))$. For each $\pi(T') \in \tilde{\Pi}(\pi(T))$, if $X \in \mathcal{F}(\pi(T'))$, then $X \in S(\pi(T'))$.

(ii) Suppose $\pi(T'), \pi(T'') \in \tilde{\Pi}(\pi(T))$. Then, $S(\pi(T')) \cap S(\pi(T'')) \subseteq S(\pi(T' \cup T''))$.

The converse of Theorem 1 (i) is false.[15] Theorem 1 (ii) states that if we consider two sub-problems $\pi(T')$ and $\pi(T'')$ of a problem $\pi(T)$, then all allocations

[13] Lemma 2 formalizes and proves this statement for any stable allocation.

[14] This is related to Proposition 3 of Hatfield and Kominers (2017). While Hatfield and Kominers (2017) consider a subset of contracts, we consider a subset of available terms to guarantee the stability of an allocation. The advantage of our approach is that we can identify the vector of terms needed for the stability of X, which is absent from their formulations.

[15] Consider $t \notin T'_h$ such that for each allocation $X \in S(\pi(T'))$, we have $(d, t) \succ_h X_h$, and $(h, t) \succ_d X_d$. Now consider T^* such that $t \in T^*_h$ and $T' \subset T^*$. An allocation with the contract $x = (d, h, t)$ blocks X.

that are stable in both sub-problems are also stable in a sub-problem where we consider the union of the set of available terms for hospitals. Although this condition is restrictive, it is necessary to ensure the stability of allocations when the set of terms offered is expanded for certain hospitals. We present an additional result on the structure of the set of stable allocations in Appendix A.

3.2 Reduction of Available Terms and Pareto Improvement

From Theorem 1 it is evident that to guarantee the stability of an allocation when the set of available terms is reduced, it is only necessary to maintain its feasibility. Hence, if an allocation is stable and some terms are *not used*, they can be removed, and in the resulting sub-problem, the allocation remains stable. Formally, given an allocation X, a term t is not used by hospital h if there exists no $x \in X$ such that $x_h = h$ and $x_t = t$. We now apply this reasoning to doctor-optimal stable allocations.[16] By withdrawing terms, some contracts that might constitute blocking allocations for some allocations are no longer feasible. It is, therefore, possible to achieve a Pareto improvement for doctors following the reduction of the set of available terms for a hospital. In Example 1, the salary of 110 is not used in $\overline{X}$. By withdrawing it, we achieve a Pareto improvement: d_1's salary increases and d_2's allocation remains unchanged under $\overline{X}'$.[17] The following corollary formalizes this point.

Corollary 1. For any problem $\pi(T) \in \Pi$, suppose there exists $h \in H$ such that h has at least one term not used in the allocation $\overline{X}$. Then, there exists $T'_h \subset T_h$ such that $\pi(T'_h, T_{-h}) \in \tilde{\Pi}(\pi(T))$, and either $\overline{X}'$ Pareto dominates $\overline{X}$, or $\overline{X} = \overline{X}'$.

The main implication of Corollary 1 is that limiting the terms available to

[16] Note that this result can be reformulated using the property of *irrelevance of unused terms* in a stable allocation. Hirata et al. (2023) define the *irrelevance of unchosen contracts* by adding contracts available to one doctor.

[17] When terms are withdrawn, doctors are no longer able to propose the associated contract to the hospital. Kesten (2010) shows, in a setting without contracts, that student applications to schools reduce the welfare of other students. Following Kesten's 2010 terminology, these contracts can be considered as an *interrupter*. However, in our model, when a term is withdrawn from a hospital's set of available terms, it is withdrawn for all doctors.

doctors can improve their welfare. In the remainder of this section, we identify the two major limitations of this approach. First, Pareto improvement is not always possible and requires the removal of specific terms. Consider Example 1 and T_{h_1}; two terms are not used: 100 and 110. By removing term 100, such that $T''_{h_1} = \{90, 110\}$, the allocation $\overline{X}'$ would not be feasible and, therefore, not stable in $\pi(T''_{h_1}, T_{-h_1})$. The following example highlights the potential impossibility of achieving a Pareto efficient allocation by removing available terms.

Example 3. Consider a problem where $D = \{d_1, d_2\}$, $H = \{h_1, h_2\}$ and $\mathcal{T} = \{t, t'\}$. The vector of terms is given by $T = (T_{h_1} = \{t, t'\}, T_{h_2} = \{t\})$. Contracts are substitutes for hospitals, and preferences are given as follows:

Doctor		
d_1	(h_1, t)	(h_1, t')
d_2	(h_1, t)	(h_2, t)

Table 5: Preference profile of doctors

Hospital			
h_1	(d_1, t')	(d_2, t)	(d_1, t)
h_2	(d_2, t)		

Table 6: Preference profile of hospitals

The doctor-optimal stable allocation of $\pi(T)$ is:

$$\overline{X} = \{(d_1, h_1, t'), (d_2, h_2, t)\}.$$

There is only one allocation X' that Pareto dominates $\overline{X}$, such that

$$X' = \{(d_1, h_1, t), (d_2, h_2, t)\}.$$

Although X' Pareto dominates $\overline{X}$ and is Pareto efficient, there is no $T' \subset T$ such that $X' \in S(\pi(T'))$. This is because the term to remove is t for hospital h_1, but removing it would prevent the feasibility of allocation X' because doctor d_1 is assigned to h_1 with term t. To maintain feasibility, adjustments would need to be made to the terms offered to each doctor. For example, doctor d_2 could be prevented from using term t at hospital h_1, while doctor d_1 retains the option. However, such adjustments are beyond the scope of our current analysis.

The second limitation is that it is not possible to improve the allocation for all doctors by removing terms. Therefore, there is no sub-problem in which all doctors are strictly better off.

Theorem 2. For any problem $\pi(T) \in \Pi$, there is no $\pi(T') \in \tilde{\Pi}(\pi(T))$ such that for each $d \in D, \overline{X}'_d \succ_d \overline{X}_d$.

The symmetrical interpretation of Theorem 2 is that expanding the set of available terms never reduces the welfare of every doctor. However, with $|D| = n$, there are problems in which, when a term is added, $n-1$ doctors are worse off. Theorem 2 follows from the weak Pareto optimality of the doctor-optimal stable allocation.[18]

4 Preference Domains

Thus far, we have focused on the set of stable allocations and the well-being of doctors. In this section, we address the conditions on the preference profile that prevent a decrease in welfare when terms are added. Our main objective is to identify the markets where this reduction in doctors' welfare does not occur.

4.1 Agent-Lexicographic Preferences

First, we consider *agent-lexicographic* preferences. We define the preferences of a hospital h as *doctor-lexicographic* if, for any two acceptable contracts $x, x' \in X$, where $x_d = x'_d = d$ and $x_h = x'_h = h$, there is no contract x'' such that $x''_d \neq d$ and $x''_h = h$ such that $x \succ_h x'' \succ_h x'$. Similarly, preferences of doctor d are *hospital-lexicographic* if, for any two acceptable contracts $x, x' \in X$ where $x_d = x'_d = d$ and $x_h = x'_h = h$, there is no contract x'' with $x''_d = d$ and $x''_h \neq h$ such that $x \succ_d x'' \succ_d x'$. For convenience, we say that the preferences of an agent

[18] See Theorem 2.27 of Roth and Sotomayor (1992).

$a \in D \cup H$ are agent-lexicographic if, for $a \in D$, the preferences $\succ_a$ are hospital-lexicographic, and for $a \in H$, the preferences $\succ_a$ are doctor-lexicographic. We denote by $\succ_{-a} \equiv (\succ_{a'})_{a' \in D \cup H \setminus \{a\}}$ the preference profile of all agents other than a.

When all agents have lexicographic preferences, terms serve as secondary criteria, with the primary focus being the assignment of doctors to hospitals. In these markets, the addition or withdrawal of terms for specific hospitals does not alter the assignment of doctors to hospitals in the doctor-optimal stable allocation.[19] We introduce a notation to denote *the set of doctors assigned to h under allocation X*, that is $D_h(X) \equiv \{d \in D : x \in X_h \text{ and } x_d = d\}$.

Lemma 1. Let $\pi(T) \in \Pi$ be a problem where each agent $a \in D \cup H$ has agent-lexicographic preferences. Then, for any $\pi(T') \in \tilde{\Pi}(\pi(T))$ we have for each $h \in H, D_h(\overline{X}) = D_h(\overline{X}')$.

This follows immediately since for each $h \in H$, T'_h is non-empty. Thus, we omit the proof. The following result establishes that if at least one agent (doctor or hospital) does not have agent-lexicographic preferences, it is always possible to construct a preference profile for the other agents, where all agents have agent-lexicographic preferences, such that adding available terms will reduce the welfare of some doctors.

Theorem 3. If $\succ_a$ are not agent-lexicographic for $a \in D \cup H$, then, there exist $\succ_{-a}$ such that for each $a' \in D \cup H \setminus \{a\}$, $\succ_{a'}$ are agent-lexicographic, and a sub-problem $\pi(T') \in \tilde{\Pi}(\pi(T))$, where for some $d \in D$, $\overline{X}'_d \succ_d \overline{X}_d$.

Lexicographic preferences for all agents impose restrictive conditions. When terms represent salaries, agent-lexicographic preferences imply that doctors are willing to work for any salary (potentially negative), and hospitals are willing to offer any salary (even excessively high) to a doctor rather than engaging another. Such conditions are rarely observed in practice. This implies that welfare reductions when terms are added may occur in any market where at least one agent does not have lexicographic preferences.

[19] In this context, the market can be viewed as one without contracts. To determine the doctor-optimal stable allocation, each doctor selects her preferred terms from those offered by the hospital to which she is assigned.

In the remainder of this section, we relax the constraint by considering only markets where hospitals have doctor-lexicographic preferences. Example 4 illustrates the additional conditions that need to be imposed to prevent a reduction in well-being.

Example 4. Consider a problem where $D = \{d_1, d_2\}$, $H = \{h_1, h_2\}$ and $\mathcal{T} = \{t, t'\}$. The vector of terms is given by $T' = (T'_{h_1} = \{t\}, T'_{h_2} = \{t\})$ and $T_{h_1} = \{t, t'\}$. Contracts are substitutes for hospitals, and hospitals' preferences are doctor-lexicographic:

Doctor			
d_1	(h_1, t')	(h_2, t)	(h_1, t)
d_2	(h_1, t)	(h_1, t')	(h_2, t)

Table 7: Preference profile of doctors

Hospital				
h_1	(d_1, t)	(d_1, t')	(d_2, t')	(d_2, t)
h_2	(d_1, t)	(d_2, t)		

Table 8: Preference profile of hospitals

Consider the sub-problem $\pi(T')$ and the problem $\pi(T_{h_1}, T'_{-h_1})$. The doctor-optimal stable allocations are given by $\overline{X}'$ and $\overline{X}$ respectively:

$$\overline{X}' = \{(d_2, h_1, t), (d_1, h_2, t)\}, \quad \overline{X} = \{(d_1, h_1, t'), (d_2, h_2, t)\}.$$

The added term is used by doctor d_1, who strictly prefers $\overline{X}$ to $\overline{X}'$, while d_2 strictly prefers $\overline{X}'$ to $\overline{X}$.[20] Intuitively, when hospital preferences are doctor-lexicographic, offering new terms can attract new doctors (doctor d_1 in Example 4), while replacing others who are now worse off (doctor d_2 in Example 4). Based on this observation, it is clear that if the set of doctors employed by hospital h

[20]Note that $\overline{X}'$ and $\overline{X}$ are Pareto efficient. Pakzad-Hurson (2023) examines the connection between efficiency and stability, identifying acyclicity and student-lexicographic preferences as the necessary and sufficient conditions for the existence of a stable and Pareto efficient allocation. In our framework, the latter corresponds to doctor-lexicographic preferences. In Example 4, hospital preferences are doctor-lexicographic and *homogeneous*. Even when imposing conditions on cycles in hospital preferences, adding terms can reduce doctors' welfare.

remains unchanged between $\overline{X}$ and $\overline{X}'$, no doctor will be worse off.[21] The following proposition formalizes this point.

Proposition 2. For any problem $\pi(T) \in \Pi$ where each hospital has doctor-lexicographic preferences, consider $h \in H$, $T'_h \subset T_h$ and the sub-problem $\pi(T'_h, T_{-h})$.

(i) If $D_h(\overline{X}) = D_h(\overline{X}')$, then there is no $d \in D$ such that $\overline{X}'_d \succ_d \overline{X}_d$.

(ii) If for a term $t \in T_h \setminus T'_h$ there exists $x \in \overline{X}_h$, such that $x_t = t$, then there exists a non-empty subset of doctors $D' \subset D$ such that for each $d' \in D', \overline{X}_{d'} \succ_{d'} \overline{X}'_{d'}$.

Proposition 2 states that for any problem where hospitals have doctor-lexicographic preferences, if the set of available terms for a hospital h is reduced and (i) the set of doctors assigned to h remains unchanged after the reduction, no doctor is better off. The symmetric interpretation is that if the set of terms is extended for hospital h and the set of doctors assigned to h is the same, then no doctor is worse off. This is related to part (i) of Proposition 1. Additionally, (ii) if at least one withdrawn term was previously used, some doctors will strictly prefer the original doctor-optimal stable allocation. Symmetrically, if one of the added terms is used, then at least one doctor is (strictly) better off. This is related to part (ii) of Proposition 1. Lemma 1, Theorem 3 and Proposition 2 (i) together provide the condition, in the maximal domain sense, that prevents the decrease in well-being when terms are added for some hospitals.

4.2 Common and Polarized Preferences

We now discuss *common and polarized preferences* for doctors and hospitals. A direct application of this domain is salaries, as illustrated in Examples 1 and 2. Doctors prefer higher salaries from a given hospital, while hospitals prefer to offer lower salaries to doctors.

We denote $t \gg_D t'$ if, for a given hospital $h \in H$, for each $d \in D, (h, t) \succ_d (h, t')$. We use the symmetric notation $t \gg_H t'$ for hospitals to indicate that for a given doctor $d \in D$, for each hospital $h \in H, (d, t) \succ_h (d, t')$. We say that *doctors'*

[21] Note that we do not impose any restrictions on the ranking of doctors for a given term. In Example 4, hospital h_1 prefers term t' to term t for doctor d_1 and term t to term t' for doctor d_2. Imposing the same ranking of terms for all doctors does not alter the results.

preferences are common if, for any $t, t' \in \mathcal{T}$, with $t \neq t'$, and any $h \in H$ we have $t \gg_D t'$. Similarly, *hospitals' preferences are common* if, for any $t, t' \in \mathcal{T}$, with $t \neq t'$, and for any $d \in D$, we have $t \gg_H t'$. The *preferences of doctors and hospitals are common and polarized* if, for any $t, t' \in \mathcal{T}$, with $t \neq t'$, $t \gg_D t'$ if and only if $t' \gg_H t$. We say that the term t is *preferred* over t' by doctors (hospitals) if doctors' (hospitals') preferences are common and $t \gg_D t'$ ($t \gg_H t'$).

Example 2 illustrates that even if an added term is used, the doctor who uses it may be worse off. To address this, we impose a stronger condition to ensure that when terms preferred by doctors are added, if any added term is used, some doctors will be better off. This condition requires that the set of doctors employed by h remains unchanged when the set of terms offered by h is extended.

Proposition 3. For any problem $\pi(T) \in \Pi$ such that the preferences of doctors and hospitals are common and polarized, consider $h \in H$, $T'_h \subset T_h$ and the sub-problem $\pi(T'_h, T_{-h})$. Then, if

(i) there exists a term $t \in T_h \setminus T'_h$ and a contract $x \in \overline{X}_h$ such that $x_t = t$,

(ii) $D_h(\overline{X}) = D_h(\overline{X}')$, and

(iii) each $t \in T_h \setminus T'_h$ is preferred by doctors to any $t' \in T'_h$,

there is no $d \in D$ such that $\overline{X}'_d \succ_d \overline{X}_d$ and there exists a non-empty subset of doctors $D' \subset D$ such that for each $d \in D', \overline{X}_d \succ_d \overline{X}'_d$.

Proposition 3 states that if the preferences of hospitals and doctors are common and polarized, then when preferred terms are added and at least one of these terms is used in the doctor-optimal stable allocation, if the set of doctors hired by h remains unchanged, some doctors strictly prefer the new doctor-optimal stable allocation, and no doctor is worse off. The same reasoning applies symmetrically when considering the withdrawal of terms.

In Example 1, conditions (ii) and (iii) are satisfied, but not (i). In Example 2, conditions (i) and (iii) are satisfied, but not (ii). The key point is that the added terms are preferred over the currently available terms. By assumption, for each $h \in H, T'_h \neq \emptyset$. Therefore, if the set of doctors assigned to h remains unchanged, it implies that at least one doctor has a preferred contract with h. Assuming that contracts are substitutable for hospitals, no doctor is worse off.

However, preference polarization is crucial to prevent a reduction in doctors' welfare. Example 5 illustrates that when preferences are common but not polarized, extending the set of available terms can reduce doctors' welfare, even if the conditions identified in Proposition 3 are satisfied.[22]

Example 5. Consider a problem where $D = \{d_1, d_2\}$, $H = \{h_1\}$ and $\mathcal{T} = \{t, t', t''\}$. The vectors of terms are given by $T' = (T'_{h_1}) = (\{t, t''\})$ and $T = (T_{h_1}) = (\{t, t', t''\})$. Contracts are substitutes for the hospital, and the preferences of both the hospital and the doctors are common:

Doctor			
d_1	(h_1, t')	(h_1, t'')	(h_1, t)
d_2	(h_1, t')	(h_1, t'')	(h_1, t)

Table 9: Preference profile of doctors

Hospital								
h_1	(d_1, t)	$((d_1, t'), (d_2, t))$	(d_1, t')	(d_2, t)	(d_2, t')	$((d_1, t''), (d_2, t''))$	(d_1, t'')	(d_2, t'')

Table 10: Preference profile of hospital h_1

The preferences for the terms are as follows: $t' \gg_D t'' \gg_D t$ and $t \gg_H t' \gg_H t''$. Consider the sub-problem $\pi(T')$ and the problem $\pi(T)$. The doctor-optimal stable allocations are given by $\overline{X}'$ and $\overline{X}$ respectively:

$$\overline{X}' = \{(d_1, h_1, t''), (d_2, h_1, t'')\}, \quad \overline{X} = \{(d_1, h_1, t'), (d_2, h_1, t)\}.$$

Doctor d_1 prefers $\overline{X}_{d_1}$ over $\overline{X}'_{d_1}$, while doctor d_2 prefers $\overline{X}'_{d_2}$ over $\overline{X}_{d_2}$. The added term t' is preferred by doctors over the currently available terms (i.e., t and t''), t' is used by d_1, and the set of doctors assigned to h_1 remains the same.

Proposition 4. For any problem $\pi(T) \in \Pi$ where the preferences of doctors and hospitals are common, consider $h \in H$, $T'_h \subset T_h$ and the sub-problem $\pi(T'_h, T_{-h})$. Then, if

(i) there exists a term $t \in T_h \setminus T'_h$ and a contract $x \in \overline{X}_h$ such that $x_t = t$,

(ii) $D_h(\overline{X}) = D_h(\overline{X}')$, and

[22]Example 5 also illustrates the proof of Proposition 3.

(iii) each $t \in T_h \setminus T'_h$ is preferred by doctors to any $t' \in T'_h$,

there exists a non-empty subset of doctors $D' \subset D$ such that for each $d \in D', \overline{X}_d \succ_d \overline{X}'_d$.[23]

Thus, the common preferences of hospitals and doctors ensure that when the conditions are met, at least one doctor is better off. However, other doctors may be worse off.

5 Hospital Incentives

Thus far, we have not considered hospitals' preferences regarding allocations, which may play a crucial role in their incentives to offer terms. Referring to Proposition 3 and the framework of common and polarized preferences, when the conditions are satisfied, the interpretation is as follows: Doctors prefer terms that are less favorable to hospitals. Hospital h offers such terms and continues to employ the same doctors. Since at least one of these terms is used, and given the substitutability of contracts for hospitals, it follows directly that h will be worse off. The following proposition formalizes the impact of offering additional terms on hospitals' preferences under the hospital-optimal stable allocation.

Proposition 5. For any problem $\pi(T) \in \Pi$, consider $h \in H$, $T'_h \subset T_h$ and the sub-problem $\pi(T'_h, T_{-h})$, then $\underline{X}_h \succeq_h \underline{X}'_h$. Furthermore, if for a term $t \in T_h \setminus T'_h$ there exists $x \in \underline{X}_h, x_t = t$, then $\underline{X}_h \succ_h \underline{X}'_h$.

Proposition 5 states that when a hospital h extends the set of terms it offers, it weakly prefers the new hospital-optimal stable allocation to the former one. The preference is strict if one of the added terms is used. Consequently, if a mechanism produces the hospital-optimal stable allocation, hospitals have an incentive to extend the set of terms they offer. Note, however, that this analysis does not consider the impact of adding terms on other hospitals' preferences or their incentives to offer terms. Roth (1985) demonstrates that hospitals can cooperate by misreporting their preferences to achieve a preferred allocation. In contrast, we

[23]The proof of Proposition 4 directly follows from the proof of Proposition 3.

show that by offering additional terms, hospitals can achieve a preferred allocation without the need for cooperation. Regarding the impact on other hospitals' preferences, three cases are possible. Consider a hospital $h' \in H \setminus \{h\}$ under the sub-problem $\pi(T'_h, T_{-h})$ and the problem $\pi(T)$:

- Case 1: $\underline{X}_{h'} \succ_{h'} \underline{X}'_{h'}$. For instance, the set of doctors employed by h may differ, and some doctors previously employed by h in $\underline{X}'_h$ may now be available to work for h'. The contracts associated with these doctors are preferred by h' over those in $\underline{X}_{h'}$.
- Case 2: $\underline{X}'_{h'} \succ_{h'} \underline{X}_{h'}$. For instance, an allocation containing a contract using an added term may block the allocation of h' in $\pi(T)$. Alternatively, doctors in $D_{h'}(\underline{X}')$ may now be included in $D_h(\underline{X})$; in other words, a doctor previously hired by h' is now hired by h.
- Case 3: $\underline{X}_{h'} = \underline{X}'_{h'}$. The allocation of h' is unchanged.

In some markets, agents' preferences are determined by external factors, such as laws or regulations. For example, priorities are considered instead of preferences in school choice or in the approach by Sönmez and Switzer (2013). While these priorities are fixed, schools may offer additional terms to attract better students.[24] By fixing preferences, we explore the potential for hospitals to manipulate their reported sets of available terms.

We define a *strategy with fixed preferences for a hospital* h as reporting a set of available terms $T_h \subseteq \mathcal{T}$. Hospital strategies naturally extend to the vector of available terms T. A mechanism φ is *term strategy-proof* if, for each $\pi(T) \in \Pi$, no hospital $h \in H$ and $\hat{T}_h \neq T_h$ exist such that $\varphi(\pi(\hat{T}_h, T_{-h})) \succ_h \varphi(\pi(T))$.

Proposition 6. Suppose that for each hospital h, preferences $\succ_h$ are fixed. The COP is not term strategy-proof.

When a hospital does not offer a term, this can be interpreted as making all contracts associated with that term unacceptable to the hospital. However, the key difference is that by offering certain terms, hospitals can achieve a preferred allocation even without using those terms. For instance, in Example 1, h_1 offers a salary of 110, does not use it, but achieves a preferred allocation.

[24]For example, military branches may offer alternative durations of service to recruit higher-quality cadets. Although Sönmez and Switzer (2013) considers only two possible terms, branches might have an incentive to offer more options.

6 Conclusion

In allocation problems without contracts, offering more alternatives benefits one side of the market. However, we show that when hospitals offer additional contract terms to doctors, it can reduce doctors' welfare. By analyzing the impact of adding terms, we demonstrate that reducing doctors' alternatives either improves welfare or preserves the doctor-optimal stable allocation. This highlights the need for regulation in the design of terms offered by hospitals.

We identify conditions on agents' preferences that prevent welfare reductions when terms are added, showing that preferences must be agent-lexicographic in the maximal domain sense. This result implies that welfare reductions can occur in many markets. Specifically, in the job market, where terms often represent salaries, we identify the conditions necessary to avoid reducing doctors' welfare. However, when these conditions are satisfied, the hospital offering new terms is worse off. We then study the incentives for hospitals to offer terms and show that when a mechanism leads to the hospital-optimal stable allocation, hospitals are incentivized to expand the set of terms they offer. In addition, even when hospital preferences are fixed, such manipulation remains possible.

Future research could explore hospital strategies under alternative stable allocations. Another approach would be to study the conditions or structure of vectors of terms that reduce the set of stable allocations to a singleton, potentially the hospital-optimal stable allocation. Our contribution also motivates empirical applications by assessing the welfare impact of adding terms.

A Additional Results

A.1 Decomposition of Stable Allocations Set

This section offers additional insights into the structure of the stable allocation set. As highlighted in Section 3.1 and Theorem 1, adding terms can render some allocations unstable. Theorem 4 identifies a condition that ensures stability. We introduce a specific *constraint* that limits the number of terms that can be used. Let $\mathcal{U}(\pi(T)) \equiv \{X : X \in \mathcal{F}(\pi(T))$ for each $h \in H,$ for any $x, x' \in X_h, x_t = x'_t\}$

denote the constraint that limits each hospital to using only one term. This constraint applies, for example, when regulations require hospitals to adopt a uniform term, such as identical salaries for all doctors. Additionally, let $\mathcal{A}(\pi(T)) \equiv \{X : X \in \mathcal{F}(\pi(T)) \text{ for each } h \in H, X_h \succ_h \varnothing\}$ represent the set of allocations *acceptable* to hospitals.

Theorem 4 states that if, for every acceptable allocation, each hospital uses at most one term (i.e., $\mathcal{A}(\pi(T)) \subseteq \mathcal{U}(\pi(T))$), the set of stable allocations can be decomposed as follows: For any problem $\pi(T)$, $S(\pi(T))$ is the intersection, for each $h \in H$, of the union of each term in T_h, fixing the set of available terms for other hospitals T_{-h}.

Theorem 4. Suppose $\pi(T) \in \Pi$ is a problem such that $\mathcal{A}(\pi(T)) \subseteq \mathcal{U}(\pi(T))$. Then,

$$S(\pi(T)) = \bigcap_{h \in H} \left(\bigcup_{t \in T_h} S(\pi(\{t\}, T_{-h})) \right).$$

The condition imposed by Theorem 4 is naturally satisfied in scenarios where each hospital is restricted to using a single term or where hospitals can only employ one doctor. By decomposing the set of stable allocations into sub-problems, it becomes possible to verify whether any stable allocation remains stable as additional terms are introduced. To illustrate Theorem 4, consider problem $\pi(T)$ in Example 1 where $T = (T_{h_1} = \{90, 100, 110\}, T_{h_2} = \{100\})$. Then $S(\pi(T)) = S(\pi(\{90\}, T_{-h_1})) \cup S(\pi(\{100\}, T_{-h_1})) \cup S(\pi(\{110\}, T_{-h_1})) \cap S(\pi(\{100\}, T_{-h_2}))$.

A.2 Modification of the Doctor-Optimal Stable Allocation

In this section, we establish a result concerning doctor-optimal stable allocations when the set of available terms is modified for a single hospital. Proposition 7 states that if the doctor-optimal stable allocation of h is the same in $\pi(T)$ and $\pi(T'_h, T_{-h})$, then the doctor-optimal stable allocation remains the same for all agents in the market.

Proposition 7. For any problem $\pi(T) \in \Pi$, consider a sub-problem $\pi(T'_h, T_{-h}) \in \tilde{\Pi}(\pi(T))$. If $\overline{X}_h = \overline{X}'_h$ then $\overline{X} = \overline{X}'$.

As discussed in Section 5, hospitals' incentives to offer terms are pivotal to doctors' welfare. Proposition 7 highlights that when the allocation of a hospital h is unaffected by a modification of terms, the allocation in the market remains unchanged.

B Proofs

B.1 Proof of Proposition 1

Proof. For the first part of (i) in Proposition 1, and (ii) we use Example 2. We use strict preference and stability for the second part of (i). We know that there exists some $d' \in D'$ such that $\overline{X}'_{d'} \succ_{d'} \overline{X}_{d'}$. Suppose, by contradiction, that for each h where $\overline{X}'_{d'} = (d', h, t)$, we have $\overline{X}'_h \succ_h \overline{X}_h$ or $\overline{X}_h = \overline{X}'_h$. Using strict preference, we know that $\overline{X}'_h \neq \overline{X}_h$ because $\overline{X}'_{d'} \succ_{d'} \overline{X}_{d'}$. We use the definition of stability to show that $\overline{X}'_h$ cannot be preferred by h to $\overline{X}_h$. If d' and h prefer $\overline{X}'$ to $\overline{X}$, then $\overline{X}$ is not stable because there exists a contract $x = (d', h, t)$ that block $\overline{X}$. Therefore, $\overline{X}_h \succ_h \overline{X}'_h$. ■

B.2 Proof of Theorem 1

Proof. (i) Consider an allocation $X \in S(\pi(T))$ such that $X \in \mathcal{F}(\pi(T'))$; Suppose, by contradiction, that $X \notin S(\pi(T'))$. Let X' denote the allocation that blocks X in $\pi(T')$ with $X \neq X'$, such that for each $h \in H$, we have $X'_h \subset C_h(X' \cup X)$. By construction of T', we know that $X' \in \mathcal{F}(\pi(T))$ and that contracts are substitutes for hospitals. Since X' blocks X in $\pi(T')$, X' also blocks X in $\pi(T)$ because there is no x such that $x \notin C_h(X')$ and $x \in C_h(X' \cup X)$ for some h. Then, $X \notin S(\pi(T))$, leading to a contradiction.

Before proving (ii), we introduce Lemma 2. The intuition behind Lemma 2 is that if an allocation is no longer stable when terms are added, then among the added terms, some are used in an allocation that blocks the stable allocation.

Lemma 2. Suppose $\pi(T')$ is a sub-problem of $\pi(T)$ such that $X' \in S(\pi(T'))$ and $X' \notin S(\pi(T))$. Then there exists $x \in X$ with $X \in \mathcal{F}(\pi(T))$ such that $x_t \in T_{x_h} \backslash T'_{x_h}$ and $x \succ_{x_d} X'_{x_d}$ and $X_{x_h} \succ_{x_h} X'_{x_h}$.

Proof. To prove Lemma 2, we consider the stability definition, the construction of $\pi(T)$ and $\pi(T')$, and that contracts are substitutes for hospitals. By construction, we know that an allocation blocks X' in the problem $\pi(T)$. We denote this allocation X. Suppose that for each contract $x \in X$, there exists $h \in H$ such that $x_t \in T'_h$. Because contracts are substitutes for hospitals, these contracts are also chosen in $\pi(T')$, and X' is not stable in $\pi(T')$. Then, there exists a term $t \in T_{x_h} \backslash T'_{x_h}$ which is used by at least one contract in X; otherwise, $X' \notin S(\pi(T'))$. ■

(ii) By contradiction, suppose there exists $X \in S(\pi(T')) \cap S(\pi(T''))$ and $X \notin S(\pi(T' \cup T''))$. By stability, there exists $X' \in \mathcal{F}(\pi(T' \cup T''))$ and $X' \notin (\mathcal{F}(\pi(T')) \cup \mathcal{F}(\pi(T'')))$; otherwise, X would not be stable in $\pi(T')$ and $\pi(T'')$ by Lemma 2. We have to show that there is no blocking set of contracts X' which could involve terms from both $T' \setminus T''$ and $T'' \setminus T'$. Suppose $x', x'' \in X$ such that $x'_t \in T'_{x'_h} \setminus T''_{x'_h}$ and $x''_t \in T''_{x''_h} \setminus T'_{x''_h}$. Since contracts are substitutes for hospitals, which rules out complementarity between contracts, the contradiction is direct: $x' \notin C_{x'_h}(X_{x'_h} \cup \{x'\})$ implies $x' \notin C_{x'_h}(X_{x'_h} \cup \{x', x''\})$ and $x'' \notin C_{x''_h}(X_{x''_h} \cup \{x''\})$ implies $x'' \notin C_{x''_h}(X_{x''_h} \cup \{x'', x'\})$. Thus, $X \in S(\pi(T' \cup T''))$. ■

B.3 Proof of Theorem 2

We use the construction of the COP to prove Theorem 2. We denote by $R_d(X) \equiv X - C_d(X)$ the set of contracts rejected by d from X. We denote by $R_h(X') \equiv X' - C_h(X')$ the set of contracts rejected by h from X'.

Cumulative Offer Process.

Step 0. Consider problem $\pi(T)$. For each $d \in D$, let $X_d(0) \equiv (X_d \cap \mathcal{F}(\pi(T))) \cup \{\varnothing\}$ be the set of contracts available for d.

Step $s \geq 1$. Each $d \in D$ chooses a contract from $X_d(s-1)$ using $C_d(\cdot)$. Let $x_d(s) \equiv C_d(X_d(s-1))$ be the chosen contract. Let $X(s) \equiv \bigcup_{d \in D} C_d(X_d(s-1))$ be the set of all chosen contracts. Each doctor d proposes to hospital $x_d(s)_h$ contract $x_d(s)$. For each $h \in H$, let $X_h(s) \equiv \{x_d(s) \in X(s) : x_d(s)_h = h\}$ be the set of contracts proposed to h. Each hospital h chooses contracts from $X_h(s)$ using $C_h(\cdot)$. Let $R(s)$ be rejected contract set in Step s. If $R(s) = \emptyset$, then stop.

Otherwise, for each $d \in D$, with $x_d(s) \in R(s)$, we let $X_d(s) \equiv X_d(s-1) \setminus \{x_d(s)\}$ and for each $d \in D$ such that $x_d(s) \notin R(s)$, we let $X_d(s) \equiv X_d(s-1)$, and proceed to Step *s+1*.

The algorithm continues until Step k, where $R(k) = \emptyset$. The final allocation is the doctor-optimal stable allocation $\overline{X}$ under the vector of terms T. Since there are finitely many doctors, hospitals, and possible contracts for each doctor-hospital pair, the COP terminates in a finite number of steps.

Proof. Suppose there exists $T' \subset T$ such that $\overline{X}'_d \succ_d \overline{X}_d$ for each $d \in D$. Since $\overline{X}' \in S(\pi(T'))$, we know from Lemma 2 that some terms in $T_h \setminus T'_h$ are used in a blocking allocation that blocks $\overline{X}'$ in $\pi(T)$. Given that preferences are strict, contracts are substitutes for hospitals, and using the construction of the COP: From Theorem 3 (b) of Hatfield and Milgrom (2005), we know that the COP converges monotonically to the highest fixed point (i.e., $\overline{X}$). Let $COP_s(\pi(T))$ denote the step s of the COP in problem $\pi(T)$. We know that at some step s, $COP_s(\pi(T)) = \overline{X}'$. Since $\overline{X}'$ is an allocation, the COP stops at step s, and $\overline{X}'$ is the doctor-optimal stable allocation of problem $\pi(T)$. Renegotiation is unnecessary. Therefore, $\overline{X}' = \overline{X}$, which concludes the proof. ■

B.4 Proof of Theorem 3

Proof. By definition of lexicographic preferences, if a hospital h does not have lexicographic preferences, then we have $|T_h| \geq 2$ and $|D| \geq 2$. Symmetrically, if a doctor d does not have lexicographic preferences, then there exists at least one hospital h such that $|T_h| \geq 2$ and $|H| \geq 2$. We may limit attention to the case with exactly two doctors and one hospital by specifying that doctors find the other hospitals to be unacceptable.

(i) Suppose $\succ_h$ is not doctor-lexicographic, and there exist some $x, x', y, y' \in \mathcal{X}$, with $x = (d_1, h, t), x' = (d_1, h, t'), y = (d_2, h, t)$ and $y' = (d_2, h, t')$. We consider four cases:

- **Case 1:** $\succ_h$: $x \succ_h y \succ_h x' \succ_h y'$. Consider $\succ_{d_1}$: $x' \succ_{d_1} x \succ_{d_1} \varnothing$ and $\succ_{d_2}$: $y \succ_{d_2} y' \succ_{d_2} \varnothing$. Let $T_h = \{t, t'\}$ and $T'_h = \{t'\}$. Since $C_h(\{x', y'\}) = \{x'\}$, it follows that $\overline{X}'_{d_1} = \{x'\}$. Similarly, $C_h(\{x', y\}) = \{y\}$ and $C_h(\{x', y, x\}) = \{x\}$, we have that $\overline{X}_{d_1} = \{x\}$. Therefore, $\overline{X}'_{d_1} \succ_{d_1} \overline{X}_{d_1}$.

- **Case 2:** $\succ_h$: $x' \succ_h y \succ_h x \succ_h y'$. Consider $\succ_{d_1}$: $x \succ_{d_1} x' \succ_{d_1} \varnothing$ and $\succ_{d_2}$: $y \succ_{d_2} y' \succ_{d_2} \varnothing$. Let $T_h = \{t, t'\}$ and $T'_h = \{t\}$. Since $C_h(\{x, y\}) = \{y\}$, it follows that $\overline{X}'_{d_2} = \{y\}$. Similarly, $C_h(\{x, y\}) = \{y\}$ and $C_h(\{x', y, x\}) = \{x'\}$, we have that $\overline{X}_{d_2} = \{\varnothing\}$. Therefore, $\overline{X}'_{d_2} \succ_{d_2} \overline{X}_{d_2}$.
- **Case 3:** $\succ_h$: $x \succ_h y \succ_h y' \succ_h x'$. Consider $\succ_{d_1}$: $x \succ_{d_1} x' \succ_{d_1} \varnothing$ and $\succ_{d_2}$: $y' \succ_{d_2} y \succ_{d_2} \varnothing$. Let $T_h = \{t, t'\}$ and $T'_h = \{t'\}$. Since $C_h(\{x', y'\}) = \{y'\}$, it follows that $\overline{X}'_{d_2} = \{y'\}$. Similarly, $C_h(\{x, y'\}) = \{x\}$ and $C_h(\{x, y, y'\}) = \{x\}$, we have that $\overline{X}_{d_2} = \{\varnothing\}$. Therefore, $\overline{X}'_{d_2} \succ_{d_2} \overline{X}_{d_2}$.
- **Case 4:** $\succ_h$: $x \succ_h y' \succ_h y \succ_h x'$. Consider $\succ_{d_1}$: $x \succ_{d_1} x' \succ_{d_1} \varnothing$ and $\succ_{d_2}$: $y' \succ_{d_2} y \succ_{d_2} \varnothing$. Let $T_h = \{t, t'\}$ and $T'_h = \{t'\}$. Since $C_h(\{x', y'\}) = \{y'\}$, it follows that $\overline{X}'_{d_2} = \{y'\}$. Similarly, $C_h(\{x, y'\}) = \{x\}$ and $C_h(\{x, y, y'\}) = \{x\}$, we have that $\overline{X}_{d_2} = \{\varnothing\}$. Therefore, $\overline{X}'_{d_2} \succ_{d_2} \overline{X}_{d_2}$.

We omit symmetric cases where contracts are reversed (x instead of x', and y instead of y') and cases where doctors are reversed (d_1 instead of d_2). This concludes the proof.

(ii) Suppose $\succ_{d_1}$ is not hospital-lexicographic, this implies that there exist $h_1, h_2 \in H$, with $h_1 \neq h_2$. Further, suppose that there exist some $x_1, x'_1, x_2, y_1, y'_1, y_2 \in \mathcal{X}$, with $x_1 = (d_1, h_1, t), x'_1 = (d_1, h_1, t'), x_2 = (d_1, h_2, t), y_1 = (d_2, h_1, t), y'_1 = (d_2, h_1, t')$ and $y_2 = (d_2, h_2, t)$. We consider two cases:

- **Case 1:** $\succ_{d_1}$: $x_1 \succ_{d_1} x_2 \succ_{d_1} x'_1 \succ_{d_1} \varnothing$. Consider $\succ_{d_2}$: $y'_1 \succ_{d_2} y_1 \succ_{d_2} y_2 \succ_{d_2} \varnothing$, $\succ_{h_1}$: $x_1 \succ_{h_1} x'_1 \succ_{h_1} y_1 \succ_{h_1} y'_1$, and h_2, which accepts all the contracts it receives. Let $T_{h_1} = \{t, t'\}$ and $T'_{h_1} = \{t'\}$. Since $C_{h_1}(\{y'_1\}) = \{y'_1\}$, it follows that $\overline{X}'_{d_2} = \{y'_1\}$. Similarly, $C_h(\{x_1, y'_1\}) = \{x_1\}$ and $C_h(\{x_1, y_1, y'_1\}) = \{x_1\}$, we have that $\overline{X}_{d_2} = \{y_2\}$. Therefore, $\overline{X}'_{d_2} \succ_{d_2} \overline{X}_{d_2}$.
- **Case 2:** $\succ_{d_1}$: $x'_1 \succ_{d_1} x_2 \succ_{d_1} x_1 \succ_{d_1} \varnothing$. Consider $\succ_{d_2}$: $y'_1 \succ_{d_2} y_1 \succ_{d_2} y_2 \succ_{d_2} \varnothing$, $\succ_{h_1}$: $x_1 \succ_{h_1} x'_1 \succ_{h_1} y_1 \succ_{h_1} y'_1$, and h_2, which accepts all the contracts it receives. Let $T_{h_1} = \{t, t'\}$ and $T'_{h_1} = \{t\}$. Since $C_{h_1}(\{y_1\}) = \{y_1\}$, it follows that $\overline{X}'_{d_2} = \{y_1\}$. Similarly, $C_h(\{x'_1, y'_1\}) = \{x'_1\}$ and $C_h(\{x'_1, y_1, y'_1\}) = \{x'_1\}$, we have that $\overline{X}_{d_2} = \{y_2\}$. Therefore, $\overline{X}'_{d_2} \succ_{d_2} \overline{X}_{d_2}$.

We omit symmetric cases where contracts are reversed (x_1 instead of x'_1, and y_1 instead of y'_1). This concludes the proof. ■

B.5 Proof of Proposition 2

Proof. Consider T and T' such that for each $h' \in H \setminus \{h\}, T_{h'} = T'_{h'}$ and for $h, T'_h \subset T_h$. Knowing that hospital preferences are doctor-lexicographic:

(i) Suppose by contradiction that $D_h(\overline{X}) = D_h(\overline{X}')$ and there exists $d \in D$ such that $\overline{X}_d \succ_d \overline{X}'_d$. We have two cases:

- **Case (i)-1:** d is employed by h in both $\overline{X}$ and $\overline{X}'$. Considering the doctor-lexicographic preferences of h and the substitutability of contracts, it is direct that no contract x' concerns doctor $x'_d = d'$ that blocks $\overline{X}'_d$. Since $\overline{X}$ is the doctor-optimal stable allocation, we know that $\overline{X}_d \succ_d \overline{X}'_d$; otherwise, d would not be assigned to h in $\overline{X}$, contradicting $D_h(\overline{X}) = D_h(\overline{X}')$.
- **Case (i)-2:** d is not employed by h in $\overline{X}$ and $\overline{X}'$. This means that an allocation blocks $\overline{X}'$ by adding terms to T'_h. In the COP, this implies that doctors have been rejected from h and d was then rejected by the hospital that employed her in $\overline{X}'$. To be rejected from h there must exist a doctor preferred by h who is employed. This contradicts the doctor-lexicographic preferences and $D_h(\overline{X}) = D_h(\overline{X}')$.

(ii) Suppose by contradiction that t is used and there is no doctor d such that $\overline{X}_d \succ_d \overline{X}'_d$. Consider $t \in T_h \setminus T'_h$ such that $x \in \overline{X}$ with $x_t = t$ and $x_d = d$. We know that no contract blocks $\overline{X}'_d$ under T' while $\overline{X}'$ is blocked under (T_h, T'_{-h}). By assumption, d is hired by h under $\overline{X}$. We have two cases:

- **Case (ii)-1:** d is hired by h under $\overline{X}'$. Then, using doctor-lexicographic preferences, we know that h continues to hire d. Contracts are substitutable, and $\overline{X}'_d$ is not blocked. Considering the doctor-optimal stable allocation, the choice of d reflects her preferred term in T_h. Since the preferences are strict and the term used is different, we have $\overline{X}_d \succ_d \overline{X}'_d$.
- **Case (ii)-2:** d is not hired by h under $\overline{X}'$. This implies that new contracts are blocking X'_d. The contradiction is straightforward since new contracts are only added for h. We know that $\overline{X}_d \succ_d \overline{X}'_d$.

These two contradictions conclude the proof. ■

B.6 Proof of Proposition 3

Proof. Let $\pi(T) \in \Pi$ be a problem such that the preferences of doctors and hospitals are common and polarized. Without loss of generality, fix a sub-problem $\pi(T'_h, T_{-h}) \in \tilde{\Pi}(\pi(T))$ such that $T' \subset T$, there exists $t \in T_h \setminus T'_h$ and a contract $x \in \overline{X}_h$ such that $x_t = t$, $D_h(\overline{X}) = D_h(\overline{X}')$, and each $t \in T_h \setminus T'_h$ is preferred by doctors to any $t' \in T'_h$. Since $D_h(\overline{X}) = D_h(\overline{X}')$, added terms are preferred by doctors, and at least one added term is used, it is direct that there exists $d \in D$ such that $\overline{X}_d \succ \overline{X}'_d$. It remains to show that there is no doctor d such that $\overline{X}'_d \succ \overline{X}_d$.

By contradiction, suppose there exists $d \in D$ such that $\overline{X}'_d \succ \overline{X}_d$. It is straightforward that $d \in D_h(\overline{X})$ otherwise $\overline{X}'_d$ is not stable in $\pi(T'_h, T_{-h})$. Since the terms that are added are preferred by doctors, we know that for each $t \in T_h \setminus T'_h$, $(h, t) \succ_d \overline{X}'_d$. By construction of the COP we know that d proposed all the contracts using the terms in $T_h \setminus T'_h$ to h and the contracts were rejected by h. $\overline{X}'_d$ is also rejected by h since $\overline{X}'_d \succ \overline{X}_d$. Among the other contracts proposed by the other doctors at h, there exists an allocation X such that $C_h(X \cup \overline{X}') = X$ and $X \subset C_D(X \cup \overline{X}')$, thus $X \succ_h \overline{X}'$. We know that there exists a doctor d' such that $\overline{X}_d \succ'_d \overline{X}'_d$ therefore $\overline{X}'_h \succ_h \overline{X}_h$, as preferences are common and polarized, this contradicts that $X \succ_h \overline{X}'_h$.

■

B.7 Proof of Proposition 5

Proof. We prove the result by contradiction. Suppose $\underline{X}'_h \succ_h \underline{X}_h$. Note that h offers more terms in $\pi(T_h, T'_{-h})$ than in $\pi(T')$. Since $\underline{X}'_h \succ_h \underline{X}_h$, we know that $C_h(\underline{X}' \cup X'') = X''$ in $\pi(T_h, T'_{-h})$, where X'' is an allocation. Since the terms have only changed for h, hospital h offers a new term such that $\underline{X}'$ is blocked. By definition of stability, X''_h is preferred to $\underline{X}'_h$ by h. We have two cases:

- **Case 1:** X'' is not stable. In this case, another allocation blocks X''. By the lattice structure (Theorem 0), the hospital-optimal stable allocation is at least as preferred by h as $\underline{X}_h$.
- **Case 2:** X'' is stable. Using the lattice structure, we know that hospitals unanimously prefer X'' to any other stable allocation. Therefore, $X''_h = \underline{X}_h$,

implying $\underline{X}_h \succeq_h \underline{X}'_h$.

By the lattice structure and considering that h is the only hospital offering new terms, it directly follows that if an added term is used, then the preference is strict. ■

B.8 Proof of Theorem 4

Proof. We prove this result in two steps:

Claim 1. If $\mathcal{A}(\pi(T)) \subseteq \mathcal{U}(\pi(T))$, then $\bigcap_{h\in H} \left(\bigcup_{t\in T_h} S(\pi(\{t\}, T_{-h}))\right) \subseteq S(\pi(T))$.

Proof. By contradiction, suppose that there exists an allocation X such that

- $X \in \bigcap_{h\in H} \left(\bigcup_{t\in T_h} S(\pi(\{t\}, T_{-h}))\right)$, and
- $X \notin S(\pi(T))$.

Since $X \notin S(\pi(T))$, there exists an allocation X' that blocks X in $\pi(T)$. Specifically, there exists h such that $C_h(X' \cup X) = X'$ and $X' \subseteq C_D(X' \cup X)$. If $X' \in \mathcal{F}(\pi(T)) \setminus \mathcal{A}(\pi(T))$, then there exists $h \in H$ such that $\varnothing \succ_h X'_h$, meaning X' does not block X. However, by definition of acceptable allocation, $X' \in \mathcal{A}(\pi(T))$. Since $\mathcal{A}(\pi(T)) \subseteq \mathcal{U}(\pi(T))$, all acceptable allocations in $\pi(T)$ are also acceptable in $\bigcup_{t\in T_h} S(\pi(\{t\}, T_{-h}))$. Therefore, X' does not block X. Thus $X \in S(\pi(T))$, leading to a contradiction. ■

Claim 2. If $\mathcal{A}(\pi(T)) \subseteq \mathcal{U}(\pi(T))$, then $S(\pi(T)) \subseteq \bigcap_{h\in H} \left(\bigcup_{t\in T_h} S(\pi(\{t\}, T_{-h}))\right)$.

Proof. By contradiction, suppose that there exists and allocation X such that

- $X \in S(\pi(T))$, and
- $X \notin \bigcap_{h\in H} \left(\bigcup_{t\in T_h} S(\pi(\{t\}, T_{-h}))\right)$.

This implies that there exists at least one hospital h for which

$$X \notin \bigcup_{t\in T_h} S(\pi(\{t\}, T_{-h})).$$

By construction, all acceptable allocations in $\pi(T)$ are also acceptable in

$$\bigcup_{t\in T_h} S(\pi(\{t\}, T_{-h})).$$

This contradict Theorem 1 (ii), implying that $X \in \bigcup_{t\in T_h} S(\pi(\{t\}, T_{-h}))$. ■

Together, Claim 1 and 2 complete the proof. ■

B.9 Proof of Proposition 7

Proof. We proceed by contradiction. Suppose $\overline{X}_h = \overline{X}'_h$ and $\overline{X} \neq \overline{X}'$. The difference between the two problems is that additional terms have been introduced exclusively for hospital h. Since $\overline{X}_h = \overline{X}'_h$, even if other doctors are tentatively assigned to h with a new term, these doctors are subsequently rejected in later steps of the COP. Upon being rejected, these doctors propose their next contracts according to their preferences, and the allocation converges to the same outcome as before. ■

References

Afacan, M. O. (2017). Some further properties of the cumulative offer process. *Games and Economic Behavior*, 104:656–665.

Aygün, O. and Sönmez, T. (2013). Matching with contracts: Comment. *American Economic Review*, 103(5):2050–2051.

Chambers, C. P. and Yenmez, M. B. (2017). Choice and matching. *American Economic Journal: Microeconomics*, 9(3):126–147.

Crawford, V. P. and Knoer, E. M. (1981). Job matching with heterogeneous firms and workers. *Econometrica: Journal of the Econometric Society*, pages 437–450.

Echenique, F. (2012). Contracts versus salaries in matching. *American Economic Review*, 102(1):594–601.

Ergin, H. I. (2002). Efficient resource allocation on the basis of priorities. *Econometrica*, 70(6):2489–2497.

Fleiner, T. (2003). A fixed-point approach to stable matchings and some applications. *Mathematics of Operations research*, 28(1):103–126.

Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15.

Gale, D. and Sotomayor, M. (1985). Some remarks on the stable matching problem. *Discrete Applied Mathematics*, 11(3):223–232.

Hatfield, J. W. and Kominers, S. D. (2015). Hidden substitutes. In *EC*, page 37.

Hatfield, J. W. and Kominers, S. D. (2017). Contract design and stability in many-to-many matching. *Games and Economic Behavior*, 101:78–97.

Hatfield, J. W. and Milgrom, P. R. (2005). Matching with contracts. *American Economic Review*, 95(4):913–935.

Hirata, D., Kasuya, Y., and Okumura, Y. (2023). Stability, strategy-proofness, and respect for improvements. *Discussion Papers, Hitotsubashi University.*

Kelso, A. S. and Crawford, V. P. (1982). Job matching, coalition formation, and gross substitutes. *Econometrica: Journal of the Econometric Society*, pages 1483–1504.

Kesten, O. (2010). School choice with consent. *The Quarterly Journal of Economics*, 125(3):1297–1348.

Kominers, S. D. (2012). On the correspondence of contracts to salaries in (many-to-many) matching. *Games and Economic Behavior*, 75(2):984–989.

Kominers, S. D. and Sönmez, T. (2016). Matching with slot-specific priorities: Theory. *Theoretical Economics*, 11(2):683–710.

Marx, K. (1867). *Das kapital.* Chapter 25.

Pakzad-Hurson, B. (2023). Stable and efficient resource allocation with contracts. *American Economic Journal: Microeconomics*, 15(2):627–659.

Roth, A. E. (1982). The economics of matching: Stability and incentives. *Mathematics of operations research*, 7(4):617–628.

Roth, A. E. (1984). Stability and polarization of interests in job matching. *Econometrica: Journal of the Econometric Society*, pages 47–57.

Roth, A. E. (1985). The college admissions problem is not equivalent to the marriage problem. *Journal of economic Theory*, 36(2):277–288.

Roth, A. E. and Sotomayor, M. (1992). Two-sided matching. *Handbook of game theory with economic applications*, 1:485–541.

Schlegel, J. C. (2015). Contracts versus salaries in matching: A general result. *Journal of Economic Theory*, 159:552–573.

Sönmez, T. (2013). Bidding for army career specialties: Improving the rotc branching mechanism. *Journal of Political Economy*, 121(1):186–219.

Sönmez, T. and Switzer, T. B. (2013). Matching with (branch-of-choice) contracts at the united states military academy. *Econometrica*, 81(2):451–488.

Yenmez, M. B. (2018). A college admissions clearinghouse. *Journal of Economic Theory*, 176:859–885.