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Abstract

Affirmative action policies, by establishing representation thresholds for protected groups, seek to balance fairness and equity in various assignment problems. Fairness is maintained by prioritizing individuals based on merit scores, while equity is ensured through guaranteed group representation. We focus on overlapping reserves, where individuals can belong to multiple groups, and introduce the *Maximal Score and Minimum Guarantee* (MSMG) choice rule, which upholds representation requirements while preserving fairness. We define the *score of an assignment* as the sum of the merit scores of the selected individuals. We demonstrate that the assignment produced by the MSMG choice rule achieves the highest possible score among all fair assignments that satisfy the given representation thresholds.

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1 Introduction

Affirmative action policies are widely employed to address the inequalities faced by disadvantaged groups, yet debates persist over how to balance fairness and equity in their implementation. These groups, often defined by gender, ethnicity, or socioeconomic status,[1] benefit from policies implemented across various domains, including firm composition, school admissions, government hiring, and legislative representation. A

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[1]The criteria are generally classified into two main categories: *vertical* and *horizontal* reservations. Vertical reservations apply to historically disadvantaged groups requiring specific quotas (e.g., caste-based or ethnic reservations), while horizontal reservations cut across all categories and apply to characteristics such as disability, veteran status, or gender, ensuring representation within each vertical category.

common approach involves setting minimum representation thresholds. However, complications arise when individuals belong to multiple groups—for instance, a worker with a disability who is also a member of an ethnic minority. In such cases, individuals may face disadvantages due to complementarities, as existing frameworks rarely account for the intersection of different characteristics. This paper contributes to this discussion by designing a choice rule that balances fairness and equity.

Two types of affirmative action policies enforcing representation thresholds are commonly implemented: *over-and-above* and *minimum guarantee* policies. Under the over-and-above approach, reserved positions are allocated exclusively to eligible individuals. If a candidate qualifies for an unreserved position based on merit (e.g., test score), their assignment does not reduce the number of reserved seats. Conversely, in the minimum guarantee approach, any assignment of a reserve-eligible candidate—whether based on merit or through a reserved position—counts toward the required representation. When individuals belong to only one group, the selection procedures in these policies differ primarily in processing order: over-and-above policies allocate unreserved positions first, followed by reserved positions (Dur et al., 2018), while minimum guarantee policies assign reserved positions first, then process unreserved positions (Echenique and Yenmez, 2015).

However, when individuals qualify for multiple reserved groups, a fundamental question arises: how should they be counted toward representation? Should they contribute to all groups they belong to, or only one? The first approach is known as *one-to-all reserve matching*, while the second is referred to as *one-to-one reserve matching* (Sönmez and Yenmez, 2020; Sönmez and Yenmez, 2022).

This paper considers the minimum guarantee framework, where identical positions are assigned to individuals.[2] Each individual is characterized by two attributes: *traits* and *scores*. Traits determine the groups to which individuals belong, and an individual may be associated with multiple groups. Minimum representation constraints are imposed following the one-to-all approach. Scores, on the other hand, reflect merit-based criteria, such as a worker's productivity in the labor market or a student's academic performance, as well as additional factors like financial need or scholarship eligibility. These two attributes together form the basis for fairness criteria.

In the absence of reservations, assignment decisions are based solely on scores, with fairness dictating that individuals with higher scores should be prioritized. However, when minimum representation constraints are introduced, selection is no longer determined solely by scores, as individuals' traits also influence the assignment process. Con-

[2] Since all positions are identical, we assume individuals are indifferent between them.

sequently, an individual with specific traits may be selected even if their score is lower than that of another unselected individual who does not qualify for reserved positions. When individuals can belong to multiple groups, assessing the fairness of an assignment becomes challenging. Sönmez and Yenmez (2019) propose a fairness notion in this context, drawing on the concept of *stability* commonly used in the matching literature.[3] In an assignment, individual j *justifiably envies individual* i if (1) j is unassigned while i is assigned (2) j possesses all the traits of i (and possibly additional ones), and (3) j has a higher score than i. An assignment is considered *fair* if no individual justifiably envies another.

When individuals can belong to multiple groups, complementarities emerge, complicating market analysis and assignment decisions. The following example illustrates this phenomenon and our approach.

Example 1. Consider five individuals w_1, w_1^d, m_1^d, m_1 and m_2. Individuals w_1 and w_1^d are women, while individuals w_1^d and m_1^d are disabled. The score ranking over individuals from the highest to the lowest is:

$$\sigma(m_1) > \sigma(m_2) > \sigma(m_1^d) > \sigma(w_1) > \sigma(w_1^d).$$

There are three available positions. Among the three selected individuals, at least one must be a woman and at least one must be a disabled individual.

Under the minimum guarantee framework, and following the horizontal choice rule introduced by Sönmez and Yenmez (2020), the disabled individual with the highest score, m_1^d, is selected first, followed by the woman with the highest score, w_1.[4] With these selections, the representation requirements are satisfied. The remaining position is then assigned to the highest-scoring individual, m_1, resulting in the assignment $\mu = \{m_1, m_1^d, w_1\}$.

Now, consider an alternative assignment $\nu = \{m_1, m_2, w_1^d\}$, which also satisfies the representation constraints. Complementarity is clear in ν, as m_2 is assigned only if w_1^d is also assigned. Both μ and ν are fair; however, they allocate positions differently, with no clear ranking criterion to determine which assignment is preferable.

To address this, we introduce the notion of *assignment score*, defined as the sum of the scores of the selected individuals. Suppose the scores are:

$$\sigma(m_1) = 100, \quad \sigma(m_2) = 90, \quad \sigma(m_1^d) = 70, \quad \sigma(w_1) = 60, \quad \sigma(w_1^d) = 55.$$

[3] Their notion is motivated by court orders in India.

[4] In this case, selecting the woman with the highest score before the disabled individual with the highest score leads to the same final assignment.

The score of assignment μ is $m(\mu) = 230$, while that of assignment ν is $m(\nu) = 245$. Since ν achieves a higher total score, we say that it *score-dominates* assignment μ. As previously mentioned, in the labor market, the score can represent an individual's productivity.[5] Therefore, assignment ν leads to higher productivity than assignment μ. Our main contribution is the design of the *Maximal Score and Minimum Guarantee* (MSMG) choice rule, which, for any given set of individuals, produces an assignment that is fair, satisfies the representation constraints, and maximizes the total score.

The MSMG choice rule consists of three distinct, ordered parts, constructing the assignment while respecting the representation thresholds, which we refer to as *requirements*.[6] Within each part, certain steps may be repeated. **Part A** begins by selecting individuals with the highest scores while ensuring that sufficient capacity is preserved to meet the representation requirements. This is analogous to the over-and-above rule, but it also considers the traits of selected individuals and adjusts the reserved capacities accordingly. Once no more individuals can be added without compromising the representation thresholds, the MSMG rule proceeds to **Part B**. In this second part, individuals are selected to balance the requirements across traits. Specifically, when the requirements are imbalanced, the choice rule prioritizes individuals with the highest unmet requirement until the thresholds are equalized.[7] Once the requirements are equalized, the rule moves to **Part C**, where complementarities between individuals are taken into account. This is done by forming pairs of individuals and comparing their combined scores. The pair with the highest combined score is selected, and this process continues until all positions are filled.

By accounting for complementarities, the idea is that selecting an individual with multiple traits can facilitate the selection of another individual with a sufficiently high score, ultimately benefiting the assignment. Similarly, if an individual with multiple traits has a sufficiently high score, they may be preferred over multiple individuals, each possessing only one trait but with higher individual scores. In Example 1, these individuals correspond to m_1^d and w_1. Our approach explicitly considers the complementarities that arise while ensuring fairness through the score-based criterion. By doing so, it can benefit individuals with multiple traits, who might not always be selected under standard choice rules, particularly the horizontal choice rule used in assignment μ.

[5]In the education context, considering the assignment score is motivated by peer effects, as students' performances are influenced by their peers. A higher total score may foster a more competitive and stimulating learning environment, enhancing outcomes (Sacerdote, 2001 and Zimmerman, 2003).

[6]In the literature, these thresholds are also referred to as *reserved seats*. However, in the one-to-all framework, this would imply that a single individual could fill multiple reserved seats.

[7]If there is only one trait, **Parts A** and **B** together coincide with the over-and-above rule.

In our approach, we limit the number of traits to two. While this may seem restrictive, it provides a practical framework for modeling a wide range of real-world scenarios. This limitation arises primarily from the increased complexity in the assignment process when multiple traits interact and create complementarities. For instance, introducing more than two traits can significantly complicate the design of the assignment mechanism, especially when trying to maintain fairness and equity. Note that Sönmez and Yenmez (2020) also consider only two traits, which makes the approach tractable. These traits can represent distinct categories, such as vertical and horizontal reservations, commonly found in affirmative action policies—such as those targeting gender and ethnicity representation. While this two-trait framework is flexible and applicable to a variety of cases, in Section 4, we discuss how the choice rule can be adapted to settings involving more than two traits, acknowledging the additional challenges this introduces.

Literature Review

This paper contributes to the growing literature on resource allocation problems under affirmative action policies. In the context of school choice, affirmative action mechanisms based on quotas have been extensively studied. Kojima (2012) identifies key challenges in designing such policies, which primarily aim to ensure fair representation of students from underrepresented backgrounds. Hafalir et al. (2013) introduce a framework where schools enforce minimum quotas for minority students. Ehlers et al. (2014) extend this analysis to settings with more than two types, considering both minimum and maximum quotas. Echenique and Yenmez (2015) provide an axiomatic characterization of choice rules that satisfy minimum representation. Kominers and Sönmez (2016) propose a general matching model with slot-specific priorities, applicable to affirmative action settings. Dur et al. (2018) examine the allocation of public school seats in Boston, highlighting how the sequencing of reserved and open seats can unintentionally weaken walk-zone priorities. Although these contributions share a similar objective to ours, they all assume that individuals belong to at most one protected group. As discussed in the introduction, in many real-world markets, individuals may belong to multiple protected groups, which introduces complementarities when minimum representation requirements are imposed. This paper contributes to the literature by incorporating these complementarities and addressing the resulting complexities.

In recent years, several contributions have considered overlaps within categories. Motivated by the design of affirmative action policies in India, Sönmez and Yenmez (2019) were the first to design choice rules in this context. They analyze the overlap of criteria by studying vertical and horizontal reservations together in the contexts of the labor market and college admissions. Under the one-to-one reserve matching, Sönmez and Yenmez (2022) characterize the horizontal envelope choice rule, which is the unique rule that maximally complies with reservations, eliminates justified envy, and is non-wasteful. Pathak et al. (2024) generalize the horizontal envelope choice rule to address the allocation of medical resources during the COVID-19 pandemic.

In our approach, we consider the one-to-all reserve matching. Sönmez and Yenmez (2020) propose a choice rule in this context that maximally complies with reservations, eliminates justified envy, and is non-wasteful. However, multiple rules satisfy these axioms. Notably, Dur and Zhang (2023) introduce a choice rule that satisfies these axioms and leads to an assignment that is not dominated in rank by any other assignment that maximally complies with reservations, eliminates justified envy, and is non-wasteful. In this paper, we introduce the MSMG choice rule, which maximally complies with reservations, eliminates justified envy, and is non-wasteful. Furthermore, we characterize it by showing that the assignment produced by the MSMG rule is never dominated in score by any other assignment that maximally complies with reservations, eliminates justified envy, and is non-wasteful.

The remainder of the paper proceeds as follows. Section 2 introduces the model and the desirable axioms for choice rules. Section 3 presents the MSMG choice rule and our results. In Section 4, we discuss applications and limitations. Proofs are collected in the Appendix.

2 Model

We consider a matching model composed of a finite set of *individuals*, denoted by $\mathcal{I}$, and q identical *positions* to allocate. Each individual prefers to be assigned to a position rather than remain unassigned and demands a single position. Since the positions are identical, each individual is indifferent among all positions.

Each individual $i \in \mathcal{I}$ is characterized by the traits she inherits. Let $\mathcal{T} = \{t_1, t_2\}$ be a finite set of *reserve eligible traits*.[8] We allow individuals to inherit multiple traits. Let $\tau(i) \subseteq \mathcal{T}$ be the *set of traits inherited by individual* i. For a given subset of individuals $I \subseteq \mathcal{I}$, let $I_t = \{i \in I : t \in \tau(i)\}$ be the *set of individuals with trait* t.

Each individual $i \in \mathcal{I}$ has a *(merit) score*, where the score function is $\sigma : \mathcal{I} \to \mathbb{R}_+$,

[8] In this paper, we consider only two reserved traits. In Section 4, we discuss the complexity of extending the model to more than two traits. As mentioned in the introduction, this constraint allows us to capture a wide range of real-life applications.

and $\sigma(i)$ is the score of individual i. We assume that for every $i, j \in \mathcal{I}$ such that $i \neq j$, we have $\sigma(i) \neq \sigma(j)$.

Let $I \subseteq \mathcal{I}$ be the set of individuals who apply for positions. An *assignment* is a subset of individuals $I' \subseteq I$ such that $|I'| \leq q$. Let $r_t \in \mathbb{N}$ denote the *minimum representation required* for trait $t \in \mathcal{T}$ in an assignment. We assume that the minimum representation required is no more than the capacity, i.e., $\sum_{t \in \mathcal{T}} r_t \leq q$.

For any set of applicants $I \subseteq \mathcal{I}$, a *choice rule* $C : 2^{\mathcal{I}} \to 2^{\mathcal{I}}$ selects an assignment, i.e., $C(I) \subseteq I$ and $|C(I)| \leq q$.

Given $I \subseteq \mathcal{I}$, minimum representations implemented on a minimum guarantee basis require that for every trait $t \in \mathcal{T}$, either at least r_t individuals with trait t are chosen or all individuals with trait t are chosen.

Definition 1. An assignment $I' \subseteq I$ *maximally complies with reservations* if it satisfies the minimum representation required for I, i.e., for every trait $t \in \mathcal{T}$, we have $|I'_t| \geq \min\{r_t, |I_t|\}$. A choice rule C maximally complies with reservations if, for every set $I \subseteq \mathcal{I}$, $C(I)$ maximally complies with reservations for I.

Note that the definition of minimum guarantee basis allows an individual i, upon admission, to count towards each of the traits that she has. This is known as *one-to-all reserve matching* (Sönmez and Yenmez, 2020).

Next, we define two axioms that a choice rule should satisfy in the context of fairness. The first axiom requires that an individual be rejected only if all positions are filled by other individuals.

Definition 2. Given a set of individuals $I \subseteq \mathcal{I}$ and an assignment $I' \subseteq I$, the assignment I' is *non-wasteful* if $|I'| = \min\{q, |I|\}$. A choice rule C is *non-wasteful* if, for every $I \subseteq \mathcal{I}$, the assignment $C(I)$ is non-wasteful.

The second axiom is the fairness notion introduced by Sönmez and Yenmez (2020).

Definition 3. Given a set of individuals $I \subseteq \mathcal{I}$ and an assignment $I' \subseteq I$, we say that individual i is *justifiably envied* by j if:

(i) $j \in I \setminus I'$, and $i \in I'$,

(ii) $\tau(i) \subseteq \tau(j)$, and

(iii) $\sigma(i) < \sigma(j)$.

Given a set of individuals $I \subseteq \mathcal{I}$, an assignment $I' \subseteq I$ *eliminates justified envy* if there does not exist a pair of individuals $(i, j) \in I \times I$ such that i is justifiably envied by j. A choice rule C *eliminates justified envy* if, for every $I \subseteq \mathcal{I}$, the assignment $C(I)$ eliminates justified envy.

In this paper, we consider the score of an assignment. Given a set of individuals $I \subseteq \mathcal{I}$, let $m(I')$ be the *score of assignment* $I' \subseteq I$ such that $m(I') = \sum_{i \in I'} \sigma(i)$. We normalize $m(\emptyset) = 0$.

Definition 4. Given a set of individuals $I \subseteq \mathcal{I}$, and two assignments $I', I'' \subseteq I$, we say that I' *score-dominates* I'' if $m(I') > m(I'')$. A choice rule C *score-dominates* a choice rule C' if, for some $I \subseteq \mathcal{I}$, $m(C(I)) > m(C'(I))$.

3 Maximal Score Choice Rule

In this section, we introduce the Maximal Score and Minimum Guarantee (MSMG) choice rule and study the axioms presented in Section 2.

Maximal Score and Minimum Guarantee (MSMG) Choice Rule C^*

Given a set of individuals $I \subseteq \mathcal{I}$, the outcome of the MSMG choice rule, denoted by $C^*(I)$, is obtained via the following three (main) parts.

Part 0: Initialization of C^*

If $|I| \leq q$, then $C^*(I) = I$, and the procedure terminates. Otherwise, set

$$r_{t_1}(0) = \min\{r_{t_1}, |I_{t_1}|\}, \quad r_{t_2}(0) = \min\{r_{t_2}, |I_{t_2}|\},$$

$$q(0) = q - r_{t_1}(0) - r_{t_2}(0), \text{ and } C^*(I)(0) = \{\emptyset\}.$$

C^* proceeds to **Part A** with these parameters.

Part A: Iterative Maximum Score Selection.

Let r_{t_1}, r_{t_2}, q and $C^*(I)$ be the given parameters, initialized as follows:

$$r_{t_1}(0) = r_{t_1}, \quad r_{t_2}(0) = r_{t_2}, \quad q(0) = q, \quad C^*(I)(0) = C^*(I).$$

Step k for $k \geq 1$:

- If $q(k-1) \leq 0$, proceed to **Part B** with $r_{t_1}(k-1)$, $r_{t_2}(k-1)$, $q(k-1)$, and $C^*(I)(k-1)$.
- Otherwise, select the $q(k-1)$ individuals with the highest scores in I, denoted by I^k. Set

$$C^*(I)(k) = C^*(I)(k-1) \cup I^k,$$

$$r_{t_1}(k) = r_{t_1}(k-1) - |I^k_{t_1}|,$$

$$r_{t_2}(k) = r_{t_2}(k-1) - |I^k_{t_2}|,$$

$$q(k) = r_{t_1}(k-1) - r_{t_1}(k) + r_{t_2}(k-1) - r_{t_2}(k).$$

If $r_{t_1}(k) = 0, r_{t_2}(k) = 0$ and $q(k) = 0$, then the procedure terminates. Otherwise, proceed to the next step.

Part B: Balancing Reserves.

Let r_{t_1}, r_{t_2}, q, and $C^*(I)$ be the given parameters, initialized as follows:

$$r_{t_1}(0) = r_{t_1}, \quad r_{t_2}(0) = r_{t_2}, \quad q(0) = q, \quad C^*(I)(0) = C^*(I).$$

Step ℓ for $\ell \geq 1$:

- If $q(\ell - 1) > 0$, proceed to **Part A**, with $r_{t_1}(\ell - 1)$, $r_{t_2}(\ell - 1)$, $q(\ell - 1)$, and $C^*(I)(\ell - 1)$.

Otherwise:

- If $r_{t_1}(\ell - 1) = r_{t_2}(\ell - 1)$, proceed to **Part C**, with $r_{t_1}(\ell - 1)$, $r_{t_2}(\ell - 1)$, $q(\ell - 1)$, and $C^*(I)(\ell - 1)$.
- If $r_{t_1}(\ell - 1) > r_{t_2}(\ell - 1)$, choose the $r_{t_1}(\ell - 1) - r_{t_2}(\ell - 1)$ individuals with trait t_1 with the highest score in $I \setminus C^*(I)(\ell - 1)$ denoted I^ℓ.
- If $r_{t_2}(\ell - 1) > r_{t_1}(\ell - 1)$, choose the $r_{t_2}(\ell - 1) - r_{t_1}(\ell - 1)$ individuals with trait t_2 with the highest score in $I \setminus C^*(I)(\ell - 1)$ denoted I^ℓ.

Set

$$C^*(I)(\ell) = C^*(I)(\ell - 1) \cup I^\ell,$$

$$r_{t_1}(\ell) = r_{t_1}(\ell - 1) - |I^\ell_{t_1}|,$$

$$r_{t_2}(\ell) = r_{t_2}(\ell - 1) - |I^\ell_{t_2}|,$$

$$q(\ell) = r_{t_1}(\ell - 1) - r_{t_1}(\ell) + r_{t_2}(\ell - 1) - r_{t_2}(\ell) - |I^\ell|.$$

If $r_{t_1}(\ell) = 0, r_{t_2}(\ell) = 0$ and $q(\ell) = 0$, then the procedure terminates. Otherwise, proceed to the next step.

Part C: Satisfying Minimum Representation.

Let r_{t_1}, r_{t_2}, q and $C^*(I)$ be the given parameters, initialized as follows:

$$r_{t_1}(0) = r_{t_1}, \quad r_{t_2}(0) = r_{t_2}, \quad q(0) = q, \quad C^*(I)(0) = C^*(I).$$

Step s for $s \geq 1$:

- If $q(s - 1) > 0$, proceed to **Part A** with $r_{t_1}(s - 1)$, $r_{t_2}(s - 1)$, $q(s - 1)$, and $C^*(I)(s - 1)$.

Otherwise,

- Choose the individual with trait t_1 with the highest score in $I \setminus C^*(I)(s-1)$, denoted $i^s_{t_1}$, and the individual with trait t_2 with the highest score in $I \setminus C^*(I)(s-1) \cup \{i^s_{t_1}\}$, denoted $i^s_{t_2}$. Set $I^s_{t_1,t_2} = \{i^s_{t_1}, i^s_{t_2}\}$.

- Choose the individual with trait t_2 with the highest score in $I \setminus C^*(I)(s-1)$, denoted $j^s_{t_2}$, and the individual with trait t_1 with the highest score in $I \setminus C^*(I)(s-1) \cup \{j^s_{t_2}\}$, denoted $j^s_{t_1}$. Set $I^s_{t_2,t_1} = \{j^s_{t_1}, j^s_{t_2}\}$.
- Choose the individual with both traits t_1 and t_2 with the highest score in $I \setminus C^*(I)(s-1)$, if such individual exists, denoted by i^s_O. Then, choose the individual with the highest score in $I \setminus C^*(I)(s-1) \cup \{i^s_O\}$, denoted by i^s. Set $I^s_O = \{i^s_O, i^s\}$. Otherwise, if there is no such individual, set $I^s_O = \{\emptyset\}$.

Choose the set of individuals that maximizes the total score:

$$I^s = I^s_{p^*}, \text{ with } p^* = \arg \max_{p \in \{(t_1,t_2),(t_2,t_1),O\}} (m(I^s_j)).$$

Set

$$C^*(I)(s) = C^*(I)(s-1) \cup I^s,$$

$$r_{t_1}(s) = r_{t_1}(s-1) - |I^s_{t_1}|,$$

$$r_{t_2}(s) = r_{t_2}(s-1) - |I^s_{t_2}|,$$

$$q(s) = r_{t_1}(s-1) - r_{t_1}(s) + r_{t_2}(s-1) - r_{t_2}(s) - |I^s|.$$

If $r_{t_1}(s) = 0, r_{t_2}(s) = 0$ and $q(s) = 0$, then the procedure terminates. Otherwise, proceed to the next step.

In words, the MSMG choice rule proceeds as follows: In **Part 0**, the rule begins by identifying the necessary reserves, which we refer to as *requirements.* The remaining capacity (excess to these requirements) is then determined. This sets up the framework for the subsequent parts. Three ordered parts follow, each containing multiple steps, which guide the selection process until the requirements are fully satisfied.

In **Part A**, if the remaining capacity from **Part 0** is positive, this capacity is filled by individuals with the highest scores. If individuals with traits are selected during this phase, the corresponding requirements are reduced accordingly. If the remaining capacity is zero or negative, the procedure moves to **Part B**.

In **Part B**, the process begins with the remaining requirements and capacity. Seats are filled incrementally to equalize the requirements for both traits. For the trait with the higher remaining requirement, C^* selects the individual with the highest score among those not yet selected. If any of the selected individuals also possess the other trait, the requirements are adjusted accordingly.

If, at this point, the requirements for both traits are equalized, the rule transitions to **Part C**. If not, the process repeats **Part A**, where individuals with the highest score are chosen until the balance is achieved.

Part C deals with complementarities between traits. After initializing the remaining requirements and capacity, C^* selects two individuals using one of three methods:

- The first method selects the individual with the highest score who has trait t_1, followed by the individual with the highest score who has trait t_2.
- The second method reverses this order, selecting first the individual with trait t_2 and then the individual with trait t_1.[9]
- The third method selects the individual with the highest score who possesses both traits, and then another individual with the highest score from the remaining pool.

After these selections, C^* compares the total scores from the three pairs of individuals and chooses the one with the highest total score. Requirements are then reduced according to the traits of the selected individuals.

In the following example, we illustrate how C^* selects individuals.

Example 2. Let $I = \{i_1, i_2, ..., i_{12}\}$, $q = 8$, $r_{t_1} = 4$, and $r_{t_2} = 2$. The distributions of traits and scores are given in the following table, where the black dots (•) represent the traits of each individual.

Individuals	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	i_{11}	i_{12}
t_1		•					•		•	•	•	•
t_2					•			•		•		•
Score	100	99	98	95	80	75	70	65	60	55	50	45

Table 1: Traits and scores.

C^* proceed as follows:

- **Part 0:** The initial requirements are given by $r_{t_1}(0) = \min\{4, 6\} = 4$, $r_{t_2}(0) = \min\{2, 4\} = 2$, and the capacity is $q(0) = 8 - 4 - 2 = 2 > 0$.
- **Part A: Step 1.** Since $q(0) > 0$, C^* chooses the two individuals with the highest score in I, namely $\{i_1, i_2\}$. Since $\tau(i_2) = \{t_1\}$ we have

$$r_{t_1}(1) = 4 - 3 \text{ and } r_{t_2}(1) = 2 - 0,$$

$$q(1) = (4 - 3) + (2 - 2) = 1 > 0.$$

[9]This step helps prevent selection effects that may arise when individuals belong to multiple groups (see Dur et al., 2018, Sönmez and Yenmez, 2020, and Sönmez and Yenmez, 2022). Example 3 illustrates this phenomenon.

- **Part A: Step 2.** C^* chooses the individual with the highest score in $I \setminus \{i_1, i_2\}$, namely $\{i_3\}$. Since $\tau(i_2) = \{\emptyset\}$ we have

$$r_{t_1}(2) = 3 - 0 \text{ and } r_{t_2}(2) = 2 - 0,$$

$$q(2) = (3 - 3) + (2 - 2) = 0.$$

- **Part B: Step 1.** Since $r_{t_1}(0) > r_{t_2}(0)$, C^* chooses the individual with the highest score in $I \setminus \{i_1, i_2, i_3\}$ with trait t_1, namely $\{i_7\}$. Since $t_2 \notin \tau(i_7)$ we have

$$r_{t_1}(1) = 3 - 1 \text{ and } r_{t_2}(1) = 2 - 0,$$

$$q(1) = (3 - 2) + (2 - 2) - 1 = 0.$$

- **Part C: Step 1.** Now $r_{t_1}(0) = r_{t_2}(0) = 2$.
  - C^* select the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_7\}$ with trait t_1, namely $\{i_9\}$, and the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_7, i_9\}$ with trait t_2, namely $\{i_5\}$, with $I^1_{t_1,t_2} = \{i_5, i_9\}$ and $m(I^1_{t_1,t_2}) = 140$.
  - C^* select the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_7\}$ with trait t_2, namely $\{i_5\}$, and the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_7, i_5\}$ with trait t_1, namely $\{i_9\}$, with $I^1_{t_2,t_1} = \{i_5, i_9\}$ and $m(I^1_{t_2,t_1}) = 140$.
  - C^* select the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_7\}$ with trait t_1 and t_2, namely $\{i_{10}\}$, and the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_7, i_{10}\}$, namely $\{i_4\}$, with $I^1_O = \{i_4, i_{10}\}$ and $m(I^1_O) = 150$.

  C^* chooses I^1_O. Since $\tau(i_4) = \{\emptyset\}$ we have

$$r_{t_1}(1) = 2 - 1 \text{ and } r_{t_2}(1) = 2 - 1,$$

$$q(1) = (2 - 1) + (2 - 1) - 2 = 0.$$

- **Part C: Step 2.** Now $r_{t_1}(1) = r_{t_2}(1) = 1$.
  - C^* select the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_4, i_7, i_{10}\}$ with trait t_1, namely $\{i_9\}$, and the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_4, i_7, i_9, i_{10}\}$ with trait t_2, namely $\{i_5\}$, with $I^2_{t_1,t_2} = \{i_5, i_9\}$ and $m(I^2_{t_1,t_2}) = 140$.
  - C^* select the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_4, i_7, i_{10}\}$ with trait t_2, namely $\{i_5\}$, and the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_4, i_5, i_7, i_{10}\}$ with trait t_1, namely $\{i_9\}$, with $I^2_{t_2,t_1} = \{i_5, i_9\}$

and $m(I^2_{t_2,t_1}) = 140$.

– C^* select the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_4, i_7, i_{10}\}$ with trait t_1 and t_2, namely $\{i_{12}\}$, and the individual with the highest score in $I \setminus \{i_1, i_2, i_3, i_4, i_7, i_{10}, i_{12}\}$, namely $\{i_5\}$, with $I^2_O = \{i_5, i_{12}\}$ and $m(I^2_O) = 125$.

C^* chooses $I^2_{t_1,t_2}$. Since $\tau(i_5) = \{t_2\}$ and $\tau(i_9) = \{t_1\}$ we have

$$r_{t_1}(2) = 1 - 1 \text{ and } r_{t_2}(2) = 1 - 1,$$

$$q(1) = (1 - 0) + (1 - 0) - 2 = 0.$$

Thus, $C^*(I) = \{i_1, i_2, i_3, i_4, i_5, i_7, i_9, i_{10}\}$.

Our first result establishes that the MSMG choice rule satisfies the axioms presented in Section 2.

Theorem 1. The MSMG choice rule C^* maximally complies with reservations, eliminates justified envy, and is non-wasteful.

Sönmez and Yenmez (2020), and later Dur and Zhang (2023) show that several rules can satisfy these axioms. However, in **Part C**, the construction of the MSMG choice rule allows for the consideration of complementarities. A key feature of this part is the selection of three pairs of individuals. The intuition behind this process is as follows: if the set I_O (which consists of the individual with the highest score who possesses both traits, followed by the individual with the highest score among those not yet selected) has the highest total score, this indicates that the individual with both traits has enabled the selection of someone who otherwise would not have been chosen. By comparing the scores across the three sets, complementarities are examined. The underlying idea is to account for potential complementarities and to allow the recruitment of individuals who possess multiple traits. If these individuals have sufficiently high scores, or if their selection enables the recruitment of an individual with a higher score, they will be chosen. The next example illustrates **Part C** of C^* and the impact of complementarities on the assignment process.

Example 3. Let $I = \{i_1, i_2, i_3, i_4\}$, $q = 2$, $r_{t_1} = 1$, and $r_{t_2} = 1$. The distributions of traits and scores are given in the following table.

In this example, C^* proceeds directly to Part C. We have the following sets and scores:

Example 3 illustrates two key points. First, the order in which individuals are selected is important, as the resulting sets differ. Second, when the two sets I_{t_1,t_2} and I_{t_2,t_1}

Individuals	i_1	i_2	i_3	i_4
t_1		•	•	
t_2		•		•
Score	100	90	80	70

Table 2: Traits and scores.

Sets	Individuals	Score
I_{t_1,t_2}	$\{i_2, i_3\}$	170
I_{t_2,t_1}	$\{i_2, i_4\}$	160
I_O	$\{i_1, i_2\}$	190

Table 3: Sets and scores.

obtained in **Part C** are distinct, the set formed by first selecting the individual with the highest score possessing both traits, followed by the individual with the highest score, yields the highest total score. The following proposition formalizes this observation.

Proposition 1. Consider **Part C** in the MSMG choice rule C^*. If for some s, $I^s_{t_1,t_2} \neq I^s_{t_2,t_1}$, then

$$O = \arg \max_{p \in \{(t_1,t_2),(t_2,t_1),O\}} m(I^s_p).$$

Proposition 1 shows that complementarity results in an increase in the assignment score when considering the one-to-all selection method.

Sönmez and Yenmez (2020) define the class of paired-admissions choice rules, which include rules that maximally comply with reservations, eliminate justified envy, and are non-wasteful. Our main result establishes that the MSMG choice rule is not score-dominated by any choice rule in this class.

Theorem 2. The MSMG choice rule C^* is not score-dominated by any choice rule that maximally complies with reservations, eliminates justified envy, and is non-wasteful.

4 Discussion

In this section, we discuss the application of our results and the computational limitations that may arise when considering more than two traits.

4.1 Applications of the MSMG Choice Rule

The MSMG choice rule balances meritocracy and fairness while maximizing the assignment score. It serves two main objectives. First, it minimizes mismatches in assignments. In the context of the labor market, where scores represent individuals' productivity, maximizing the assignment score is akin to optimizing economic output, benefiting both firms and employees. Second, the rule enhances opportunities for individuals with multiple protected traits. As highlighted in the introduction, traditional affirmative action policies often fail to address the complexities of supporting individuals who belong to multiple disadvantaged groups. By accounting for complementarities and incorporating individuals' scores, this approach increases the attractiveness of candidates with multiple traits. As a result, firms can more efficiently meet diversity quotas, gaining greater flexibility in their hiring strategies.

A key application of this framework is in the design of compensation structures for individuals with multiple protected traits. By incorporating individual scores, this method offers a systematic procedure for determining the necessary bonuses or incentives to ensure the selection of such individuals. In a labor market setting, this approach allows for the precise identification of the level of subsidy a firm should receive to incentivize hiring while maintaining meritocratic principles. This methodology is particularly relevant for public sector hiring, university admissions, and corporate diversity initiatives, where structured affirmative action policies aim to balance fairness and efficiency. For example, universities implementing intersectional affirmative action programs can use this approach to allocate financial aid more effectively, ensuring that students from multiple underrepresented backgrounds receive equitable support.

4.2 Limitations and Computational Challenges

This paper makes a twofold contribution to the literature. First, we introduce a novel approach to handling complementarities while adhering to fairness criteria. Previous research has highlighted the computational challenges involved in constructing fair assignments when there is overlap between groups. While we acknowledge these challenges, conventional methods often rely on either restrictive axiomatic frameworks or brute-force computations to determine optimal assignments.[10] In contrast, we propose a choice rule that is both intuitive and practical to implement.

[10] As discussed by Dur and Zhang (2023), some approaches require computing all assignments that meet the representation thresholds, followed by applying a choice rule to these assignments to determine fairness.

Secondly, we provide insights into extending the MSMG choice rule to scenarios involving more than two protected traits. For example, with three traits, **Part C** of C^* must evaluate multiple assignment configurations: selecting the individual with the highest score possessing all three traits, selecting individuals with two of the three traits along with another individual possessing the remaining trait, and so on. The emergence of such complementarities in a setting with more than two traits introduces significant computational complexities.

These challenges align with findings from (Sönmez and Yenmez, 2020), who establish that no paired-admissions choice rule satisfies the substitutes condition (Proposition 5 of Sönmez and Yenmez, 2020). Consequently, while our approach offers a structured solution for designing multi-trait affirmative action mechanisms, future research should focus on addressing the computational scalability of these methods in settings with more than two traits.

A Proofs

Before proving our results, we introduce a lemma for Theorem 1. Lemma 1 establishes that in **Part C** of C^*, if the pair with the highest score is the one obtained by considering the set I^s_O, and the second individual has at least one trait, then I^s_O coincides with at least one of the two other sets under consideration.

Lemma 1. Consider **Part C** in the MSMG choice rule C^*. If for some s, $O = \arg\max_{p \in (t_1,t_2),(t_2,t_1),O} m(I^s_p)$ and for some $t \in \mathcal{T}$, $|(I^s_O)_t| > 1$, then there exists $p \in \{(t_1, t_2), (t_2, t_1)\}$ such that $I^s_O = I^s_p$.

Proof. We know that $m(I^s_O) \geq m(I^s_{t_1,t_2})$ and $m(I^s_O) \geq m(I^s_{t_2,t_1})$. Let i_O denote the first individual selected from I^s_O and j the second. We know that j possesses a trait (i.e., $\tau(j) \neq \emptyset$). Without loss of generality, assume that $t_1 \in \tau(j)$. Since j was selected as the individual with the highest score among those not yet chosen, we have two possible cases:

- If $\sigma(i_O) > \sigma(j)$, then $I^s_{t_2,t_1} = \{i_O, j\}$ because $t_2 \in \tau(i_O)$ and i_O is the individual with the highest score. Additionally, j is the individual with the trait t_1 and the highest score among those not yet chosen. Therefore, $I^s_O = I^s_{t_2,t_1}$.
- If $\sigma(j) > \sigma(i_O)$, then $I^s_{t_1,t_2} = \{i_O, j\}$, as $m(I^s_O) \geq m(I^s_{t_1,t_2})$, and j has the trait t_1, with $t_2 \in \tau(i_O)$. Thus, $I^s_O = I^s_{t_1,t_2}$.

■

A.1 Proof of Theorem 1

Proof.

Lemma 2. C^* maximally complies with reservations.

Proof. Consider an arbitrary set of individuals $I \subseteq \mathcal{I}$. By the definition of the choice rule C^*, in **Part 0**, $r_{t_1}(0) = \min\{r_{t_1}, |I_{t_1}|\}$ and $r_{t_2}(0) = \min\{r_{t_2}, |I_{t_2}|\}$. This construction follows from Definition 1.

In **Part B**, individuals are selected based on the reserves. It is straightforward that individuals with the trait with the highest $r_t(\ell)$, $t \in \{t_1, t_2\}$, are chosen until $r_{t_1}(\ell) = r_{t_2}(\ell)$. C^* then proceeds to **Part C**, where individuals are selected in pairs. The rule remains in **Part C** unless, within a selected pair, one individual possesses both traits, while the other has at least one trait. In such cases, individuals are selected in a manner that restores the equality $r_{t_1}(\ell) = r_{t_2}(\ell)$. The rule continues until $r_{t_1}(\ell) = r_{t_2}(\ell) = 0$. ■

Lemma 3. C^* eliminates justified envy.

Proof. Suppose, for contradiction, that C^* does not eliminate justified envy. That is, there exist a set of individuals $I \subseteq \mathcal{I}$, an individual $i \in C^*(I)$ and an individual $j \in I \setminus C^*(I)$ such that $\sigma(j) > \sigma(i)$ and

$$|\,[(C^*(I) \setminus \{i\}) \cup \{j\}]_{t_1}\,| \geq \min\{r_{t_1}, |I_{t_1}|\}, \quad |\,[(C^*(I) \setminus \{i\}) \cup \{j\}]_{t_2}\,| \geq \min\{r_{t_2}, |I_{t_2}|\}.$$

Since i is chosen and j is rejected even though j has a higher score than i, i could not be chosen in **Part A** of C^*. Therefore, it must be the case that i possesses a trait that j does not, according to the construction of $C^*(I)$ in **Part B** or **Part C**. Without loss of generality, let t_1 be this trait. Since

$$|\,[(C^*(I) \setminus \{i\}) \cup \{j\}]_{t_1}\,| \geq \min\{r_{t_1}, |I_{t_1}|\},$$

there must be at least $r_{t_1} + 1$ individuals in $C^*(I)$ who have trait t_1. We now consider two cases based on whether i has trait t_2 or not.

- **Case 1:** $t_2 \in \tau(i)$, so $\tau(i) = \{t_1, t_2\}$. Individual i must have been selected in **Part B** or **Part C** for her trait t_2 or because i has both traits. We further consider two possibilities:

  (i) Suppose that there are at least $r_{t_2} + 1$ individuals with trait t_2 in $C^*(I)$. We know that at least one individual with trait t_1 and one individual with trait t_2 must have been chosen in **Part A** at the end of the procedure. Let

$\bar{i}$ be the last individual with trait t_1 chosen, and $\underline{i}$ the last individual with trait t_2 chosen. We know that $\sigma(\bar{i}) > \sigma(j)$ and $\sigma(\underline{i}) > \sigma(j)$. Let $\bar{\underline{i}}$ be the last individual with both traits to be selected by C^*. This individual exists because there is at least one individual with both traits chosen, as $i \in C^*(I)$. If $\bar{\underline{i}}$ was chosen in **Part A** the contradiction is direct since j has not been chosen. Thus it follows that $\sigma(\bar{\underline{i}}) > \sigma(j)$. Since i has been chosen before $\bar{\underline{i}}$, we know that $\sigma(i) \geq \sigma(\bar{\underline{i}}) > \sigma(j)$ a contradiction. Therefore, suppose that $\bar{\underline{i}}$ has been chosen in **Part B** or **Part C**.

If $\bar{\underline{i}}$ was chosen in **Part B**, we know that for a trait $t \in \{t_1, t_2\}$ there was a higher reserve requirement. The individual $\bar{\underline{i}}$ was therefore chosen for one of her traits, and as the individual with this trait having the highest score. Thus, $\bar{\underline{i}}$ has a higher score than $\bar{i}$ if the trait was t_1 and $\underline{i}$ if the trait was t_2. Since at least one of the two statements must be true, it follows that $\sigma(\bar{\underline{i}}) > \sigma(j)$.

If $\bar{\underline{i}}$ has been chosen in **Part C**, at a step s we have three possibilities, depending on which set maximizes the score. If $\bar{\underline{i}} \in I^s_{t_1,t_2}$, this means that i has been chosen for one of her traits, and it follows that $\bar{\underline{i}}$ has a higher score than $\bar{i}$ if the trait was t_1 and $\underline{i}$ if the trait was t_2. The second case where $\bar{\underline{i}} \in I^s_{t_2,t_1}$ is symmetrical. Finally, if $\bar{\underline{i}} \in I^s_O$ we know that

$$m(I^s_O) \geq m(I^s_{t_2,t_1}) \text{ and } m(I^s_O) \geq m(I^s_{t_1,t_2}).$$

If the individual chosen for her score has a trait, then by Lemma 1, we know that $m(I^s_O) = m(I^s_{t_1,t_2})$ or $m(I^s_O) = m(I^s_{t_2,t_1})$, and therefore $\bar{\underline{i}}$ was chosen for her trait, and that $\sigma(\bar{\underline{i}}) > \sigma(j)$. Thus, this individual has no trait. By construction of C^*, the requirement for the trait only decreased by 1, and $q(s) = 0$, meaning it was not possible to proceed to **Part A**. Consequently, the individuals $\underline{i}$ and $\bar{i}$ could not have been selected, which leads to a contradiction as there are at least $r_{t_1} + 1$ individuals with trait t_1 in $C^*(I)$, and $r_{t_2} + 1$ individuals with trait t_2 in $C^*(I)$.

(ii) Suppose that there are at most r_{t_2} individuals with trait t_2 in $C^*(I)$. In this case, j must also have t_2 since $|\,[(C^*(I) \setminus \{i\}) \cup \{j\}]_{t_2}\,| \geq \min\{r_{t_2}, |I_{t_2}|\}$. Therefore, $\tau(j) = \{t_2\}$. Similar to Case 1 (i), consider the last individual with the trait t_2 chosen in C^*, denoted $\underline{i}$. We know that $\sigma(\underline{i}) \geq \sigma(j)$ since there must be at least $r_{t_1} + 1$ individuals with trait t_1 in $C^*(I)$. If $\underline{i}$ also has the trait t_1, then by construction we have that $\sigma(i) \geq \sigma(\underline{i})$ and therefore

$\sigma(i) \geq \sigma(j)$, a contradiction. Suppose that $\underline{i}$ does not have trait t_1. We consider the last individual $i^\dagger$ with trait t_1 to be chosen. Since there are at least $r_{t_1}+1$ individuals in $C^*(I)$ who have trait t_1, we know that $i^\dagger$ is chosen in **Part A**. By construction of C^*, the choice of traits is made in **Part B** and **Part C**, and $i^\dagger$ must have both traits t_1 and t_2. Therefore, $\sigma(i) \geq \sigma(i^\dagger)$, which implies that $\sigma(i) \geq \sigma(j)$, a contradiction.

- **Case 2:** $t_2 \notin \tau(i)$, so $\tau(i) = \{t_1\}$. Since $|\,[(C^*(I) \setminus \{i\}) \cup \{j\}]_{t_1}\,| \geq \min\{r_{t_1}, |I_{t_1}|\}$, we know that the last individual with trait t_1 has been choosen in **Part A**. Let $i^\dagger$ denote this individual. Since $\sigma(i^\dagger) > \sigma(j)$, and by the construction of C^*, we know that i has been selected in **Part B** or **Part C** for her trait. Therefore, $\sigma(i) \geq \sigma(i^\dagger) > \sigma(j)$, a contradiction.

■

Lemma 4. C^* is non-wasteful.

Proof. Consider an arbitrary set of individuals $I \subseteq \mathcal{I}$ and q. In **Part 0**, if $|I| \leq q$ then all individuals in I are selected, and we have $C^*(I) = I$, ensuring that $C^*(I)$ is non-wasteful. We only need to consider the case where $|I| > q$. In **Part 0**, the requirement for trait t_1 is given by $r_{t_1}(0) = \min\{r_{t_1}, |I_{t_1}|\}$, which guarantees that there are individuals with trait t_1 present in I. Symmetrically with t_2. At the end of **Part A**, the selection process ensures that the total number of remaining unassigned individuals is at most $r_{t_1}(k) + r_{t_2}(k)$. For **Part B** and **Part C**, if more traits are filled than the number of individuals (i.e., at least one individual with both traits is chosen), then the process returns to **Part A** concluding that C^* is non-wasteful. ■

■

A.2 Proof of Proposition 1

Proof. To prove Proposition 1, we first show that if for some s, $I^s_{t_1,t_2} \neq I^s_{t_2,t_1}$ then $I^s_{t_1,t_2} \cap I^s_{t_2,t_1} \neq \emptyset$.

Let i_1 be the first individual selected in $I^s_{t_1,t_2}$ and i_2 the second, where i_1 has trait t_1 and i_2 has trait t_2. Let j_2 be the first individual selected in $I^s_{t_2,t_1}$ and j_1 the second, where j_2 has trait t_2 and j_1 has trait t_1. By contradiction, suppose $I^s_{t_1,t_2} \cap I^s_{t_2,t_1} = \emptyset$. We know that $\sigma(i_1) > \sigma(j_1)$ since $i_1 \in I^s_{t_1,t_2}$, and $\sigma(i_2) > \sigma(j_2)$ as $i_2 \in I^s_{t_1,t_2}$ or $j_2 \notin I \setminus \{i_1\}$. Similarly, we also know that $\sigma(j_2) > \sigma(i_2)$ since $j_2 \in I^s_{t_2,t_1}$, and $\sigma(j_1) > \sigma(i_1)$ or $i_1 \notin I \setminus \{j_2\}$. This leads to a direct contradiction, and we must conclude that $i_1 = j_2$. By construction we know that $\tau(i_1) = \{t_1, t_2\}$. Consider now the construction of I^s_O. Let

i_O be the first individual selected in I^s_O, and j_O the second. It follows that $i_O = i_1 = j_2$, since if $i_O \neq i_1$, it would contradict the fact that $\sigma(i_O) > \sigma(i_1)$, and $\tau(i_O) = \{t_1, t_2\}$, which would imply that i_O must be selected first in $I^s_{t_1,t_2}$. Since j_O is the individual with the highest score in $I \setminus \{i_1\}$, we have $\sigma(j_O) \geq \sigma(i_2)$ and $\sigma(j_O) \geq \sigma(j_1)$. Therefore, $m(I^s_O) \geq m(I^s_{t_1,t_2})$ and $m(I^s_O) \geq m(I^s_{t_2,t_1})$ which conclude the proof. ■

A.3 Proof of Theorem 2

Proof. Consider a set of individuals $I \subseteq \mathcal{I}$. If $|I| \leq q$, then $C^*(I) = I$ and I is the only assignment that maximally complies with reservations and which is non-wasteful. Thus, it suffices to focus on the case where $|I| > q$. Since C^* is non-wasteful, we have $|C^*(I)| = q$.

Let $J \subseteq I$ be an assignment that maximally complies with reservations, eliminates justified envy, and is non-wasteful, such that $C^*(I) \neq J$. We must show that $m(C^*(I)) \geq m(J)$. Suppose by contradiction that $m(J) > m(C^*(I))$.

By construction of C^*, it is clear that the difference in assignment scores cannot be made at **Part A**, since $C^*(I)$ selects individuals with the highest score. We consider three cases:

- **Case 1:** If $r_{t_1} = r_{t_2} = 0$, then C^* selects the individuals in I with the highest score up to q in **Part A**. Since J is non-wasteful and eliminates justified envy, it is not possible that $m(J) > m(C^*(I))$.
- **Case 2:** If $r_{t_1} > 0$ and $r_{t_2} = 0$, then C^* proceeds to **Parts A** and **B** only. In **Part A**, C^* chooses the $q - r_{t_1}$ individuals with the highest score. Requirements are reduced if some of these individuals have the trait t_1, and other individuals with the highest score are chosen. When, for some k, $q(k) = 0$, C^* proceeds to **Part B**, where it selects individuals with the highest scores among those with the trait t_1. This construction follows that of the Over-and-Above Choice Rule (Dur et al., 2018). Thus, if J maximally complies with reservations, it is not possible for $m(J) > m(C^*(I))$. Without loss of generality, the same reasoning holds if $r_{t_1} = 0$ and $r_{t_2} > 0$.
- **Case 3:** If $r_{t_1} > 0$ and $r_{t_2} > 0$, then, C^* begins with **Part A**, selecting the $q - r_{t_1} - r_{t_2}$ individuals with the highest scores. The requirements are adjusted if some of these individuals possess trait t_1 or t_2, and other individuals with higher scores are chosen. When, for some k, $q(k) = 0$, C^* proceeds to **Part B**. Up to this point, it is clear that the selected individuals are those with the highest scores.
  In **Part B**, assume, without loss of generality, that $r_{t_1}(l = 1) > r_{t_2}(l = 1)$. C^*

selects the $r_{t_1}(l=1) - r_{t_2}(l=1)$ individuals with trait t_1 who have the highest scores. Since J maximally complies with the representation constraints, it follows directly that the representation of individuals with trait t_1 in J must be at least $r_{t_1}(l=1) - r_{t_2}(l=1)$. By selecting individuals with the highest scores, it is impossible for the individuals in J with trait t_1 to have a higher score than those already chosen by C^*.

If some of the selected individuals also possess trait t_2, then $r_{t_2}(l=1)$ is adjusted accordingly. C^* then proceeds to **Part A**, selecting individuals with the highest scores.

If $r_{t_2}(k) = 0$ or $r_{t_1}(k) = 0$, we consider Case 2 and conclude that it is not possible for $m(J) > m(C^*(I))$. If $r_{t_2}(k) = r_{t_1}(k) = 0$, we consider Case 1 and conclude that it is not possible for $m(J) > m(C^*(I))$.

Otherwise, **Parts A** and **B** are repeated until $r_{t_1}(l=1) = r_{t_2}(l=1)$, at which point C^* proceeds to **Part C**. In **Part C**, three sets are selected, and only the one with the highest total score is chosen. We consider the next claim.

Claim 1. If $m(I^s_{t_1,t_2}) > m(I^s_O)$ $(m(I^s_{t_2,t_1}) > m(I^s_O))$, then there is no $i \in I^s_{t_1,t_2}$ $(i \in I^s_{t_2,t_1})$ such that $\tau(i) = \{t_1, t_2\}$.

Proof. Suppose, by contradiction, that $m(I^s_{t_1,t_2}) > m(I^s_O)$ and there exists $i \in I^s_{t_1,t_2}$ such that $\tau(i) = \{t_1, t_2\}$. Without loss of generality, assume that i was selected for its trait t_1. By construction, we know that i is also selected in I^s_O since i has both traits, and is the individual with trait t_1 who has the highest score. The second individual selected in I^s_O is the one with the highest score among those who have not yet been chosen. Therefore, $m(I^s_O) \geq m(I^s_{t_1,t_2})$, leading to a contradiction. ■

In **Part C**, if $I^s_{t_1,t_2}$ (or $I^s_{t_2,t_1}$) is chosen, then we know that no individual with both traits (Claim 1) is included. Individuals are selected first from those with trait t_1 (t_2) who have the highest score, and then from those with trait t_2 (t_1) who have the highest score. The requirements decrease by 1 at each iteration. By construction, it is therefore not possible to select individuals with a higher score while still maximally complying with the reservation constraints.

If I^s_O is the set that maximizes the score, then there are three possible cases:

- If both individuals in I^s_O with both traits, then the one with the lowest score was selected because they were the individual with the highest score among those not yet chosen. The requirements decrease by 2, and the algorithm proceeds. If we had $r_{t_1}(l) = r_{t_2}(l) = 1$, the individual with the lowest score in I^s_O is still the individual with the highest score among those not yet selected, making it impossible to choose an individual with a higher score.

- If one individual in I_O^s has both traits while the other has none, then the requirements decrease by 1, and the algorithm continues. Since I_O^s is the set that maximizes the score, we know that it is not possible to consider a set with a higher score while still maximally complying with the reservation constraints.

- If the individual selected for their score in I_O^s with a single trait, then the requirements decrease by 2 and by 1 accordingly. Since I_O^s is the set that maximizes the score, we know that it is not possible to consider a set with a higher score while still maximally complying with the reservation constraints. At the next step, if $r_{t_1}(l+1) \neq r_{t_2}(l+1)$, then C^* proceeds to **Part A** and chooses the individuals with the highest scores among those not yet selected.

Following this reasoning, it is not possible that $m(J) > m(C^*(I))$, which concludes the proof. ■

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