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Strategic formation of production networks

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Abstract

We provide a strategic model of the formation of production networks that subsumes the standard general equilibrium approach. The objective of firms in our setting is to choose their supply relationships so as to maximize their profit at the general equilibrium that unfolds. We show that this objective is equivalent to the maximization by the firms of their eigenvector centrality in the production network. As is common in network formation games based on centrality, there are multiple Nash equilibria in our setting. We have investigated the characteristics and the social efficiency of these equilibria in a stylized version of our model representing international trade networks. We show that the impact of network structure on social welfare is firstly determined by a trade-off between costs of increasing process complexity and positive spillovers on productivity induced by the diversification of the input mix. We further analyze a variant of our model that accounts for the risks of disruption of supply relationships. In this setting, we characterize how social welfare depends on the structure of the production network, the spatial distribution of risks, and the process of shock aggregation in supply chains. We finally show that simple trade policies characterized by sets of links that are either prevented or catalyzed can be a powerful equilibrium selection device.

JEL Classification: D85, C65, D83

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1 Introduction

Decentralized network formation processes are key mechanisms in shaping global socio-economic structures. A number of recent episodes have shown that substantial negative externalities can be associated with these processes. Behaviours aligned with micro-level objectives can lead to the emergence of risk at the macro-level: risk of polarization in social networks [Levy, 2021], risk of epidemic spreading in human networks [Pastor-Satorras and Vespignani, 2001, Antràs et al., 2023], systemic risk in financial networks [Acemoglu et al., 2015], macro-economic risk in production networks [Acemoglu et al., 2012, Elliott et al., 2022]. Against this background, the oversight of network formation processes is becoming politically relevant in a number of fields. First and foremost, deglobalization is at the forefront of the policy agenda in a range of developed economies [Witt, 2019, Kornprobst and Paul, 2021]. It is partly conceived as a policy response to raising global risks [Razin, 2020, Irwin, 2020].

Despite the policy relevance of the issue, there exists few formal models of the interplay between decentralized network formation processes and macro-level social objectives. This paper aims to advance the literature in this direction by providing a model of the strategic formation of production networks, investigating the social efficiency of Nash equilibria and identifying potential welfare improving policies in this setting. Our model subsumes the standard general equilibrium approach. We consider that the production technologies of firms are determined by their choice of suppliers. Accordingly, firms choose strategically their supply relationships in view of the profits the corresponding technologies will yield at a general equilibrium of the economy emerging as a Nash equilibrium of this production network formation game. In other words, firms aim to insert themselves efficiently in the network of exchanges, assuming markets are competitive.

We show that the objective of firms can equivalently be expressed, in network terms, as the maximization of their eigenvector centrality in the production network. In such centrality maximization games, the set of Nash equilibria is typically very large [Catalano et al., 2022]. This induces substantial indeterminacy, and potentially major inefficiency, of economic outcomes. We investigate this issue more specifically in a setting where supply relationships can be disrupted by exogenous shocks, due e.g. to natural hazards. In this setting, socially efficient production networks shall balance the structure of risk and the distribution of productivity to deliver the largest expected supply. On the contrary, the decentralized supply choices of firms, which focus on their own centrality, neglect the risk they transmit through their supply chains. This can lead to substantial inefficiencies. Namely, we show that the price of anarchy can become infinitely large as the number of firms grow. Against this background, we investigate policy measures that can nudge firms towards efficient equilibria by restricting the set of admissible supply relationships, e.g. through preferential trade agreements. Such measures can be very efficient: a few targeted interventions can reduce substantially the indeterminacy of equilibrium and the scope of potential inefficiencies.

1.1 Related literature

Our contribution builds on the recent thread of literature that analyses general equilbrium economies through the prism of network theory in order to generate new macro-economic insights [Acemoglu et al., 2012, Gualdi and Mandel, 2016, Moran and Bouchaud, 2019, Carvalho et al., 2021, Dessertaine et al., 2022]. It is more specifically related to [Gualdi and Mandel, 2019] and [Acemoglu and Azar, 2020], which subsume a standard general equilibrium model into a dynamic network formation process. Yet, these contributions focus in the interplay between network formation and economic growth and do not consider strategic behaviour of firms in their choice of supply relationships. Notably, in [Acemoglu and Azar, 2020], firms choose their suppliers so as to minimize production costs and thus somehow acts as the agents of the representative household, limiting the potential scope of negative externalities.

Our focus on the role of production networks in the propagation of risks is linked to a range of contributions about network-based amplification of micro-economic risks [Bak et al., 1993, Battiston et al., 2007, Acemoglu et al., 2012, Barrot and Sauvagnat, 2016, Carvalho et al., 2021]. Yet, these contributions generally aim at quantifying the amplification of micro-economic shocks through network effects whereas we are concerned with the micro-economic and behavioral determinants of the riskiness of the network. In this respect, our approach is closely related to Elliott et al. [2022] that shows endogenous formation of supply networks can be conductive to fragility. However the approach of Elliott et al. [2022] is less directly related to general equilibrium than ours, analyzes specifically tree-like supply chains, and focuses on fragility per se rather than on the interplay between strategic behaviour and social efficiency.

Its emphasis on social efficiency relates our work to previous contributions that have investigated, in different contexts, negative externalities induced by network formation processes. Fagiolo [2005] investigates a model of network formation where externalities become negative as the size of an agent's neighborhood grows. Carayol et al. [2008] analyze network formation in a spatialized version of the connections model of [Jackson and Wolinsky, 1996] and show that emergent networks are insufficiently dense and should be more structured around central agents. Morrill [2011] considers a model where any new relationship imposes a negative externality on the rest of the network and shows that socially efficient and stable networks generally diverge. Finally, Buechel and Hellmann [2012] analyze systematically the sources of inefficiency in network formation and relate situations of positive externalities with stable networks that cannot be too dense, while situations with negative externalities tend to induce "too dense" networks.

The remaining of this paper is organized as follows. Section 2 presents our model of production network formation and its connection with general equilibrium theory. Section 3 investigates the welfare properties of the Nash equilibrium of the model in a context where links can be disrupted by exogenous shocks. Section 4 identifies policy measures that can nudge firms towards efficient equilibria by restricting the set of admissible supply relationships. An extended appendix contains the set of proofs for our results.

2 Strategic formation of general equilibrium networks

2.1 Economic framework

We consider an economy with a representative household, indexed by 0, and a finite number of firms $M := \{1, \cdots, m\} \subset \mathbb{N}^*$ producing differentiated goods. We let $N =$

$M \cup \{0\}$. The representative household supplies one unit of labor, receives the profits from the firms and is characterized by a Cobb-Douglas utility function of the form

$$u(x_{0,1}, \cdots, x_{0,n}) := \prod_{j \in M} x_{0,j}^{a_{0,j}},$$

where $a_0 \in \mathbb{R}_+^M$, $a_{0,i}$ is the share of firm i in the household's consumption expenditure, and $\sum_{j \in M} a_{0,j} = 1$.

As in Acemoglu and Azar [2020], we assume that the firms have the ability to choose their suppliers and that these endogenous choices define their production technology. More specifically, we assume that goods are grouped in a set L of categories/sectors, and we denote by M_ℓ the set of firms producing goods of type ℓ (we consider 0 to denote a specific category for labor, and define $M_0 = 0$). Each firm i is then characterized at a "meta-technological" level by a vector of requirements $b_i = (b_{i,0}, \cdots, b_{i,L}) \in \mathbb{R}_+^L$ such that $\sum_{\ell \in L} b_{i,\ell} \leq 1$. In this setting, $b_{i,\ell}$ represents the proportion of inputs that firm i must source from sector ℓ. Firm i must then choose its suppliers under these sectoral constraints, i.e. it must choose input shares $(a_{i,j})_{j \in M} \in \mathbb{R}_+^M$ towards each agent such that for all $\ell \in L$,

$$\sum_{j \in M_\ell} a_{i,j} = b_{i,\ell}.$$

Its actual production technology is then given by a Cobb-Douglas production function of the form

$$f_{a,i}(x_{i,0}, \cdots, x_{i,m}) := \lambda_i(a_i) \prod_{j \in N} x_{i,j}^{a_{i,j}},$$

where $\lambda_i(a_i) \in \mathbb{R}_{++}$ is a productivity parameter that depends on the choice of suppliers.

The choice by each firm i of its vector of suppliers, a_i, together with the given of the household's consumption shares a_0 defines a general equilibrium economy. Given that the utility and the production functions are Cobb-Douglas, the economy is completely characterized by the productivity functions λ, the consumption shares a_0, and the production network matrix $A := (a_{i,j})_{i \in M, j \in M} \in \mathbb{R}_+^{M \times M}$. We shall thus denote this economy as $\mathcal{E}(\lambda, a)$. A general equilibrium of the economy $\mathcal{E}(\lambda, a)$ is standardly defined as follows:

Definition 1 *A general equilibrium of the economy $\mathcal{E}(\lambda, a)$ is a collection of prices $\overline{p} \in \mathbb{R}_+^N$, (final and intermediary) consumption choices $\overline{x} \in \mathbb{R}_+^{N \times N}$, and output $\overline{y} \in \mathbb{R}_+^N$ (with $\overline{y}_0 = 1$) such that*

(i) Each firm $i \in M$ maximizes its profit under the technological constraints, i.e. $(\overline{y}_i, \overline{x}_i)$ is solution of the following problem:

$$(\mathcal{P}) : \begin{cases} \max & \overline{p}_i y_i - \overline{p} \cdot x_i \\ s.t & y_i = f_{a,i}(x_i) \end{cases}.$$

(ii) The household maximizes its utility under its budget constraint, i.e. $\overline{x}_0$ is solution of the following problem:

$$(\mathcal{C}) : \begin{cases} \max & u(x_0) \\ s.t & \overline{p} \cdot x_0 \leq p_0 + \sum_{j \in M} \overline{p}_j \overline{y}_j - \overline{p} \cdot \overline{x}_j \end{cases}.$$

(iii) All markets clear, i.e. for all $i \in N$, one has

$$\sum_{j \in N} \overline{x}_{j,i} = \overline{y}_i.$$

2.2 Existence of a general equilibrium

Let us recall that in a Cobb-Douglas setting, for all $i, j \in M$, $a_{i,j}$ represents the proportion of firm i's expenses directed towards firm j. Furthermore, if $\sum_{k \in N} a_{i,k} \leq 1$, the profit rate of a profit-maximizing firm i is given by the degree of decreasing returns to scale $\varepsilon_i = (1 - \sum_{j \in N} a_{i,j}) = (1 - \sum_{\ell=0}^{L} b_{i,\ell})$. Hence, if p_0 is the price of labor and $(v_i)_{i \in M}$ are the revenues of the firms at equilibrium, one must have, for all $j \in M$,

$$v_j = a_{0,j} p_0 + a_{0,j} \sum_{i \in M} v_i \varepsilon_i + \sum_{i \in M} a_{i,j} v_i, \tag{1}$$

where $a_{0,j} p_0$ corresponds to consumption of good j based on labor income, $a_{0,j} \sum_{i \in M} v_i \varepsilon_i$ corresponds to consumption of good j based on profit income, and $\sum_{i \in M} a_{i,j} v_i$ corresponds to intermediary consumption of good j (see the proof of Proposition 1 for details). Overall, equilibrium financial flows in the economy can be captured by the following matrix:

$$\tilde{A} := \begin{pmatrix} 0 & a_{1,0} & \cdots & a_{m,0} \\ a_{0,1} & a_{1,1} + \varepsilon_1 a_{0,1} & \cdots & a_{m,1} + \varepsilon_m a_{0,1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{0,i} & a_{1,i} + \varepsilon_1 a_{0,i} & \cdots & a_{m,i} + \varepsilon_m a_{0,i} \\ \vdots & \vdots & \vdots & \vdots \\ a_{0,m} & a_{1,m} + \varepsilon_1 a_{0,m} & \cdots & a_{m,m} + \varepsilon_m a_{0,m} \end{pmatrix}.$$

Thus, one has $\begin{pmatrix} p_0 \\ v \end{pmatrix} = \tilde{A} \times \begin{pmatrix} p_0 \\ v \end{pmatrix}$ if and only if Equation (1) holds for all $j \in M$.

Remark 1 *If there are constant returns to scale, one simply has* $\tilde{A} = \begin{pmatrix} 0 & a_0^T \\ a_{\cdot,0} & A^T \end{pmatrix}$, *where X^T denotes the transpose of X.*

It is straightforward to check that $\tilde{A}$ is column-stochastic. The following conditions then imply that it is aperiodic and irreducible.

Assumption 1 *The model parameters satisfy the following conditions:*

(i) The representative household consumes every good, i.e. $a_0 \in \mathbb{R}^M_{++}$.

(ii) All firms use labor as input, i.e., $b_{i,0} = a_{i,0} > 0$ *for all* $i \in M$.

(iii) At least one firm uses another input than labor in its production process, i.e. there exists $i_0 \in M$ *and* $\ell \in L/\{0\}$ *such that* $b_{i_0,\ell} > 0$.

Lemma 1 *Under Assumption 1, the matrix $\tilde{A}$ is aperiodic and irreducible.*

This suffices to show the existence and to provide a characterization of the general equilibrium of $\mathcal{E}(\lambda, a)$

Proposition 1 *There exists a general equilibrium in the economy $\mathcal{E}(\lambda, a)$ which is unique up to price normalization. Assuming $p_0 = 1$, we further get that*

(i) equilibrium revenues $\overline{v}_i := \overline{p}_i \overline{y}_i$ are such that

$$\begin{pmatrix} 1 \\ v \end{pmatrix} = \tilde{A} \times \begin{pmatrix} 1 \\ v \end{pmatrix},$$

which means that Equation (1) holds and $\sum_{i \in M} a_{i,0} v_i = 1$;

(ii) equilibrium profits are such that

$$\overline{\pi}_i = (1 - \sum_{j \in N} a_{i,j}) \overline{v}_i = (1 - \sum_{\ell \in L} b_{i,\ell}) \overline{v}_i;$$

(iii) equilibrium prices are such that

$$\log(\overline{p}) = (A - I)^{-1} u + (A - I)^{-1} D \log(\overline{v}), \tag{2}$$

where $u_i = \log(\lambda_i) + \sum_{j \in N} a_{i,j} \log(a_{i,j})$, $D = \mathrm{diag}(\sum_{j \in N} a_{i,j} - 1)$, $\log(\overline{p})_i = \log(\overline{p}_i)$, and $\log(\overline{v})_i = \log(\overline{v}_i)$.

2.3 Characterization of social welfare

The characterization of Equilibrium in Proposition 1 allows to provide a closed-form expression for the consumer's welfare associated to a production network structure. Namely, one has the following result:

Lemma 2 *The welfare of the representative consumer at the equilibrium $(\overline{x}, \overline{p}, \overline{y})$ of the economy $\mathcal{E}(\lambda, a)$ is given by*

$$V(a, \lambda) = \log(u(\overline{x})) = \log(v_0) + \sum_{i \in M} a_{0,i} \log(a_{0,i}) + (a_0)^T (I - A)^{-1} u + (a_0)^T (I - A)^{-1} D \log(\overline{v}),$$

where $u_i = \log(\lambda_i) + \sum_{j \in N} a_{i,j} \log(a_{i,j})$ and $D = \mathrm{diag}(\sum_{j \in N} a_{i,j} - 1)$.

This expression further simplifies in the case of constant returns to scale.

Lemma 3 *If for all $i \in M$, $\varepsilon_i = 0$, or equivalently $\sum_{\ell=0}^{L} b_{i,\ell} = \sum_{k \in N} a_{i,k} = 1$, then we have*

$$V(a, \lambda) = \log(p_0) + \sum_{i \in M} a_{0,i} \log(a_{0,i}) + (a_0)^T (I - A)^{-1} u,$$

where $u_i = \log(\lambda_i) + \sum_{j \in N} a_{i,j} \log(a_{i,j})$.

2.4 Strategic framework

The economic framework we consider is very similar to that of Acemoglu and Azar [2020], where firms' suppliers choice also define a general equilibrium economy. However, Acemoglu and Azar [2020] focus on the emergence of endogenous growth through increasing product variety in a setting with constant returns to scale, and where firms choose their suppliers in view of minimizing production costs. In this setting, the incentives of firms are aligned with these of the household, and firms thus somehow act as the agents of the household. We are rather concerned by the strategic behavior of profit maximizing firms and the negative externalities it can induce on the household.

As a benchmark representation of this situation, we consider the normal-form game $\mathcal{G}(a_0, b)$ in which each firm $i \in M$

(i) chooses as a strategy a vector of input weights a_i in the set of admissible technological configurations $S_i(b) := \{a_i \in \mathbb{R}^N_+ \mid \sum_{j\in M_\ell} a_{i,j} = b_{i,\ell}\}$ and

(ii) receives as payoff the equilibrium profit in the hence defined economy $\mathcal{E}(\lambda, a)$, i.e.

$$\pi_i(a_i, a_{-i}) = (1 - \sum_{j\in N} a_{i,j})\overline{v}_i = (1 - \sum_{\ell\in L} b_{i,\ell})\overline{v}_i,$$

where $\overline{v}$ is such that $\overline{v} := \tilde{A}^T\overline{v}$ and $\sum_{j\in M} a_{j,0}\overline{v} = 1$ by Proposition 1.

A Nash equilibrium in the game $\mathcal{G}(a_0, b)$ is defined as follows:

Definition 2 *A Nash equilibrium of the game $\mathcal{G}(a_0, b)$ consists in the choice by each firm $i \in M$ of a vector of input weights $\overline{a}_i$ in the set of admissible technological configurations $S_i(b)$ such that for all $i \in \mathcal{M}$ and all $a_i \in S_i(b)$, one has*

$$\pi_i(\overline{a}_i, \overline{a}_{-i}) \geq \pi_i(a_i, \overline{a}_{-i}).$$

Hence, we represent the formation of production networks as a network formation game in which each firm chooses its suppliers with the aim of maximizing profit at the corresponding economic equilibrium. In economic terms, firms choose their position in the global supply chain, assuming that supply relationships will remain stable and that competition will occur through substitution between suppliers.

Remark 2 *Given the weights b, it is straightforward that $\mathcal{G}(a_0, b)$ is equivalent to the game with strategy space $S_i(b)$ where the payoff of player i is determined by the invariant probability distribution of the Markov chain defined by the transition matrix $\tilde{A}^T$. In other words, the payoff is described by $\mu(a)$ such that $\sum \mu_i(a) = 1$ and $\tilde{A}\mu(a) = \mu(a)$.*

The existence of a Nash equilibrium in $\mathcal{G}(a_0, b)$ then follows from the observation that the game is an *ordinal potential* (see [Monderer and Shapley, 1996]), as in the closely related network formation games considered in [Catalano et al., 2022] and [Cominetti et al., 2022].

Proposition 2 *The game $\mathcal{G}(a_0, b)$ is ordinal potential and admits at least one Nash equilibrium.*

Remark 3 *Following Proposition 1, the profits in the general equilibrium economy $\mathcal{E}(\lambda, a)$ and, consequently, the payoffs in the game $\mathcal{G}(a_0, b)$ are independent of the productivity functions λ.*

2.5 Network-based characterization of equilibrium behavior

From a network-theoretic perspective, Remark 2 implies that $\mathcal{G}(a_0, b)$ is equivalent to the game where firms choose their outgoing network connections to maximize their eigenvector centrality in the network with the adjacency matrix $\tilde{A}$. Given the outgoing links from the household, the eigenvector centrality of firms can be characterized in terms of walks emanating from the household. Namely, let us associate to a link (i, j) the weight $\tilde{a}_{j,i} := a_{i,j} + \varepsilon_i a_{0,j}$ corresponding to the share of revenues of i that are routed towards j, either directly through intermediary consumption $a_{i,j}$ or through the consumer via the spending of profit-related income $\varepsilon_i a_{0,j}$. Accordingly, given a walk of length k from $i \in M$ to $j \in M$ denoted as $p = (h_1, h_2, \ldots, h_k) \in M^k$ with $h_1 = i$ and $h_k = j$, let us define the associated weight as $w_a(p) := \prod_{i=1}^{k-1}(a_{h_i,h_{i+1}} + \varepsilon_{h_i} a_{0,h_{i+1}}) = \prod_{i=1}^{k-1} \tilde{a}_{h_{i+1},h_i}$, representing the share of revenues of i reaching j through the corresponding walk. We then denote by $\mathcal{P}_{j,i}$ the set of walks from $j \in M$ to $i \in M$ and by $P_{j,i}(a) = \sum_{p \in \mathcal{P}_{j,i}} w_a(p)$ the sum of weights of walks in $\mathcal{P}_{j,i}$. One can then express the profit of firms in terms of the weights of network walks.

Lemma 4 *For all $i \in M$,*

$$\pi_i(a) = \varepsilon_i (I - \tilde{A}_M)_i^{-1} a_0 = \varepsilon_i \sum_{j \in M} a_{0,j} P_{j,i}(a).$$

In order to characterize more precisely the profit maximizing behavior of the firm, it is useful to decompose network walks further into direct walks and cycles. A walk $p = (h_1, h_2, \ldots, h_k)$ is called a direct walk from j to i if $h_j \neq i$ for all $j \in \{2, \ldots, k-1\}$ and a direct walk from i to i is called a direct cycle. We use $\mathcal{D}_{j,i}$ to denote the set of all direct walks from j to i, (in particular, $\mathcal{D}_{i,i}$ is the set of direct cycles around i) and $D_{j,i}(a)$ to denote the sum of weights of such walks. One shall note that $D_{j,i}(a)$ is independent of a_i as it contains no outgoing link from i. The profit of firm i can be expressed in terms of the direct paths to i and the direct cycles around i as follows:

Lemma 5 *The profit function of firm $i \in M$ is given by*

$$\pi_i(a) = \varepsilon_i P_{0,i}(a) = \varepsilon_i \frac{a_{0,i} + \sum_{j \in M/\{i\}} a_{0,j} D_{j,i}(a)}{1 - D_{i,i}(a)},$$

or equivalently,

$$\pi_i(a_i, a_{-i}) = \varepsilon_i \frac{a_{0,i} + \sum_{j \in M/\{i\}} a_{0,j} D_{j,i}(a_{-i})}{1 - \tilde{a}_{i,i} - \sum_{k \in M/\{i\}} \tilde{a}_{k,i} D_{k,i}(a_{-i})}.$$

Moreover, π_i is continuous and quasi-concave in a_i.

Remark 4 *Being row-stochastic, $\tilde{A}^T$ is the transition matrix of a Markov chain, and $P_{j,i}$ can be interpreted in the context of this Markov chain as the probability of reaching j from i before reaching 0 (considering only walks with nodes in M).*

Hence, the revenues of firm i correspond to the flow of consumption spending (based on labor income) that reaches i. The profit of firm i corresponds to a share ε_i of these revenues. Lemmas 4 and 5 hence highlight that the firms aim to maximize incoming connectivity from the consumer. As the firms only choose their suppliers, they can only affect this connectivity indirectly by maximizing the term $D_{i,i}(a)$, i.e. by ensuring that the largest possible share of incoming value from the consumer remains in their supply chain. Thus, they must choose suppliers that link back towards them, directly or indirectly. In other words, firms have incentives to form clusters of suppliers with integrated supply chains. In particular, firms will always choose to keep production integrated internally if this is an available option, i.e. a firm does not have an external supplier for its own category of product.

Proposition 3 *For every Nash equilibrium $\overline{a}$ of $\mathcal{G}(a_0, b)$, one has for all $i \in M$,*

$$i \in M_\ell \Rightarrow \overline{a}_{i,i} = b_{i,\ell}.$$

2.6 Network-based characterization of the social welfare

Following Lemmas 2 and 3, given firm linkage choices a, the social welfare is given by

$$V(a, \lambda) = \log(u(\overline{x})) = \log(v_0) + \sum_{i \in M} a_{0,i} \log(a_{0,i}) + (a_0)^T (I - A)^{-1} u + (a_0)^T (I - A)^{-1} D \log(\overline{v}),$$

which simplifies in the case of constant returns to scale ($\varepsilon_i = 0$) to

$$V(a, \lambda) = \log(p_0) + \sum_{i \in M} a_{0,i} \log(a_{0,i}) + (a_0)^T (I - A)^{-1} u,$$

where for all $i \in M$, $u_i = \log(\lambda_i) + \sum_{j \in N} a_{i,j} \log(a_{i,j})$.

In the latter case, maximizing $V(a, \lambda)$ with respect to $(a_i)_{i \in M}$ amounts to maximizing

$$W(A, \lambda) := (a_0)^T (I - A)^{-1} u. \quad (3)$$

Given the regularity of $V(a, \lambda)$, $W(A, \lambda)$ can be used, in a first-order approximation, as a measure of welfare if the ε_is are small-enough, i.e. if all firms have small enough decreasing returns to scale.

Equation (3) highlights two determinants of social welfare. First, the vector u whose coordinates $u_i = \log(\lambda_i) + \sum_{j \in N} a_{i,j} \log(a_{i,j})$ correspond to the difference between the log productivity of firm i and the Shannon entropy of its distribution of input weights across suppliers. The derivation of Equation (13) in the proof of Proposition 1 highlights how this "entropy corrected productivity" emerges as a condensed measure of the production costs of firm i combining productivity per se (measured by λ_i) and the complexity of the production process (measured by the Shannon entropy $\sum_{j \in N} a_{i,j} \log(a_{i,j})$). The larger the u_i, the lower the production cost and the equilibrium price of good i shall be (all other things being equal, see the proof of Proposition 1). Second, social welfare depends on the connectivity between the consumer and firms with high "entropy corrected productivity". Indeed, Equation (3) can be written as a weighted sum of walks in the production networks

$$W(A, \lambda) := \sum_{i \in M} \sum_{j \in M} a_{0,i} P_{i,j}(a) u_j.$$

Accordingly, the first-order conditions for welfare maximization can be expressed in terms of network structure as follows:

Proposition 4 *If the production network A is maximizing the welfare measure $W(a, \lambda)$ among the set of technological configurations $\{A \in \mathbb{R}_+^{M \times M} \mid \forall i \in M,\ \sum_{j \in M_\ell} a_{i,j} = b_{i,\ell}\}$, one has for all $i \in M$, for all $\ell \in L$, and for all $j \in M_\ell$,*

$$a_{i,j} > 0 \Rightarrow \frac{\partial W(A, \lambda)}{\partial a_{i,j}}(a) = P_{0,i} \left(\sum_{k \in M} P_{j,k} u_k + \log(a_{i,j}) + 1 \right) = \max_{k \in M_\ell} \frac{\partial U}{\partial a_{i,k}}(a).$$

Hence, the marginal contribution of link (i, j) to social welfare depends on the connectivity between the household and firm i on one hand, and the connectivity between firm j and other firms weighted proportionally to (entropy-corrected) productivity, on the other hand. Accordingly, at a social optimum, only production links that maximize the connectivity between the household and high-productivity firms shall be enabled.

Overall, one observes major differences between the determinants of the social welfare and these of firms' profitability. The social welfare depends on connectivity from high (entropy-corrected) productivity firms towards the household whereas firms' profitability is independent of productivity and depends on the connectivity from the household towards the firms. In other words, the analysis of efficiency in the game $\mathcal{G}(a_0, b)$ shall be based on a measure of social welfare defined independently from the individual payoff functions. This discrepancy between individual and social objectives can be observed in a number of network formation games, e.g. when individuals choose their peers in social networks to maximize influence but hence favor polarization [see e.g. Bolletta and Pin, 2020] or when financial institutions form lending relationships to diversify risk but thereby create structures prone to systemic risk [see e.g. Allen and Gale, 2000].

3 Social welfare in strategic production networks

In this section, we explore the potential inefficiencies induced in our framework by the discrepancy between micro-level incentives and social objectives. Considering potential applications to international trade within the context of the globalization/deglobalization debate [Witt, 2019, Kornprobst and Paul, 2021], we concentrate our analysis on a class of replicate economies that can be viewed as a model of monopolistic competition à la Armington [Armington, 1969, Dixon et al., 2016]. In this setting, where the determinants of final demand and social welfare are given by the representative utility function, we investigate how strategic choices of firms shape trade in intermediaries and global supply chains. We further investigate the welfare impacts of these strategic choices in the presence of risk and heterogeneous productivity. In this section, we measure welfare simply by $W(A, \lambda)$ since we mainly focus on the situation where the ε_is are small enough.

3.1 Equilibria in a replicate economy

In this subsection, we investigate in details the structure of the set of equilibria in the setting of a replicate economy [see e.g. Debreu and Scarf, 1963]. Every replication can be interpreted e.g. as a country in a context of international trade.

Let us first define a simple game as a game in which $M = L$ so that there exists exactly one firm in each category. Such a game is characterized by a share of consumption expenditures $a_0 \in \mathbb{R}_+^L$ and technological requirements $b_\ell \in \mathbb{R}^L$ for each firm $\ell \in L = M$. In the corresponding game $\mathcal{G}(a_0, b)$, the strategy sets are singletons and thus the unique Nash equilibrium is characterized by $\overline{a}_\ell = b_\ell$ for all $\ell \in M$.

We then define $\mathcal{G}^n(a_0, b)$ as the n-fold replicate of $\mathcal{G}(a_0, b)$, where there are exactly n firms in each category ℓ, all with technology requirements b_ℓ, and the household spending share on each firm in category ℓ is exactly $a_{0,\ell}/n$. More formally,

Definition 3 *The n-fold replicate of the game $\mathcal{G}(a_0, b)$, denoted by $\mathcal{G}^n(a_0, b)$, is the game $\mathcal{G}(a_0^n, b^n)$ such that*

(i) the set of firms is $M = \{1, \dots, nL\}$;

(ii) the set of firms in category $\ell \in L$ is $M_\ell = \{(\ell - 1)n + 1, \dots, \ell n\}$;

(iii) the consumption shares a_0^n are such that for all $\ell \in L$ and all $i \in M_\ell$, $a_{0,i}^n = \frac{a_{0,\ell}}{n}$;

(iv) the technology requirement b^n are such that for all $i \in M_\ell$, $b_i^n = b_\ell$.

Such a replicate economy can be considered as the model of an economy with n countries, where firms of country c are labeled as $c, n + c, \dots, (L - 1)n + c$. Each firm must choose in which country it sources its different inputs. Two polar cases can naturally be considered in this setting: one of an autarkic/island economy where each firm only sources inputs domestically, and one of a globalized economy where each firm sources inputs uniformly from each country. More broadly, one can consider partitions of the set of countries into economically integrated regions where firms source inputs only within the region. Let $\mathcal{Q} := \{Q_1, \dots, Q_K\}$ be a partition of $\{1, \dots, n\}$. We define $Q_{\ell,k} = \{(\ell - 1)n + i : i \in Q_k\}$ the set of replications of input type ℓ in the cluster Q_k. In an n-fold replicate game $\mathcal{G}^n(a_0, b)$, we define a $\mathcal{Q}$-clustered network/production structure $a^{\mathcal{Q}}$ such that for all $i \in Q_{\ell,k}$,

$$\begin{cases} a_{i,i}^{\mathcal{Q}} = b_{i,\ell}, \\ a_{i,j}^{\mathcal{Q}} = 0 \text{ for } j \in Q_{\ell,k} \setminus \{i\}, \\ a_{i,j}^{\mathcal{Q}} = \dfrac{b_{i,\ell'}}{|Q_k|} \text{ for } j \in Q_{\ell',k} \text{ with } \ell' \neq \ell, \\ a_{i,j}^{\mathcal{Q}} = 0 \text{ otherwise.} \end{cases} \tag{4}$$

It can be demonstrated that every $\mathcal{Q}$-clustered network/production structure is a Nash equilibrium of the n-fold replicate game $\mathcal{G}^n(a_0, b)$.

Proposition 5 *If for all $i \in M$, ε_i is sufficiently small, then every $\mathcal{Q}$-clustered network/production structure $a^{\mathcal{Q}}$ is a Nash equilibrium of the normal-form game $\mathcal{G}^n(a_0, b)$.*

Remark 5 *It has been previously observed that network formation games based on centrality can generate a large number of Nash equilibria [see e.g. Catalano et al., 2022, Cominetti et al., 2022].*

Proposition 5 implies that there is substantial indeterminacy about the network structure that can emerge from strategic interactions in the formation of global supply chains. Certain equilibrium configurations might lead to substantial inefficiencies. In order to investigate this issue, we shall quantify, in the following, these inefficiencies using the notion of price of anarchy. In our setting, it is defined as follows:

Definition 4 *The price of anarchy $POA(b)$ is defined as the ratio between the largest equilibrium social welfare in the class of economies $\mathcal{E}(A, \lambda)$ with technology configurations in $S(b)$ and the lowest equilibrium social welfare in an economy that is a Nash equilibrium of the game $\mathcal{G}(a_0, b)$.*

Among the potential equilibrium network configurations presented in Proposition 5, two polar cases will be of particular interest: (i) the *islands economy* such that $|\mathcal{Q}| = n$, in which firms are only connected to firms in the same cluster/country, and (ii) the *fully connected economy* such that $|\mathcal{Q}| = 1$, in which firms are uniformly connected across clusters.

3.2 Social welfare and productivity

In the simple setting of the game $\mathcal{G}^n(a_0, b)$, the impact of network structure on social welfare is mostly determined through the productivity characteristics of technologies, which are captured by the productivity functions λ. A benchmark case is given by Hicks-neutral productivity $\overline{\lambda}$ such that for all $(a_i) \in \mathbb{R}_+^M$,

$$\overline{\lambda}(a_i) = \frac{1}{\prod_{j \in M} a_{i,j}^{a_{i,j}}}.$$

Indeed, Hicks-neutral productivity exactly compensate productivity losses from increasing process complexity through positive spillovers on productivity. If all firms have Hicks-neutral productivity, social welfare is independent of network structure.

Proposition 6 *If all firms have Hicks-neutral productivity, i.e. for all $i \in M$, $\lambda_i = \overline{\lambda}$, the the social welfare is equal at all the $\mathcal{Q}$-clustered network equilibria of the normal form game $\mathcal{G}^n(a_0, b)$.*

The comparative statics between equilibria in the general case then depend on the relative strengths of technological spillovers on productivity with respect to the Hicks-neutral case. Namely, we say that a firm has increasing (resp. decreasing) *returns to diversification* if positive spillovers from input diversification on productivity are greater (resp. smaller) than in the Hicks-neutral case.

Definition 5 *Firm i has increasing returns to diversification if for all $(a_i), (a'_i) \in \mathbb{R}_+^M$,*

$$\overline{\lambda}(a_i) \geq \overline{\lambda}(a'_i) \Rightarrow \frac{\lambda_i(a_i)}{\lambda_i(a'_i)} \geq \frac{\overline{\lambda}(a_i)}{\overline{\lambda}(a'_i)}.$$

Respectively, firm i has decreasing returns to diversification if for all $(a_i), (a'_i) \in \mathbb{R}_+^M$,

$$\overline{\lambda}(a_i) \geq \overline{\lambda}(a'_i) \Rightarrow \frac{\lambda_i(a_i)}{\lambda_i(a'_i)} \leq \frac{\overline{\lambda}(a_i)}{\overline{\lambda}(a'_i)}.$$

Remark 6 *It is, in particular, the case that a firm has decreasing returns to diversification if its productivity is constant and independent of the network structure.*

In a setting where there are increasing returns to diversification, more interconnected network structures will be more socially efficient, and conversely in the case where there are decreasing returns to diversification. In particular, one has an exact comparative static result comparing the islands and the fully connected economy:

Proposition 7 *In the game* $\mathcal{G}^n(a_0, b)$,

- *if all the firms have increasing returns to diversification, social welfare in the fully connected economy is greater than social welfare in the islands economy;*
- *if all the firms have decreasing returns to diversification, social welfare in the islands economy is greater than social welfare in the fully connected economy.*

These differences in welfare can be quite substantial from a quantitative standpoint, as highlighted by the following example:

Proposition 8 *In the game* $\mathcal{G}^2(a_0, b)$ *where all firms have constant productivity* λ, *the welfare difference between the island and the fully connected economy is of the form*

$$K \log(n)$$

with $K > 0$. *Accordingly, the price of anarchy tends to infinity as* n *tends towards infinity.*

Overall, the results in this subsection emphasize that the optimal network structure depends on productivity spillovers. If there are strong productivity spillovers to input diversification, then more interconnected, globalized network structures are more socially efficient. If these spillovers are weak, more localized networks structures are more efficient. Note however that our setting à la Armington is likely to underestimate the benefits of globalized networks because its monopolistic competition structure limits the possibilities of substitution between domestic and global goods.

Our results also complement those in Acemoglu and Azar [2020] giving necessary conditions for sustained economic growth via endogenous formation of production networks. Indeed, if there are increasing returns to diversification, total output and social welfare at the fully connected equilibrium of $\mathcal{G}^n(a_0, b)$ will grow as n grows. Hence, as in Acemoglu and Azar [2020], the ability of firms to update their supply relationships following the entry of new firms, suffices to sustain economic growth. There is the choice between multiple suppliers, only those such that this is maximal shall be selected (or all suppliers shall have same marginal contribution towards household).

3.3 Social welfare and risk

The recent literature has emphasized (emerging) risks as potential major drivers of change in the structure of global supply chains [see e.g. Razin, 2020, Irwin, 2020]. In order to investigate this issue, we extend our model by considering that each link (i, j) can be, independently, disrupted with a probability $r_{i,j} \in [0, 1]$. These exogenous shocks on supply relationships impact are assumed to impact the production process and thus the output of the firms. We formalize these impacts by considering that if the set of links K is disrupted in the network A, the production function of firm i is altered to

$$f_i(x_{i,0}, \cdots, x_{i,n}) := (1-\rho)^{\phi_i(K,A)} \lambda_i(a_i) \prod_{j \in N} x_{i,j}^{a_{i,j}},$$

where ρ is a parameter determining the average magnitude of shocks and ϕ is a disruption function defining the impact on output through aggregation of the shocks occurring in the supply chain.

One can remark that these disruptions of productivity do not impact the profits of firm at general equilibrium and thus do not modify the set of Nash equilibria of the game $\mathcal{G}^n(a_0, b)$. However, these shocks induce substantial modifications of social welfare, which is now given by the expected utility of the representative household given the distribution of risk, i.e.

$$\hat{V}(a, \lambda) = \sum_{K \subset M \times M} [\prod_{\{(i,j) \in K^c\}} (1 - r_{i,j}) \prod_{\{(h,k) \in K\}} r_{h,k}] V(a, (1-\rho)^{\phi(K,A)} \otimes \lambda), \tag{5}$$

and

$$\hat{W}(a, \lambda) = \sum_{K \subset M \times M} [\prod_{\{(i,j) \in K^c\}} (1 - r_{i,j}) \prod_{\{(h,k) \in K\}} r_{h,k}] W(a, (1-\rho)^{\phi(K,A)} \otimes \lambda), \tag{6}$$

where $\otimes$ denotes multiplication coordinatewise.

Equations (5) and (6) highlight that in this extended setting, social welfare depend on the structure of the production network, the "spatial" distribution of risks given by r, and the disruption functions ϕ. Socially efficient networks shall both minimize the risk of disruption and ensure resilience in case a disruption occurs. One can characterize more precisely these efficient networks once the distribution of risks and the disruption functions are specified.

In order to disentangle between risk and productivity related effects in the game $\mathcal{G}^n(a_0, b)$ where all firms have Hicks-neutral productivity functions, we further assume that utility weights are independent of the category/country, i.e. for all ℓ and for all $i, j \in M_\ell$, $a_{0,i} = a_{0,j}$. In this setting, we consider two polar cases for the disruption functions:

- Firstly, there is a fixed-cost to disruption independently of the number of links affected in each category. It is characterized by $\phi^{\min}$ such that for all $i \in M$,

$$\phi_i^{\min}(K, A) = \sum_{\ell \in L} \min_{j \in M_\ell | (i,j) \in K \text{ and } a_{i,j} > 0} a_{i,j}.$$

- Secondly, the impacts on the suppliers cumulate additively. It is characterized by ϕ^{sum} such that for all $i \in M$,

$$\phi_i^{\text{sum}}(K, A) = \sum_{j|(i,j)\in K \text{ and } a_{i,j}>0} a_{i,j}.$$

Furthermore, we consider two polar cases for the spatial distribution of risks:

- First, a case where risk is homogeneous, i.e. for all $i, j \in M$ $r_{i,j} = r$ for some $r \in [0, 1]$.
- Second, a case where risk depends on the distance between firms, i.e. for $i \in M_\ell$ and $j \in M_{\ell'}$, $r_{i,j} = r\frac{|\ell-\ell'|+1}{n}$ for some $r \in [0, 1]$.

In this setting, one can provide a complete comparative static for the $\mathcal{Q}$-clustered equilibria of the game $\mathcal{G}^n(a_0, b)$.

Proposition 9 *In the normal form game $\mathcal{G}^n(a, b)$ with risk,*

1) if $\phi^i(K, A) = \phi_i^{\min}(K, A)$, then the maximum (resp. minimum) social welfare is achieved in the fully connected (resp. island) economy, irrespective of whether risk is homogeneous or increases with distance;

2) if $\phi^i(K, A) = \phi_i^{sum}(K, A)$ and risk is homogeneous, then the social welfare is the same for all $\mathcal{Q}$-clustered equilibria;

3) if $\phi^i(K, A) = \phi_i^{sum}(K, A)$ and risk is increasing with distance, then the maximum (resp. minimum) social welfare is achieved in the the island (resp. fully connected) economy.

Despite the stylized nature of the examples considered, Propostion 9 highlights a range of results relevant for applications. When supply disruptions are not cumulative (Case 1), diversification allows to mitigate risk and a fully connected network structure is always preferable, independently of the spatial distribution of risk. When supply disruptions cumulate additively and the spatial distribution of risks is homogeneous (Case 2), the benefits of diversification are exactly compensated by the increase costs of risks, so that every equilibrium network structure is equivalent. When supply disruptions cumulate additively and risk is proportional to spatial distance (Case 3), the benefits of diversification are more than offset by the increase in risk so that an autarkic organization is preferable.

3.4 Welfare improving network policies

The previous results of this section highlight that multiple production networks can emerge as strategic equilibria of decentralized network formations processes. Certain equilibria might be very inefficient from the social welfare point of view. In particular, firms may fail to account for the productivity gaps and/or the risks associated to certain supply relationships.

However, the clustered nature of these equilibria put forward in Proposition 5 hint at relatively simple ways to restrict the set of possible equilibria in order to coordinate behavior on efficient equilibria. Indeed, in order to prevent the formation of a given cluster at equilibrium, it suffices to prevent the formation of a single link in that cluster. Conversely, to opt for certain clusters at equilibrium, it suffices to ensure *à priori* that one of the links of the cluster will be formed. In practice, the creation or prevention of links can be enforced by a range of tools such as trade agreements, tariffs, and non-tariff barriers.

Formally, we define a trade policy by a pair of subgraphs $(\mathcal{P}, \mathcal{C}) \in M^2 \times M^2$ where $\mathcal{P}$ is the set of links that are "prevented" by the policy and $\mathcal{C}$ the set of links that are "catalyzed" by the policy. It is assumed that a prevented link can not be present in an equilibrium network while a subsidized link must be present in an equilibrium network. More precisely, we say that a $\mathcal{Q}$-clustered equilibrium network is compatible with a trade policy $(\mathcal{P}, \mathcal{S})$ if it contains none of the links in $\mathcal{P}$ and all the links in $\mathcal{S}$. A trade policy can efficiently coordinate behavior on certain equilibria in the sense that given the clustered nature of equilibria, one requires to prevent and/or catalyze only a few links to select specific equilibria. Namely, one has the following proposition:

Proposition 10 *In any n-fold replicate game $\mathcal{G}^n(a_0, b)$,*

- *if a trade policy $(\mathcal{P}, \mathcal{C})$ prevents at least one link from category ℓ to category ℓ', then category ℓ and category ℓ' are in separate clusters at every compatible $\mathcal{Q}$-clustered equilibrium network;*
- *if a trade policy $(\mathcal{P}, \mathcal{C})$ catalyzes at least one link from category ℓ to category ℓ', then category ℓ and category ℓ' are the same cluster at every compatible $\mathcal{Q}$-clustered equilibrium network.*

4 Conclusion

We have developed a strategic model of the formation of production networks. Our models subsumes standard general equilibrium models as, once firms have chosen their supply relationships, the emerging outcome is a general equilibrium of the economy with monopolistic competition among firms. Accordingly, the objective of firms in the network formation game is to choose their supply relationships so as to maximize their profit at the resulting equilibrium. In other words, firms aim to insert themselves efficiently in the network of exchanges. We have also shown that the objective of the firms is equivalent to the maximization of their eigenvector centrality in the production network.

As is common in network formation games based on centrality, there generally are multiple Nash equilibria in our setting. We have investigated the characteristics and the social efficiency of these equilibria in a stylized version of our model representing international trade networks. We show that the impact of network structure on social welfare is firstly determined by a trade-off between costs of increasing process complexity and positive spillovers on productivity induced by the diversification of the input mix. If the latter effect dominates, strongly interconnected, globalized, network structures are socially efficient. Conversely, if the cost from increasing process complexity dominates, island-based, localized, network structures are socially efficient.

We further analyze a variant of our model that accounts for the risks of disruption of supply relationships. In this setting, we characterize how social welfare depends on the structure of the production network, the spatial distribution of risks, and the process of shock aggregation in supply chains. Optimal network structure is hence dependent on the riskiness environment in which the economy operates. In particular, risky supply relationships might be profitable for the firm but generate substantial negative externalities at the macro-level. We finally show that simple trade policies characterized by sets of links that are either prevented or catalyzed can be a powerful equilibrium selection device.

Our results highlight that the validity of standard welfare theorems is jeopardized when one subsumes general equilibrium models into a broader model of the formation of production networks including strategic considerations. From the theoretical point of view, our results are in line with the network theoretic literature that has shown that decentralized network formation processes could lead to inefficient macro-level outcomes in a range of socio-economic settings such as polarization in social networks, epidemic spreading in human networks or systemic risk in financial networks. From the policy perspective, our results speak to the current debate about de-globalization by highlighting that the characteristics of efficient production and trade networks might depend on contextual factors such as environmental or geopolitical risks.

5 Acknowledgments

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Appendix A proofs

Proof (of Lemma 1) *Let us begin by noting that for all $i \in M$, the condition $b_{i,0} > 0$ implies $a_{i,0} > 0$. Moreover, if $b_{i_0,\ell} > 0$ for some $\ell \in L/0$ holds, it implies the existence of $j_0 \in M$ such that $a_{i_0,j_0} > 0$. The irreducibility of $\tilde{A}$ is evident since for all $i, j \in M$, we have $a_{0,i} > 0$, $a_{j,0} > 0$, and consequently, $a_{j,0}a_{0,i} > 0$. The state 0 is aperiodic due to the conditions $a_{0,i_0}a_{i_0,0} > 0$ and $a_{0,i_0}a_{i_0,j_0}a_{j_0,0} > 0$.*

Proof (of Proposition 1) *Given a price vector $p \in \mathbb{R}^N_+$, the first-order condition for profit maximization by firm i implies that for all $j \in N$,*

$$p_j x_{i,j} = a_{i,j} p_i \lambda_i \prod_{k \in N} x_{i,k}^{a_{i,k}} = a_{i,j} p_i y_i.$$

This implies that the output of firm i is determined by

$$y_i := f_i(x_{i,0}, \cdots, x_{i,n}) = \left[\lambda_i p_i^{\sum_{j \in N} a_{i,j}} \prod_{j \in N} \left(\frac{a_{i,j}}{p_j} \right)^{a_{i,j}} \right]^{\frac{1}{1 - \sum_{j \in N} a_{i,j}}}, \tag{7}$$

and the profit of firm i is expressed as

$$\pi_i := p_i y_i (1 - \sum_{j \in N} a_{i,j}).$$

The first-order conditions for utility maximization yield that for all $j \in M$,

$$p_j x_{0,j} = a_{0,j} p_0 + a_{0,j} \sum_{i=1}^{n} p_i y_i (1 - \sum_{k \in N} a_{i,k}).$$

The market clearing condition for good $j \in M$ is given by

$$y_j = \frac{a_{0,j} p_0 + a_{0,j} \sum_{i=1}^{n} p_i y_i (1 - \sum_{k \in N} a_{i,k})}{p_j} + \sum_{k \in M} \frac{a_{k,j} p_k y_k}{p_j},$$

or equivalently,

$$p_j y_j = a_{0,j} p_0 + a_{0,j} \sum_{i=1}^{n} p_i y_i (1 - \sum_{k \in N} a_{i,k}) + \sum_{k \in M} a_{k,j} p_k y_k. \tag{8}$$

Furthermore, the market clearing condition for the labor market yields

$$1 = \sum_{j \in M} \frac{a_{j,0} p_j y_j}{p_0},$$

which can be expressed as

$$p_0 = \sum_{j \in M} a_{j,0} p_j y_j. \tag{9}$$

Considering $j \in M$ and defining the revenue of firm j as $v_j = p_j y_j$, Equations (8) and (9) establish that for every $j \in M$, the following relationships hold:

$$v_j = a_{0,j} v_0 + \sum_{k \in M} a_{k,j} v_k, \tag{10}$$

where v_0 represents household revenues defined by

$$v_0 = p_0 + \sum_{k \in M} (1 - \sum_{i \in N} a_{i,k}) v_k = \sum_{k \in M} (a_{k,0} + (1 - \sum_{i \in N} a_{i,k})) v_k = \sum_{k \in M} (1 - \sum_{k \in M} a_{i,k}) v_k. \tag{11}$$

Equations (10) and (11) can be expressed in matrix form as

$$\tilde{A}^T v = v, \tag{12}$$

where $\tilde{A}^T$ denotes the transpose of $\tilde{A}$. Since $\tilde{A}^T$ is column-stochastic, aperiodic, and irreducible, the Perron-Frobenius theorem straightforwardly implies the existence of $v \in \mathbb{R}_+^N$ such that Equation (12) holds. Furthermore, v is unique up to price normalization and is entirely determined by the choice of p_0 in Equation (9), which now becomes $\sum_{j \in M} a_{j,0} \overline{v}_j = 1$.

Finally, to demonstrate the existence and characterize equilibrium prices, Equation (7) is employed, yielding that for all $i \in M$,

$$v_i = \left[\lambda_i p_i \prod_{j\in N}\left(\frac{a_{i,j}}{p_j}\right)^{a_{i,j}}\right]^{\frac{1}{1-\sum_{j\in N} a_{i,j}}}. \tag{13}$$

By taking logarithms on both sides of Equation (13), we obtain

$$\log(v_i) = \frac{1}{1-\sum_{j\in N} a_{i,j}}[\log(\lambda_i) + \log(p_i) + \sum_{j\in N} a_{i,j}\log(a_{i,j}) - \sum_{j\in N} a_{i,j}\log(p_j)]$$
$$\iff (1-\sum_{j\in N} a_{i,j})\log(v_i) = [\log(\lambda_i) + \log(p_i) + \sum_{j\in N} a_{i,j}\log(a_{i,j}) - \sum_{j\in N} a_{i,j}\log(p_j)].$$

Assuming $p_0 = 1$, the expression can be reformulated as

$$\sum_{j\in M} a_{i,j}\log(p_j) - \log(p_i) = \log(\lambda_i) + \sum_{j\in N} a_{i,j}\log(a_{i,j}) - (1-\sum_{j\in N} a_{i,j})\log(v_i),$$

which can be compactly represented in matrix form

$$(A-I)\log(p) = u + D\log(v),$$

where $u_i = \log(\lambda_i) + \sum_{j\in N} a_{i,j}\log(a_{i,j}$, $D = \text{diag}(\sum_{j\in N} a_{i,j} - 1)$, $\log(p)_i = \log(p_i)$ and $\log(v)_i = \log(v_i)$. Since for all i, $a_{i,0} > 0$, it is evident that the spectral radius of A is less than one, making $(A-I)$ invertible. Consequently, we can derive

$$\log(p) = (A-I)^{-1}u + (A-I)^{-1}D\log(v).$$

Proof (of Lemma 2) *By Proposition 1, the equilibrium consumption for the consumer is such that for all $i \in M$,*

$$\overline{x}_{0,i} = \frac{a_{0,i}\overline{v}_0}{\overline{p}_i},$$

where $\overline{v}_0 = p_0 + \sum_{i\in M}\varepsilon_i\overline{v}_i$ and $\varepsilon_i = 1 - \sum_{j\in N} a_{i,j}$. The utility of the consumer is thus

$$u(\overline{x}) = \prod_{i\in M}\left(\frac{a_{0,i}\overline{v}_0}{\overline{p}_i}\right)^{a_{0,i}}.$$

Taking the logarithm of this utility level, one obtains

$$\log(u(\overline{x})) = \log(\overline{v}_0) + \sum_{i\in M} a_{0,i}\log(a_{0,i}) - (a_0)^T\log(\overline{p}).$$

Combining with the equation representing the equilibrium price $\overline{p}$ (Equation (2)), one gets

$$\log(u(\overline{x})) = \log(v_0) + \sum_{i\in M} a_{0,i}\log(a_{0,i}) + (a_0)^T(I-A)^{-1}u + (a_0)^T(I-A)^{-1}D\log(\overline{v}),$$

where $u_i = \log(\lambda_i) + \sum_{j\in N} a_{i,j}\log(a_{i,j})$ and $D = \text{diag}(\sum_{j\in N} a_{i,j} - 1)$.

Proof (of Lemma 3) *For the first part of the lemma, it suffices to check that if for all $i \in M$, $\varepsilon_i = 0$, one has $\overline{v}_0 = p_0$ and $D = 0$.*

For the second part of the lemma, it suffices to show that p_0 is fixed by normalization and that $\sum_{i \in M} a_{0,i} \log(a_{0,i})$ is independent of $(a_i)_{i \in M}$.

Proof (of Proposition 2) *By Remark 2, one can consider the equivalent game where the payoff of agent i are given by the invariant probability of $i \in M$, π_i, for the Markov chain defined by $\tilde{A}$. Now, following the Markov chain tree theorem ([see e.g. Kemeny and Snell, 1976]), π_i can be characterized as follows:*

- *For each $i \in M$, denote T_i as the set of i-rooted spanning trees. These trees are defined as acyclic graphs with a node set of M, where node i has no outgoing edges, and every other node $j \in M \setminus i$ possesses an out-degree of 1.*
- *The weight of a tree $t \in T_i$ is defined as $w(t) := \prod_{(j,k) \in t} \tilde{A}_{j,k}$, where the notation (j,k) denotes the edges of t. Notably, $j \neq i$ always holds, ensuring that $w(t)$ remains independent of a_i for $t \in T_i$.*
- *The weight of T_i can then be defined as $w(T_i) := \sum_{t \in T_i} w(t)$, and w_i is given by*

$$\pi_i := \frac{w(T_i)}{\sum_{j \in M} w(T_j)}.$$

Since $w(T_i)$ is independent of a_i, each agent i seeks to maximize $\phi(a) := \dfrac{1}{\sum_{j \in M} w(T_j)}$. Notice that this function is an ordinal potential function for the game, which ends the proof.

Proof (of Lemma 4) *Using Equation (8) and the fact that $P_0 = 1$, one obtains directly for $i \in M$,*

$$v_j = a_{0,j} + a_{0,j} \sum_{i \in N} v_i \varepsilon_i + \sum_{k \in M} a_{k,j} v_k.$$

This implies that

$$v = a_0 + \tilde{A}_M v.$$

The matrix $\tilde{A}_M$ is non-negative. Moreover, as $\tilde{A}$ is column-stochastic and for all $j \in M$, $\tilde{a}_{0,j} > 0$, it is clear that for all $j \in M$, $\sum_{i \in M} \alpha_{i,j} < 1$. Thus, the largest eigenvalue of $\tilde{A}_M$ is less than one, and $(I - \tilde{A}_M)$ is invertible. One then has

$$v = (I - \tilde{A}_M)^{-1} a_0.$$

Furthermore, we can write

$$v = \sum_{n=0}^{+\infty} (\tilde{A}_M)^n a_0,$$

which, in turn, yields

$$v = \sum_{j \in M} a_{0,j} P_{j,i}.$$

Finally, it is sufficient to substitute these expressions into $\pi_i(a) = \varepsilon_i v_i$ to conclude the proof.

Proof (of Lemma 5) *Following Lemma 4, one has* $\pi_i(a) = \varepsilon_i \sum_{j\in M} a_{0,j} P_{j,i}$. *It thus suffices to prove that for all* $i \neq j \in M$,

$$P_{j,i} = \frac{D_{j,i}(a)}{1 - D_{i,i}(a)}$$

and

$$P_{i,i} = \frac{1}{1 - D_{i,i}(a)}.$$

For any path $p \in \mathcal{P}_{j,i}$ *with* $j \neq i$, *it is clear that either the path is a direct path or it can be (uniquely) decomposed into a directed path* $d \in \mathcal{D}_{j,i}$ *and a cycle* c *around* i. *We use* $\mathcal{C}_i$ *to denote the set of all cycles around* i. *Then it is straightforward to get*

$$P_{j,i} = D_{j,i}(1 + C_i),$$

where $C_i := \sum_{c\in\mathcal{C}_i} w_a(c)$ *is the sum of weights of cycles around* i.

Now, let us define the multiplicity of a cycle around i. *We say that a cycle around* i *has a multiplicity of* $k \geq 1$ *if node* i *appears* $k+1$ *times in the cycle. We denote the set of all cycles around* i *of multiplicity* k *as* $\mathcal{C}_i^k$ *and remark that* $\mathcal{C}_i^1 = \mathcal{D}_{i,i}$. *We note that* $\mathcal{C}_i = \cup_{k=1}^{\infty} \mathcal{C}_i^k$, *i.e. every cycle around* i *of multiplicity* k *can be decomposed into* k *cycles around* i *of multiplicity 1, and the decomposition is unique. This implies that*

$$C_i = \sum_{k=1}^{+\infty} (D_{i,i})^k.$$

Thus,

$$P_{j,i} = D_{j,i}(1 + C_i) = D_{j,i} \sum_{k=0}^{+\infty} (D_{i,i})^k = \frac{D_{j,i}}{1 - D_{i,i}},$$

where one has used Remark 4 which ensures that $D_{i,i} \leq P_{i,i} < 1$. *A similar reasoning shows that*

$$P_{i,i} = \frac{1}{1 - D_{i,i}}.$$

Finally, the linearity of $1 - \tilde{a}_{i,i} - \sum_{k\in M/\{i\}} \tilde{a}_{k,i} D_{k,i}(a_{-i})$ *in* $\tilde{a}_{\cdot,i}$, *and thus in* a_i, *implies that the profit of firm* i *is continuous and quasi-concave in* a_i.

Proof (of Proposition 3) *It follows from Lemma 5 that at a Nash Equilibrium, profit maximization amounts to maximizing*

$$\tilde{a}_{i,i} + \sum_{k\in M/\{i\}} \tilde{a}_{k,i} D_{k,i}(a_{-i}) = a_{i,i} + \varepsilon_i a_{0,i} + \sum_{k\in M/\{i\}} (a_{i,k} + \varepsilon_i a_{0,k}) D_{k,i}(a_{-i}).$$

Furthermore, by Remark 4, one necessarily has $D_{k,i}(a) \leq P_{k,i}(a) < 1$. *In this setting, it is straightforward that if* $\overline{a}_i$ *maximizes profit given* $\overline{a}_{-i}$, *then* $\overline{a}_{i,i}$ *must be as large as possible, i.e. one must have* $\overline{a}_{i,i} = b_{i,\ell}$.

Proof (of Proposition 4) *It suffices to show that*

$$\frac{\partial W(A,\lambda)}{\partial a_{i,j}}(a) = (a_0)^T[(I-A)^{-1}U_{i,j}(I-A)^{-1}]u + (a_0)^T(I-A)^{-1}u_{i,j},$$

where $u_{i,j}$ is a column vector

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \log(a_{i,j})+1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The proposition follows by expressing the coefficients of $(I-A)^{-1} = \sum_{n=0}^{+\infty} A^n$ in terms of network paths and applying Karush-Kuhn-Tucker conditions.

Remark that in the $\mathcal{Q}$-cluster economy, for every $j, j' \in M_{\ell'}$ and for every input ℓ, $b_{j,\ell} = b_{j',\ell}$. Therefore, from now on, in the $\mathcal{Q}$-cluster economy, we denote $b_{\ell',\ell}$ the input weight $b_{j,\ell}$ for some firm $j \in M_{\ell'}$. To simplify of the notation, let us denote $\mathcal{Q}_k = \bigcup_{\ell \in L} Q_{\ell,k}$ the set of all firms in the cluster Q_k.

Proof (of Proposition 5) *Consider $\mathcal{Q} = \{Q_1, \ldots, Q_K\}$ and the associated vector of input weights for firm i, $a_i^{\mathcal{Q}}$ defined in system (4). We aim to prove that for every firm i, an optimal vector of input weights is $a_i^{\mathcal{Q}}$ provided that the vector of input weights of all other firm j is $a_j^{\mathcal{Q}}$.*

Let us consider firm i producing goods of type ℓ in a cluster Q_k, i.e. $i \in Q_{\ell,k}$. Applying Proposition 3, the optimal choice of input type ℓ is using his own output as the input. Therefore, the optimal weights for the input of type ℓ is $a_{i,i}^{\mathcal{Q}} = b_{\ell,\ell}$, $a_{i,j}^{\mathcal{Q}} = 0$ for all $i \neq j \in M_\ell$.

For input of type $\ell' \neq \ell$, let us look at the profit function of firm i at the vector of input weights $a = (a_i, a_{-i}^{\mathcal{Q}})$, which is given by Lemma 5

$$\pi_i(a_i, a_{-i}^{\mathcal{Q}}) = \varepsilon_i \frac{a_{0,i} + \sum_{j \in M/\{i\}} a_{0,j} D_{j,i}(a_{-i}^{\mathcal{Q}})}{1 - \tilde{a}_{i,i} - \sum_{k \in M/\{i\}} \tilde{a}_{k,i} D_{k,i}(a_{-i}^{\mathcal{Q}})}.$$

Recall that $D_{j,i}(a_{-i}^{\mathcal{Q}})$ is the sum of weights of all direct paths from j to i in the adjusted matrix $\tilde{A}$ in Section 2.5. There are two possible cases:

Case 1: *Assume that $j \in Q_{\ell',k}$. We have $D_{j,i}(a_{-i}^{\mathcal{Q}}) \geq \dfrac{b_{\ell',\ell}}{|Q_k|}$.*

Case 2: *Assume that $j \notin Q_{\ell',k}$. We remark that for any $i, j \in M$, the sum of weights of all paths from j to i is at most $\max_{i \in M} \dfrac{1}{1 - \varepsilon_i - b_{i,0}}$, which is at most $C = \max_{i \in M} \dfrac{1}{1 - b_{i,0}} > 0$. Now, we consider a particular direct path p from j to i as follows:*

$$\begin{cases} p = (h_1, h_2, \ldots, h_q), \\ h_1 = j, h_q = i, h_{q'} \notin \mathcal{Q}_k \text{ for all } q' \in \{2, \ldots, q-1\}. \end{cases}$$

Intuitively, this path contains all firms that are not in the clusters Q_k except the terminal firm $h_q = i \in Q_{\ell,k}$. If we denote $D_{j,i}(a^{\mathcal{Q}}_{-i}, Q_k)$ the sum of weights of such direct paths from j to i, then we have

$$D_{j,i}(a^{\mathcal{Q}}_{-i}, Q_k) = \sum_{j' \notin \mathcal{Q}_k} D_{j,j'}(a^{\mathcal{Q}}_{-i}) \tilde{a}^{\mathcal{Q}}_{j',i}.$$

Note that since $j' \notin \mathcal{Q}_k$, one gets $\tilde{a}^{\mathcal{Q}}_{j',i} = \varepsilon_i a_{0,j'} \leq D = \max_{i\in M} a_{0,i} \times \max_{i \in M} \varepsilon_i$. Moreover, there are at most $(n - |Q_k|) \times L$ such j', which follows

$$D_{j,i}(a^{\mathcal{Q}}_{-i}, Q_k) \leq C \times (n - |Q_k|) \times L \times D.$$

Therefore, one obtains

$$\begin{aligned} D_{j,i}(a^{\mathcal{Q}}_{-i}) = \sum_{i' \in Q_k} D_{j,i'}(a^{\mathcal{Q}}_{-i}, Q_k) \times D_{i',i}(a^{\mathcal{Q}}_{-i}) &\leq |Q_k| \times C \times (n - |Q_k|) \times L \times D \times C \\ &< C^2 \times n^2 \times L \times D. \end{aligned}$$

Note that C, n, and L are fixed, and $D \to 0$ when $\max_{i\in M} \varepsilon_i \to 0$. Therefore, $D_{j,i}(a^{\mathcal{Q}}_{-i}) \to 0$ when $\max_{i\in M} \varepsilon_i \to 0$. So, if ε_i is small enough for all i, then $D_{j_1,i}(a^{\mathcal{Q}}_{-i}) > D_{j_2,i}(a^{\mathcal{Q}}_{-i})$ for any $j_1 \in Q_{\ell',k}$ and $j_2 \notin Q_{\ell',k}$. Therefore, the optimal weights of firm i to firm $j \notin Q_{\ell',k}$ is 0.

If $j_1, j_2 \in Q_{\ell',k}$, then $D_{j_1,i}(a^{\mathcal{Q}}_{-i}) = D_{j_2,i}(a^{\mathcal{Q}}_{-i})$ due to the fact that j_1 and j_2 have the same vector of input weights since they are replication of each other. Then $a^{\mathcal{Q}}_{i,j} = \frac{b_{\ell,\ell'}}{|Q_k|}$ for all $j \in Q_{\ell',k}$ are clearly optimal weights for input type $\ell' \neq \ell$, which ends of the proof.

The proofs of Propositions 6 and 7 rely on the following Lemma.

Lemma 6 *Let A^{Q_k} be the production network matrix associated to the network structure $a^{\mathcal{Q}}$ restricted to the cluster Q_k and define $(I - A^{Q_k})^{-1} = C^{Q_k} = [c_{i,j}]_{(|Q_k|L)\times(|Q_k|L)}$, then for $i \in Q_{\ell,k}$, one has*

$$\begin{cases} c_{i,i} = \frac{C_{\ell,\ell}}{|Q_k|} + \frac{|Q_k|-1}{|Q_k|}\frac{1}{1-b_{\ell,\ell}}, \\ c_{i,j} = \frac{C_{\ell,\ell}}{|Q_k|} - \frac{1}{|Q_k|}\frac{1}{1-b_{\ell,\ell}} \text{ for } j \in Q_{\ell,k} \setminus \{i\}, \\ c_{i,j} = \frac{C_{\ell,\ell'}}{|Q_k|} \text{ for } j \in Q_{\ell',k} \text{ with } \ell' \neq \ell, \end{cases}$$

where $C_{\ell,\ell'}$ does not depend on the choice of $\mathcal{Q}$-cluster.

Proof (of Lemma 6) *Let $|Q_k| = m_k$, and let $I - A^{Q_k} = [d_{i,j}]_{m_k L \times m_k L}$. Considering $i \in Q_{\ell,k}$, we have*

$$\sum_{j \in \mathcal{Q}_k} d_{i,j} c_{j,h} = \begin{cases} 0 \text{ if } i \neq h, \\ 1 \text{ if } i = h. \end{cases} \tag{14}$$

Now, for all $\tilde{\ell}$*, we denote* $C_{i,\tilde{\ell}} = \sum_{j \in Q_{\tilde{\ell},k}} c_{i,j}$*. Summing up Equation (14) over* $h \in Q_{\ell,k}$ *and over* $h \in Q_{\ell',k}$ *for every* $\ell' \neq \ell$*, we get*

$$\sum_{h \in Q_{\ell,k}} \left(\sum_{j \in \mathcal{Q}_k} d_{i,j} c_{j,h} \right) = 1,$$
$$\sum_{h \in Q_{\ell',k}} \left(\sum_{j \in \mathcal{Q}_k} d_{i,j} c_{j,h} \right) = 0.$$

By interchanging the indices, this is equivalent to

$$\sum_{j \in \mathcal{Q}_k} d_{i,j} C_{j,\ell'} = \begin{cases} 0 \text{ if } \ell' \neq \ell, \\ 1 \text{ if } \ell' = \ell. \end{cases} \tag{15}$$

Now, since $i \in M_\ell$*, we have*

$$\begin{aligned} d_{i,i} &= 1 - b_{\ell,\ell}, \\ d_{i,j} &= 0 \text{ for } j \in M_\ell \setminus \{i\}, \\ d_{i,j} &= -\frac{b_{\ell,\ell'}}{m_k} \text{ for } j \in Q_{\ell',k} \text{ with } \ell \neq \ell'. \end{aligned}$$

Equation (15) can be written as follows:

$$(1 - b_{\ell,\ell}) C_{i,\ell'} + \sum_{\tilde{\ell} \neq \ell} \left(\sum_{j \in Q_{\tilde{\ell},k}} \left(-\frac{b_{\ell,\tilde{\ell}}}{m_k} \right) C_{j,\ell'} \right) = 0 \text{ for all } \ell' \neq \ell,$$
$$(1 - b_{\ell,\ell}) C_{i,\ell} + \sum_{\tilde{\ell} \neq \ell} \left(\sum_{j \in Q_{\tilde{\ell},k}} \left(-\frac{b_{\ell,\tilde{\ell}}}{m_k} \right) C_{j,\ell} \right) = 1.$$

This implies that for all $i, j \in Q_{\ell,k}$ *and for all* ℓ'*,* $C_{i,\ell'} = C_{j,\ell'}$*. Thus, for* $i \in Q_{\ell,k}$*, we can denote* $C_{i,\ell'} = C_{\ell,\ell'}$*. Equation (15) now becomes*

$$(1 - b_{\ell,\ell}) C_{\ell,\ell'} + \sum_{\tilde{\ell} \neq \ell} \left(-b_{\ell,\tilde{\ell}} C_{\tilde{\ell},\ell'} \right) = 0 \text{ for all } \ell' \neq \ell,$$
$$(1 - b_{\ell,\ell}) C_{\ell,\ell} + \sum_{\tilde{\ell} \neq \ell} \left(-b_{\ell,\tilde{\ell}} C_{\tilde{\ell},\ell} \right) = 1.$$

Denote $C_\ell = (C_{\ell',\ell})_{\ell' \in L} \in \mathbb{R}^L$ *and denote* $D = I - B$*, where* $B = [b_{\ell',\ell}]_{L \times L}$*. Equations above implies that*

$$D (C_\ell)^T = (e_\ell)^T,$$

where $e_\ell = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^L$ *is the* ℓ*-th canonical basis. Thus,* $C_\ell = e_\ell (D^T)^{-1}$*, which does not depend on the choice of* $\mathcal{Q}$*-cluster economy. Note that due to symmetry* *of cluster Q_k, it is easy to see that $c_{i,j} = c_{i,j'}$ for all i, j, j' such that $i \in Q_{\ell,k}$, $j, j' \in Q_{\ell',k}$ with $\ell \neq \ell'$. Therefore, one obtains*

$$c_{i,j} = \frac{C_{\ell,\ell'}}{m_k} \text{ for } j \in Q_{\ell',k} \text{ with } \ell' \neq \ell.$$

Moreover, we have

$$\sum_{j \in \mathcal{Q}_k} c_{i,j} d_{j,h} = \begin{cases} 0 \text{ if } i \neq h, \\ 1 \text{ if } i = h. \end{cases}$$

Taking $h \in Q_{\ell,k} \setminus \{i\}$ and $h = i$, we get

$$\sum_{\tilde{\ell} \neq \ell} \left(\sum_{j \in Q_{\tilde{\ell},k}} \frac{C_{\ell,\tilde{\ell}}}{m_k} \left(-\frac{b_{\tilde{\ell},\ell}}{m_k} \right) \right) + c_{i,h}(1 - b_{\ell,\ell}) = 0,$$
$$\sum_{\tilde{\ell} \neq \ell} \left(\sum_{j \in Q_{\tilde{\ell},k}} \frac{C_{\ell,\tilde{\ell}}}{m_k} \left(-\frac{b_{\tilde{\ell},\ell}}{m_k} \right) \right) + c_{i,i}(1 - b_{\ell,\ell}) = 1.$$

Therefore, $c_{i,i} - c_{i,h} = 1/(1 - b_{\ell,\ell})$ for all $h \in Q_{\ell,k} \setminus \{i\}$, implying that $c_{i,h} = c_{i,h'}$ for all $h, h' \in Q_{\ell,k} \setminus \{i\}$. Combining with $\sum_{h \in Q_{\ell,k}} c_{i,h} = C_{\ell,\ell}$, one gets

$$\begin{cases} c_{i,i} = \frac{C_{\ell,\ell}}{m_k} + \frac{m_k - 1}{m_k} \frac{1}{1 - b_{\ell,\ell}}, \\ c_{i,j} = \frac{C_{\ell,\ell}}{m_k} - \frac{1}{m_k} \frac{1}{1 - b_{\ell,\ell}} \text{ for } j \in Q_{\ell,k} \setminus \{i\}. \end{cases}$$

Proof (of Proposition 6) *Let $\mathcal{Q} = \{Q_1, \dots, Q_K\}$ be a partition of $\{1, \dots, n\}$. Matrix $(I - A)^{-1}$ can be described as follows:*

$$\begin{pmatrix} C^{Q_1} & & & & \\ & \ddots & & & \\ & & C^{Q_i} & & \\ & & & \ddots & \\ & & & & C^{Q_K} \end{pmatrix}.$$

Thus, we have for $i \in Q_{\ell,k}$,

$$\begin{aligned} \left((a_0)^T (I - A)^{-1}\right)_i &= \sum_{h \in \bigcup_\ell Q_{\ell,k}} a_{0,h} c_{h,i} \\ &= \sum_{\ell' \in L} \frac{a_{0,\ell'}}{n} C_{\ell',\ell} = \frac{1}{n} \sum_{\ell' \in L} a_{0,\ell'} C_{\ell',\ell}, \end{aligned}$$

which is independent of the choice of $\mathcal{Q}$.

If for all i, $\lambda_i = \bar{\lambda}$, then $u_i = u_j = a_{i,0} \log a_{i,0}$ which is also independent of the structure of network. So the welfare $W(A, \lambda) = (a_0)^T (I - A)^{-1} u$ are the same for all $\mathcal{Q}$-cluster economy.

Proof (of Proposition 7) *We first show that among all the $\mathcal{Q}$-clustered network equilibria, $\overline{\lambda}(a_i)$ achieves the highest (resp. lowest) value in the fully connected (resp. islands) economy. Indeed, assume that $\mathcal{Q} = \{Q_1, \dots, Q_K\}$ and $i \in Q_{\ell,k}$. One gets*

$$\log(\overline{\lambda}(a_i)) = -\sum_{j \in Q_k} a_{i,j} \log(a_{i,j}).$$

Replacing a_i defined in system (4), we have

$$\log(\overline{\lambda}(a_i)) = -b_{\ell,\ell} \log(b_{\ell,\ell}) - \sum_{\ell' \neq \ell} b_{\ell,\ell'} \log\left(\frac{b_{\ell,\ell'}}{|Q_k|}\right).$$

It is easy to see that if $0 < c < 1$, function $x \log(c/x)$ is decreasing with respect to $x \geq 1$. Thus, the value of $b_{\ell,\ell'} \log(b_{\ell,\ell'}/|Q_k|)$ is highest (resp. lowest) when $|Q_k| = 1$ (resp. $|Q_k| = n$). Therefore, $\overline{\lambda}(a_i)$ achieves the highest (resp. lowest) value when $|Q_k| = n$ (resp. $|Q_k| = 1$).

If we denote a_i^F (resp. a_i^I) the vector of input weights of firm i in the fully connected (resp. islands) economy, then $\overline{\lambda}(a_i^F) > \overline{\lambda}(a_i^I)$. Thus, if all firms have increasing returns to diversification, then one has

$$\frac{\lambda_i(a_i^F)}{\overline{\lambda}(a_i^F)} > \frac{\lambda_i(a_i^I)}{\overline{\lambda}(a_i^I)}.$$

Now, note that for all i, $u_i = \log(\lambda_i(a_i)) - \log(\overline{\lambda}(a_i)) + b_{\ell,0} \log(b_{\ell,0})$. Therefore,

$$\begin{aligned} u_i^F &= \log(\lambda_i(a_i^F)) - \log(\overline{\lambda}(a_i^F)) + b_{\ell,0} \log(b_{\ell,0}) > u_i^I \\ &= \log(\lambda_i(a_i^I)) - \log(\overline{\lambda}(a_i^I)) + b_{\ell,0} \log(b_{\ell,0}). \end{aligned}$$

Applying the similar computation as in Proof of Proposition 6, the welfare of the fully connected economy $W(A^F, \lambda) = (a_0)^T (I - A_F)^{-1} u^F$ is greater than the welfare of the islands economy $W(A^I, \lambda) = (a_0)^T (I - A_I)^{-1} u^I$, where A^F (resp. A^I) is the production network associated to the fully connected (resp. islands) economy. The result for decreasing returns to diversification proceeds in the same way.

Proof (of Proposition 8) *Defining $\sum_{\ell' \in L} a_{0,\ell'} C_{\ell',\ell} = Q_\ell$, same computation as in the proof of Proposition 6 gives that for all $i \in M_\ell$,*

$$\left((a_0)^T (I - A)^{-1}\right)_i = \frac{1}{n} Q_\ell.$$

The welfare $W(A, \lambda) = (a_0)^T (I - A)^{-1} u$ can be written as follows:

$$\begin{aligned} W(A, \lambda) &= \sum_i \left((a_0)^T (I - A)^{-1}\right)_i u_i \\ &= \sum_{\ell \in L} \frac{1}{n} Q_\ell \left(\sum_{i \in M_\ell} u_i\right). \end{aligned}$$

In the fully connected economy, we have for all $i \in M_\ell$,

$$u_i^F = \log(\lambda) + b_{\ell,0} \log(b_{\ell,0}) + b_{\ell,\ell} \log(b_{\ell,\ell}) + \sum_{\ell' \neq \ell} b_{\ell,\ell'} \log\left(\frac{b_{\ell,\ell'}}{n}\right).$$

In the islands economy, we have for all $i \in M_\ell$,

$$u_i^I = \log(\lambda) + b_{\ell,0}\log(b_{\ell,0}) + b_{\ell,\ell}\log(b_{\ell,\ell}) + \sum_{\ell' \neq \ell} b_{\ell,\ell'}\log(b_{\ell,\ell'}).$$

Thus, the difference between the welfare of the fully connected economy and the islands one is

$$\begin{aligned} W(A^I,\lambda) - W(A^F,\lambda) &= \sum_{\ell\in L}\frac{1}{n}Q_\ell\Bigg(\sum_{i\in M_\ell}(u_i^I - u_i^F)\Bigg) \\ &= \sum_{\ell\in L}Q_\ell\Bigg(\sum_{\ell'\neq\ell}b_{\ell,\ell'}\log(n)\Bigg) \\ &= \log(n)\sum_{\ell\in L}Q_\ell\Bigg(1 - b_{\ell,\ell} - b_{\ell,0}\Bigg). \end{aligned}$$

Clearly, $Q_\ell(1 - b_{\ell,\ell} - b_{\ell,0}) > 0$ for all ℓ. Thus, denoting $K = \sum_{\ell\in L}Q_\ell(1 - b_{\ell,\ell} - b_{\ell,0})$ ends the proof.

Proof (of Proposition 9) *The following lemmas are useful for the proof:*

Lemma 7 *Let $p_1,\ldots,p_n$ are real numbers. Then*

$$\sum_{k=1}^{n} k\left(\sum_{1\leq i_1<\ldots<i_k\leq n} p_{i_1}\cdots p_{i_k}\prod_{j\neq i_1,\ldots,i_k}(1-p_j)\right) = \sum_{j=1}^{n}p_j.$$

Proof (of Lemma 7) *Denoting $N = \{1,\ldots,n\}$, we have for every $1 \leq i \leq n$,*

$$\begin{aligned} p_i &= p_i\sum_{k=0}^{n-1}\left(\sum_{\substack{1\leq i_1<\ldots<i_k\leq n\\ i_1,\ldots,i_k\in N\setminus\{i\}}} p_{i_1}\cdots p_{i_k}\prod_{\substack{j\neq i_1,\ldots,i_k\\ j,i_1,\ldots,i_k\in N\setminus\{i\}}}(1-p_j)\right) \\ &= \sum_{k=0}^{n-1}\left(\sum_{\substack{1\leq i_1<\ldots<i_k\leq n\\ i_1,\ldots,i_k\in N\setminus\{i\}}} p_i p_{i_1}\cdots p_{i_k}\prod_{\substack{j\neq i_1,\ldots,i_k\\ j,i_1,\ldots,i_k\in N\setminus\{i\}}}(1-p_j)\right). \end{aligned}$$

By summing up over i, one can obtain the desired result by noting that for fixing $1 \leq i_1 < \ldots < i_k \leq n$, the term $p_{i_1}\cdots p_{i_k}\prod_{j\neq i_1,\ldots,i_k}(1-p_j)$ appears exactly k times.

Lemma 8 *Let $a_1,\ldots,a_n, b_1\ldots,b_m$ be real numbers. Then*

$$\frac{\sum_{1\leq i\neq j\leq n}|a_i - a_j|}{n} + \frac{\sum_{1\leq i\neq j\leq m}|b_i - b_j|}{m} \leq \frac{\sum_{\substack{1\leq i\leq n\\ 1\leq j\leq m}}|a_i - b_j|}{m+n}. \tag{16}$$

Proof (of Lemma 8) *To prove Inequality (16), we will prove the following equivalent inequality:*

$$\frac{m}{n}\left(\sum_{1\le i\ne j\le n}|a_i-a_j|\right)+\frac{n}{m}\left(\sum_{1\le i\ne j\le m}|b_i-b_j|\right)\le \sum_{\substack{1\le i\le n\\1\le j\le m}}|a_i-b_j|.$$

Without loss of generality, we assume that all points belong to the segment $[0,1]$ and some two of the points are endpoints of the segment. Denote $A=\{a_1,\dots,a_n\}$, $B=\{b_1,\dots,b_m\}$, and $x=\{a_1,\dots,a_n,b_1\dots,b_m\}$.

Define function $f:[0,1]^{m+n}\to\mathbb{R}$ as follows:

$$f(x)=\frac{m}{n}\left(\sum_{1\le i\ne j\le n}|a_i-a_j|\right)+\frac{n}{m}\left(\sum_{1\le i\ne j\le m}|b_i-b_j|\right)-\left(\sum_{\substack{1\le i\le n\\1\le j\le m}}|a_i-b_j|\right).$$

It is clear that f is a continuous function on a compact set, so it attains the minimum at some points x. Consider such a point with the smallest number of different values of coordinates. Then these different values belongs to $\{0,1\}$.

Indeed, assume that there is a group of coordinates whose value is strictly between 0 and 1. We remark that if we simultaneously increase (or decrease) these coordinates in its small neighborhood containing no other values of coordinates of x, the value of f changes monotonically. Therefore, if we move this group of coordinates in the right direction which does not increase the value of f, we reach to other point x', at which we attain minimum of f and the number of different values of coordinates of x' is smaller. This is a contradiction.

Thus, there is a minimum point x^* of f where there are p numbers from A and q numbers from B equal to 0, and $n-p$ numbers from A and $m-q$ numbers from B equal to 1. At x^*, we have

$$\begin{aligned}f(x^*)&=\frac{m}{n}p(n-p)+\frac{n}{m}q(m-q)-(p(m-q)+q(n-p))\\&=-\frac{(mp-nq)^2}{mn}\le 0,\end{aligned}$$

which concludes the proof.

Let us proceed with the proof of the proposition. We first observe that we can write the disruption functions $\phi_i^{min}(K,A)$ and $\phi_i^{sum}(K,A)$ as follows:

$$\begin{aligned}\phi_i^{\min}(K,A)&=\sum_{\ell\in L}\phi_{i,\ell}^{\min}(K,A),\\\phi_i^{sum}(K,A)&=\sum_{\ell\in L}\phi_{i,\ell}^{sum}(K,A),\end{aligned}$$

where

$$\begin{aligned}\phi_{i,\ell}^{\min}(K,A)&=\min_{j\in M_\ell|(i,j)\in K\text{ and }a_{i,j}>0}a_{i,j},\ \text{ and}\\\phi_{i,\ell}^{sum}(K,A)&=\sum_{j\in M_\ell|(i,j)\in K\text{ and }a_{i,j}>0}a_{i,j}.\end{aligned}$$

At the moment, we drop the superscript and write

$$\phi_i(K,A) = \sum_{\ell \in L} \phi_{i,\ell}(K,A).$$

Since the productivity factor λ_i *is Hick-neutral, the expected (log) social welfare becomes*

$$Const \; + (a_0)^T (I-A)^{-1} u^r,$$

where

$$\begin{aligned} u_i^r &= \log(1-\rho) \sum_{K \subset M \times M} \left[\prod_{\{(a,b) \in K^c\}} (1-r_{a,b}) \prod_{\{(c,d) \in K\}} r_{c,d} \phi_i(K,A) \right] \\ &= \log(1-\rho) \sum_{\ell \in L} \left(\sum_{K \subset M \times M} \left[\prod_{\{(a,b) \in K^c\}} (1-r_{a,b}) \prod_{\{(c,d) \in K\}} r_{c,d} \phi_{i,\ell}(K,A) \right] \right). \end{aligned}$$

Assume that $\mathcal{Q} = \{Q_1, \ldots, Q_K\}$*. Due to the special structure of* $\mathcal{Q}$*-cluster economy, the term* $\phi_{i,\ell}(K,A)$ *depends only on the cardinal of set* $K \cap \{(i,j) \mid j \in Q_{\ell,k}\}$*, and the links* (i,j) *for* $j \in M_\ell \setminus Q_{\ell,k}$ *are irrelevant for computing probability. It is trivial to see that the term* $\sum_{K \subset M \times M} \prod_{\{(a,b) \in K^c\}} r_{a,b} \prod_{\{(c,d) \in K\}} (1-r_{c,d}) \phi_{i,\ell}(K,A)$ *is*

$$\Phi_{i,\ell}(A) = \mathbb{P}(\textit{Some links among } |Q_k| \textit{ links } (i,j)_{j \in Q_{\ell,k}} \textit{ is disrupted }) \phi_{i,\ell}(.,A),$$

where $\phi_{i,\ell}(.,A)$ *is the value of* $\phi_{i,\ell}(K,A)$ *for some* K *containing those disrupted link. Thus,*

$$u_i^r = \log(1-\rho) \sum_{\ell \in L} \Phi_{i,\ell}(A).$$

Now, let m_k *be the cardinality of* Q_k *and let us consider a firm* i *of input type* $\bar{\ell}$ *in cluster* Q_k*, i.e.,* $i \in Q_{\bar{\ell},k}$*.*

We remark that in both two disruption functions, $\phi_{i,\bar{\ell}}(A) = b_{\bar{\ell},\bar{\ell}}$*, which is independent of structure of* $\mathcal{Q}$*, so we focus on* $\Phi_{i,\ell}(A)$ *where* $\ell \neq \bar{\ell}$*. It is easy to see that*

$$\Phi_{i,\ell}(A) = \sum_{q=1}^{m_k} \mathbb{P}(q \textit{ links among } |Q_k| \textit{ links } (i,j)_{j \in Q_{\ell,k}} \textit{ are disrupted }) \phi_{i,\ell}(q,A),$$

where $\phi_{i,\ell}(q,A) = \phi_{i,\ell}(K,A)$ *for some* K *such that* $|K \cap \{(i,j) \mid j \in Q_{\ell,k}\}| = q$*.*

If $\phi_i = \phi_i^{\min}$*, we have*

$$\phi_{i,\ell}^{\min}(q,A) = \min_{j \in M_\ell | (i,j) \in K \textit{ and } a_{i,j} > 0} a_{i,j} = \frac{b_{\bar{\ell},\ell}}{m_k}.$$

Thus,

$$\begin{aligned} \Phi_{i,\ell}^{\min}(A) &= \Big(1 - \mathbb{P}(\textit{All links } (i,j), \textit{ where } j \in Q_{\ell,k}, \textit{ are not disrupted})\Big) \frac{b_{\bar{\ell},\ell}}{m_k} \\ &= b_{\bar{\ell},\ell} \frac{1 - \prod_{j \in Q_{\ell,k}} (1-r_{i,j})}{m_k}. \end{aligned}$$

If $\phi_i = \phi_i^{sum}$*, we have*

$$\phi_{i,\ell}^{sum}(q, A) = \sum_{j \in M_\ell | (i,j) \in K \text{ and } a_{i,j} > 0} a_{i,j} = q \frac{b_{\bar{\ell},\ell}}{m_k}.$$

Thus,

$$\Phi_{i,\ell}^{sum}(A) = \frac{b_{\bar{\ell},\ell}}{m_k} \sum_{q=1}^{m_k} q \times \mathbb{P}(q \text{ links among } |Q_k| \text{ links } (i,j)_{j \in Q_{\ell,k}} \text{ are disrupted}).$$

The probability of exact q *disrupted links* (i, j_1)*,...,*(i, j_q)*, where* $j_1, \ldots, j_q \in Q_{\ell,k}$*, is*

$$r_{i,j_1} \cdots r_{i,j_q} \prod_{\substack{j \neq j_1, \ldots, j_q \\ j \in Q_{\ell,k}}} (1 - r_{i,j}).$$

Applying Lemma 7, we have

$$\Phi_{i,\ell}^{sum}(A) = \frac{b_{\bar{\ell},\ell}}{m_k} \sum_{j \in Q_{\ell,k}} r_{i,j}.$$

Same computation as in the proof of Proposition 6 gives that for all $i \in M_{\bar{\ell}}$*,*

$$\left((a_0)^T (I - A)^{-1}\right)_i = \frac{1}{n} \sum_{\ell \in L} a_{0,\ell} C_{\ell,\bar{\ell}}.$$

As in the proof of Proposition 8, we define $\sum_{\ell' \in L} a_{0,\ell'} C_{\ell',\bar{\ell}} = Q_{\bar{\ell}}$*. Thus*

$$\begin{aligned}
(a_0)^T (I - A)^{-1} u^r &= \frac{1}{n} \sum_{\bar{\ell} \in L} Q_{\bar{\ell}} \Big(\sum_{i \in M_{\bar{\ell}}} u_i^r \Big) \\
&= \frac{1}{n} \log(1 - \rho) \sum_{\bar{\ell} \in L} Q_{\bar{\ell}} \left(\sum_{i \in M_{\bar{\ell}}} \Big(\sum_{\ell \in L} \Phi_{i,\ell}(A) \Big) \right) \\
&= \frac{1}{n} \log(1 - \rho) \sum_{\bar{\ell} \in L} Q_{\bar{\ell}} \left(\sum_{\ell \in L} \Big(\sum_{i \in M_{\bar{\ell}}} \Phi_{i,\ell}(A) \Big) \right).
\end{aligned}$$

We then focus on the term $\sum_{i \in M_{\bar{\ell}}} \Phi_{i,\ell}(A)$*, which is*

$$\sum_{k=1}^{K} \Big(\sum_{i \in Q_{\bar{\ell},k}} \Phi_{i,\ell}(A) \Big). \tag{17}$$

If $\ell = \bar{\ell}$*, then this sum equals to* $n \times r_{i,i} \times b_{\bar{\ell},\bar{\ell}}$ *which is independent of the choice of* $\mathcal{Q}$*-cluster in either homogeneous risk or increasing risks. Therefore, the only concern arises when* $\ell \neq \bar{\ell}$*.*

Now, we are ready to prove the main result.

1) *If* $\phi_i(K,A) = \phi_i^{\min}(K,A)$ *and risk is homogeneous, we have* $r_{i,j} = r$ *for all links* (i,j) *and* $\Phi_{i,\ell}^{\min}(A) = b_{\bar{\ell},\ell}\dfrac{1-(1-r)^{m_k}}{m_k}$. *Thus,*

$$\sum_{k=1}^{K}\Big(\sum_{i\in Q_{\bar{\ell},k}}\Phi_{i,\ell}^{\min}(A)\Big) = b_{\bar{\ell},\ell}\sum_{k=1}^{K}\Big(1-(1-r)^{m_k}\Big).$$

Applying the inequality $(1-x)+(1-y)\geq 1-xy$ *for all* $0\leq x,y\leq 1$, *one gets the highest value of* $\sum_{i\in M_{\bar{\ell}}}\Phi_{i,\ell}(A)$ *is when* $|\mathcal{Q}|=1$ *(*$|\mathcal{Q}|=n$*), which concludes our desired result.*

2) *If* $\phi_i(K,A) = \phi_i^{sum}(K,A)$ *and risk is homogeneous, we have* $r_{i,j} = r$ *for all links* (i,j) *and* $\Phi_{i,\ell}^{sum}(A) = b_{\bar{\ell},\ell}\times r$. *Our desired result is followed by*

$$\sum_{k=1}^{K}\Big(\sum_{i\in Q_{\bar{\ell},k}}\Phi_{i,\ell}^{sum}(A)\Big) = n\times b_{\bar{\ell},\ell}\times r.$$

3) *If* $\phi_i(K,A) = \phi_i^{sum}(K,A)$ *and risk increases with distance, we have*

$$\begin{aligned}\sum_{k=1}^{K}\Big(\sum_{i\in Q_{\bar{\ell},k}}\Phi_{i,\ell}^{sum}(A)\Big) &= b_{\bar{\ell},\ell}\sum_{k=1}^{K}\Big(\frac{\sum_{i\in Q_{\bar{\ell},k}}\sum_{j\in Q_{\ell,k}} r_{i,j}}{m_k}\Big)\\ &= b_{\bar{\ell},\ell}\frac{r}{n}\left(n+\sum_{k=1}^{K}\Big(\frac{\sum_{i,j\in Q_k}|i-j|}{m_k}\Big)\right).\end{aligned}$$

Applying Lemma 8 several times, we can conclude that the maximal (minimal) value of $\sum_{i\in M_{\bar{\ell}}}\Phi_{i,\ell}(A)$ *attains when* $K=n$ *(K=1).*

Proof (of Proposition 10) *The proof is straightforward by enumeration of the set of* $\mathcal{Q}$*-clustered equilibrium network.*

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