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Lockdowns and the aggregate dynamics of a pandemic: mind the rebound

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Lockdowns and the aggregate dynamics of a pandemic: mind the rebound.

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Abstract

We study optimal lockdown decisions taken by a policymaker facing a pandemic modelled according to the standard SIRD deterministic model. The policymaker trades off the economic costs and the mortality record of the pandemic which depend on the severity and duration of the lockdown. We contrast the shortsightedness versus the farsightedness of the policymaker. Policy-related peaks and rebounds are characterized and explain why a no-accommodating policy is self-defeating. A farsighted policy should not be too severe so as to avoid a rebound. The shortest duration consistent with a given health goal is not the less costly. There exists an optimal pair of duration and lockdown severity resulting in herd immunity.

Keywords: Pandemic; lockdown policy; Covid-19.

JEL classification: D61; H51; I18.

Highlights:

- We analytically characterize the impact of lockdown policy on the dynamics of a pandemic.

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- We show the importance of foreseeing the post-lockdown evolution of the pandemic.
- We explain how rebounds are linked to lockdown policy.
- There exists an optimal policy combining duration and lockdown severity, consistent with herd immunity.

1 Introduction

A pandemic such as the Covid-19 pandemic raises the issue of the best way to tackle it and in particular the extent of lockdown policy as the way to stem the dissemination of the virus within a given population.

There are two main strategies with respect to lockdown: suppression or mitigation. The suppression strategy (aka “zero Covid”) does not compromise and aims at eradicating the pandemic by means of an extreme lockdown policy disregarding the immediate economic costs so generated. The mitigation strategy (aka “living with Covid”) aims at finding a compromise between the objective of limiting the number of fatal casualties generated by the pandemic and the objective of mitigating the economic negative consequences of lockdown measures. In this paper we investigate the mitigation strategy applied to a pandemic from a theoretical point of view. Explicitly solving the model we use, we are able to fully characterize the dynamics of the pandemic depending on the assumed lockdown policy adopted by the policymaker and squarely address the issue of the optimal policy when the aims of the policymaker are expressed by means of a social payoff function. In other words, from a positive point of view, we use the model to understand the consequences over the dynamics of a lockdown policy; from a normative point of view, we tackle the determination of the optimal lockdown policy to be decided by a policymaker confronted with a pandemic. Understanding the properties of lockdown policies, both positively and normatively, is important because in this matter as in others, the possibility of policy mistakes with dire consequences cannot be ruled out. As far as lockdown decisions are concerned, two sorts of mistake are particularly important: policymakers may be myopic and unable (or unwilling) to look at the health costs beyond the lockdown period, and they may be unable (or unwilling) to endogenize the length of lockdown, taking into account the economic as well as the health cost generated by a pandemic. On the whole, the approach developed in this paper allows us to better understand the negative consequences over time of inadequate policy strategies and suggest important characteristics of the optimal ones: adopting a farsighted perspective, anticipating the rebound process, and supporting a mitigation strategy, as a longer and milder lockdown policy dominates a suppression policy.

A pandemic (like smallpox or tuberculosis), unlike an endemic (like the flu), is generated by a virus which is not susceptible of re-infection. It is eradicated once herd immunity is achieved: no one may be infected. In the case of an endemic, since living humans or animals can be re-infected, eradication is impossible. Accordingly, a pandemic is modelled according to a variant of the canonical SIR model, whereas an endemic is better modelled using a SIRS model.[1] In this paper, focusing on pandemics, we analytically solve a SIRD model augmented with a reduced-form equation summarizing the economic losses incurred because of the pandemic and we search for the optimal policy to be applied. We emphasize the importance of the time horizon of a policymaker dealing with a pandemic. Indeed a policymaker who myopically solely focuses on the severity of a lockdown and is not able to take into consideration the duration of a lockdown makes a serious mistake. In particular it is likely to lead to rebounds which ultimately worsen the fatality record of the pandemic. On the contrary a farsighted policymaker prefers to have long but mild lockdowns than short severe ones.

We first study the dynamics of a pandemic in the absence of a lockdown policy using a Susceptible-Infected-Recovered-Dead model (SIRD). This configuration provides us with a benchmark: our analysis crucially hinges on the result that the peak of a pandemic is attained when the proportion of susceptible agents is equal to the inverse of the "natural" reproduction number. Then we address the issue of an active lockdown policy, defined as increasing social distancing during a certain time period (duration)

[1]See Kermack & McKendrick (1927), Murray (2007). The dynamics generated by the various models used in epidemiology are qualitatively very different. These differences are often neglected or misperceived whereas they condition the adequacy of the health policy to be applied in each case.

by restricting freedom of behavior, including freedom of moves: wearing masks, forbidding certain act, limiting access to some activities. Social distancing limits individual interactions and thus the dissemination of the virus. A lockdown policy therefore consists in choosing the reproduction number of the pandemic over a given period. The policymaker faces a standard dilemma between economic and health objectives. On the one hand, a lockdown inflicts economic losses that the government wants to limit; on the other hand, it reduces social interaction, thus the spreading of the virus and the induced loss of lives. This dilemma is formalized by a welfare function depending on these two arguments where the relative weight given to the mortality argument captures the implicit "value of life" as assessed by the policymaker.

We contrast a short-term and a long-term perspective. We define a short-term perspective as a single setting of a policy-chosen reproduction number when the time horizon of the policymaker is limited to the duration of the lockdown policy. In contrast, a long-term perspective takes into account the future consequences of a fixed-duration lockdown policy. The first one is in line with the behavior of some policymakers when they have a limited time horizon in mind (e.g. a political term). The second one is consistent with intertemporal rationality. We characterize the dynamics of the pandemic in both cases. In particular we study the impact of the "value of life" parameter on the chosen reproduction number.

For each perspective a series of propositions illuminates the solution of the policy dilemma. The importance of the various timings related to lockdown, the timing of action, the termination date of the pandemic and their interplay with the marginal impact of the degree of lockdown, is highlighted. Considering the policy duration as given, we show that the optimal lockdown degree (the optimal reproduction number chosen by the policymaker) negatively depends on the "value of life" and the fatality parameters.

Using the long-term perspective we show that a post-policy pandemic rebound may happen if the lockdown policy has been too strict and/or its duration too short: it happens when the controlled pandemic has not passed its peak, that is when the end-of-policy proportion of susceptible agents is above the inverse of the natural reproduction number. This explains why an extreme eradication policy such as a zero-Covid policy is self-defeating. Realistically considering that it cannot last forever, there will be a rebound once this policy is lifted and this rebound may lead to a very high number of deaths and a low proportion of end-of-time susceptible agents. Taking into account the possibility of rebounds explains why the optimal reproduction number chosen in the long-term perspective may be higher than the reproduction number chosen in the short-term perspective.

Tackling the role of the duration of a lockdown policy, we show that the shortest policy able to reach a given mortality number, implying a stricter lockdown policy (the choice of a lower reproduction number), is not economically the less costly: a less severe lockdown policy extending over a longer period of time generates less economic losses. Addressing the normative issue of the optimal policy, relaxing the assumption of a given lockdown duration and endogenizing both the extent of lockdown and duration, we prove that there exists an optimal couple of duration and reproduction number solving the policy trade-off. Such a policy is consistent with herd immunity.

The propositions resulting from this analytical effort provide illuminating insights on the interplay between the dynamics of a pandemic and the policy measures adopted to control it, such as lockdowns. Useful recommendations on the strictness and duration of a lockdown may be derived from these propositions and should be useful for policymakers confronted with a pandemic such as the Covid-19. Results derived from a model which does not overly simplify the dynamics laws of a pandemic by eliminating state variables or making extreme assumptions appear more general and robust than those obtained when simpler models of a pandemic's dynamics are used or when simulation exercises based on calibrated versions of the standard SIR model are performed. On the whole, they support the view that a mitigation lockdown policy ("living with Covid-19") is welfare-improving compared to a drastically severe policy (that is, the desire to fight the dissemination of the virus by means of a very strict lockdown policy applied over an unknown duration). The latter policy is unsustainable as the epidemic will surge again sooner or later: there are rebounds, once the extreme lockdown measures are lifted, with letality consequences which can be severe. It thus cannot appear as an optimal policy.

Literature review

Most papers focusing to lockdown policies, in relation with the Covid-19 pandemic[2], have an empirical objective of understanding what to do and resort to a calibration approach based on the SIR epidemiology model and aim at better understanding the stakes of lockdown policies. Eichenbaum, Rebelo, and Trabandt (2020a) solve computationally a SIR model that features agents' optimization and search for the best consumption tax policy. Eichenbaum, Rebelo, and Trabandt (2020b) account also for behavioral adjustments, as in their setting infected individuals become more reckless if tested which makes testing optimal only when combined with quarantines. Alvarez, Argente, and Lippi (2020) calibrate a SIR model using data of the WHO, in order to find the optimal containment policy. Piguillem and Shi (2020) tackle an optimal control problem in a similar SIR model augmented with a intertemporal welfare function based on two arguments, consumption-related utilities and the mortality records of the pandemic. They calibrate their model on Italian data, compare the outcomes of various lockdown policies and assess the relevance of a testing policy. Akbarpour et al. (2020) simulate an agent-based model calibrated to a rich set of micro-level data and analyze the impact of various policies. Acemoglu et al. (2021) analyze a multi-group SIR model, including a more general transmission function. They show that differential lockdown policies defined by the social planner are preferable to a undifferentiated lockdown policy. Farboodi, Jarosch, and Shimer (2020) take into account the individual choices with respect to social activity in a pandemic and search, using a calibrated model based on US data, the optimal policy with respect to social distancing.

Loertscher & Muir (2021) is closer to our methodology as they analyze a lockdown policy in a theoretical SIR model which maximizes output subject to the constraint that contagion is contained so that total hospitalizations do not exceed the health capacity constraint both with a homogeneous population and an heteregenous population. However such a policy does not correspond to the solving of a trade-off between economic

[2]The Covid-19 pandemic has generated a flurry of papers aiming at finding the proper lockdown policy by means of calibration exercises. For a survey on the economics of Covid-19, cf. Brodeur *et al.* (2020).

and health costs encompassed into a welfare function which is the subject of this paper. Camera & Gioffré (2021) analytically study the economic impact of a sequence of short-lived but extreme lockdowns in a model based on the theory of random matching, which makes explicit how epidemics spread through economic activity. They do not study the extent of lockdown and therefore do not address the issue of optimal lockdowns. Bliman & Duprez (2021) study the link between lockdowns of fixed duration and the final death toll of a pandemic, but do not address the dilemma between economic and health costs related to lockdowns. Britton & Leskelä (2023) consider the minimization of the total number of infected individuals. They show that an optimal intervention strategy implies a single constant-level lockdown (and not a continuously varying optimal control strategy). Similarly, Bliman & Rapaport (2023) show that for the problem of minimizing the epidemic final size in the SIR model through social distancing, there is no benefit in splitting interventions on several disjoint time periods. Caulkins *et al.* (2021) use simulation techniques to investigate optimal lockdown strategies within a SIR model but do not tackle the analytical solution of the problem. Lastly Gonzalez-Eiras & Niepelt (2020b) resort to two simplified and tractable versions of the modified SIR model developed by Bohner *et al.* (2019) in order to study an optimal lockdown policy.[3] The first one neglects the death burden and the distinction between infected and recovered. The second one rests on the assumption of full mortality for the infected and of equal productivity of the susceptible and the infected. The lockdown variable adjusts continuously. These models are calibrated. In a companion paper, Gonzalez-Eiras & Niepelt (2020a) building upon Bailey (1975) simplify the SIR model and restrict it to a single state variable, eliminating the possibility of recovery after infection and thus mortality; this model is calibrated searching for the optimal lockdown trajectory. These various simplifying assumptions drastically reduce the scope of a lockdown policy. Our paper tackles the issue of characterizing analytically the optimal policy as a compromise between economic and health objectives when the laws of motion of a pandemic as formalized by a standard SIR model are explicitly

[3] They justifiy their choice by writing: "SIR models of various flavours feature two endogenous epidemiological state variables; this makes it difficult to embed economic choices in those frameworks without sacricing analytical tractability, transparency, and generality."

taken into account. In Aspri *et al.* (2021) the social planner minimizes a loss function which combines production and number of Covid deaths during a given period, in a SEAIRD model. They prove the existence and uniqueness of an optimal control policy (via a compacity argument). It is not characterized and there is no mention of epidemic rebounds. Examples of contrained optimal policies are given via parameter selection. Morris *et al.* (2021) study in a standard SIR model the theoretically optimal strategy to reduce the peak prevalence, with time limited intervention. There are no economic considerations and the social planner's objective is to reduce the height of the epidemic peak. It is shown that at the optimum there will be twin peaks, i.e. after the first peak a rebound of the same height, since herd immunity is not attained after the first peak. The optimal strategy is "maintain" (keeping the number of infected constant) "then suppress" (fixing the reproduction number to 0), which cannot be viewed as a realistic policy. Grigorieva & Khailov (2014) study the optimal control problem of minimizing the total number of the infected on a given time interval, with the use of the Pontryagin Maximum Principle. The optimal non pharmaceutical intervention (quarantine or lockdown) is shown to be constant on a time interval (see also Grigorieva *et al.* (2016)). Kruse & Strack (2020) study an optimal social distancing strategy with continuously varying lockdown intensity in the presence of a trade-off between health and social costs. But they do not tackle the case of rebounds as they assume comprehensive vaccination of the population at the end of the lockdown policy interval. Andersson *et al.* (2022) tackle a similar trade-off as ours but use a simpler model than the SIR model, with neither death nor recovery. Moreover they consider a myopic planner and do not address the after-policy dynamics of an epidemic.

2 The model.

We consider a closed society which is affected by a pandemic. There is no shock in this setting and the evolution of the pandemic can be described by a deterministic SIRD model with mass action incidence, i.e. with an incidence $-\frac{dS}{dt}$ of the form of Equation

(1) below[4]. Martcheva (2015); **?** explains that the "mass action incidence is used in diseases for which disease-relevant contact increases with an increase in the population size. For instance, in influenza and SARS, contacts increase as the population size (and density) increase". The same is true for COVID-19.[5]

The dynamics of the pandemic is given by the following set of equations for any $t \in \mathbb{R}$:

$$\frac{dS}{dt} = -b_0 I(t) S(t) \tag{1}$$

$$\frac{dI}{dt} = b_0 I(t) S(t) - \gamma I(t) \tag{2}$$

$$\frac{dRec}{dt} = (1-\delta)\gamma I(t) \tag{3}$$

$$\frac{dD}{dt} = \delta\gamma I(t) \tag{4}$$

where $S(t)$ is the number of individuals susceptible of being infected in the population at time t, $I(t)$ the number of infected individuals, $Rec(t)$ the number of recovered individuals, and $D(t)$ the number of deaths due to the pandemic. After the "infected" stage, individuals are said to be "removed". The number of removed individuals is $R(t)$ with $R(t) = Rec(t) + D(t)$. A fraction δ of the "removed" (i.e. after infection) dies from the pandemic. This parameter δ is the infection fatality rate. The fraction $(1-\delta)$ recovers. Let N be the total number of individuals at time $t = 0$ in the society. We have $S(t) + I(t) + Rec(t) + D(t) = N$ at every instant t.

Now, we can define the following proportions:

$s(t) = \frac{S(t)}{N}$ is the proportion of individuals susceptible of being infected in the population at a given instant t, $i(t) = \frac{I(t)}{N}$ the proportion in the population of infected individuals and $r(t) = \frac{Rec(t)+D(t)}{N} = \frac{R(t)}{N}$ the proportion of removed individuals, with $s(t) + i(t) + r(t) = 1$ at every instant t. Setting $\beta_0 = Nb_0$, this formally leads to the

[4] Gallic *et al.* (2022) uses a SIRD model to analyze the lockdowns in Europe and their optimality for COVID-19 pandemics, Lin *et al.* (2010) uses a SIRD model to study non-pharmaceutical interventions against pandemic influenza.

[5] On mass action incidence with a SIRD model, see Osemwinyen & Diakhaby (2015) for Ebola virus, and Martianova *et al.* (2020) for COVID-19.

following standard SIR model:

$$\frac{ds}{dt} = -\beta_0 i(t) s(t) \quad (5)$$

$$\frac{di}{dt} = \beta_0 i(t) s(t) - \gamma i(t) \quad (6)$$

$$\frac{dr}{dt} = \gamma i(t) \quad (7)$$

We set $\mathcal{R}_0 \equiv \frac{\beta_0}{\gamma}$ the natural (initial) reproduction number.[6] The parameter β_0 refers to social interactions and controls the spreading of the pandemic as it affects the variation of the proportion of "susceptible" agents. It is specific to a pandemic and captures the physical impact of social interactions within society on the dynamics of the pandemic. This structural parameter is related to social habits and collective mores. The parameter γ is positive and corresponds to the rate of infected individuals recovering in a given unit of time. "Removed" means either returning to perfect health (recovered) or death. It is assumed here that once someone recovers from the virus, he or she is never infected again: recovery is permanent.[7] We shall return to this point in the conclusion. This model has been used by Rowthorn and Maciejovski (2020) for simulation exercises related to the Covid-19 pandemic. Here we shall analytically solve it, under various lockdown policy configurations. The "natural" reproduction number may capture the rearrangement of the production process such as teleworking, and more generally, the changes of voluntary behavior induced by the advent of the pandemic. We abstract from investigating this issue and take it for given.

We first study the dynamics of the pandemic when there is no lockdown policy imposed by a public authority. The pandemic develops freely according to the reproduction number $\mathcal{R}_0$ and dies away when a sufficient fraction of the population has recovered and does not transmit the virus any more. This policy has been dubbed a

[6]aka "basic" reproduction number. As noted by Avery *et al.* (2020), this number "embodies both the underlying biological ability of the pathogen to jump from person to person in various types of interactions as well as the number of interactions of each type that people have in the ordinary course of their daily lives" (p.84) and may partially result from self-interested voluntary measures of social distancing.

[7]This assumption doen not properly reflect the Covid-19 pandemic. Yet the model captures its basic characteristics when new variants are neglected. Variants can be introduced in the model at the cost of increasing complexity.

"collective immunity" strategy. In this case, the pandemic eventually vanishes through herd immunity: the number of recovered people is large enough so that the virus does not find a significant number of "susceptible" individuals and does not reproduce itself anymore. We shall use this configuration as a benchmark against which the various lockdown policies may be compared. We assume $\mathcal{R}_0 > 1$, otherwise the pandemic cannot start. Following Kröger & Schlickeiser (2020), we assume the following boundary conditions: $s(-\infty) = 1$, $i(-\infty) = 0$ and $r(-\infty) = 0$.

Collective immunity is reached when the pandemic is extinct. Given the deterministic nature of the model, it is reached at the "end of time". Formally it is defined as $(s_\infty, 0, r_\infty)$: there are no more infected people, the proportion of susceptible s_∞ is positive and the proportion of recovered r_∞ is equal to $1 - s_\infty$. We refer to s_∞ as the end-of-pandemic (or "terminal") susceptible proportion. It corresponds to herd immunity.

Assuming that the reproduction number does not vary over time, we have the following lemma on the dynamics of the SIR model, given the "natural" laws of motion of the three key variables $s(t), i(t)$ and $r(t)$ of the pandemic (when driven by $\mathcal{R}_0$).[8]

Lemma 1. **(i)** *The dynamics of the pandemic for $t \in \mathbb{R}$ is given by*

$$r(t) = -\frac{1}{\mathcal{R}_0} \ln s(t) \tag{8}$$

$$i(t) = 1 - s(t) + \frac{1}{\mathcal{R}_0} \ln s(t) \tag{9}$$

$$\int_{s(t)}^{s(0)} \frac{1}{\beta_0 s \left[1 - s + \frac{1}{\mathcal{R}_0} \ln s\right]} ds = t \ . \tag{10}$$

$t \mapsto s(t)$ is a decreasing function, and $t \mapsto r(t)$ is an increasing function.

(ii) *The proportion of infected $i(t)$ is first increasing, then decreasing. It is maximal when $s(t) = \frac{1}{\mathcal{R}_0}$ and equal to $i_{\max} = 1 - \frac{1}{\mathcal{R}_0}\left[1 + \ln(\mathcal{R}_0)\right]$.*

[8]For a similar result see Harko *et al.* (2014), p.187.

(iii) *At the end of the pandemic, we have* $(s, i, r) = (s_\infty, 0, r_\infty)$*, with* $r_\infty = 1 - s_\infty$ *and* s_∞ *given by*

$$\mathcal{R}_0 = -\frac{\ln(s_\infty)}{1 - s_\infty}, 0 < s_\infty < 1. \tag{11}$$

Proof. See Appendix A.1. □

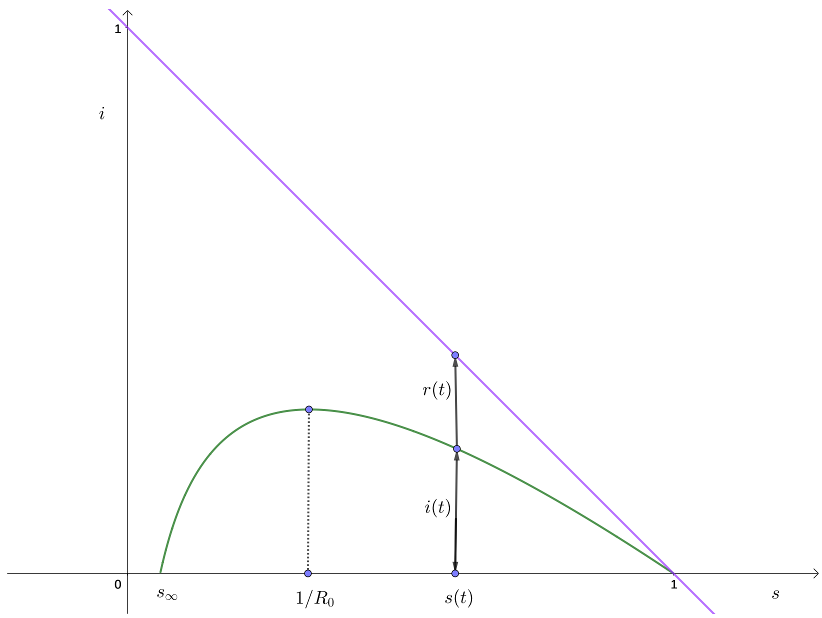

Figure 1: Variations of i with respect to s in the plane (s, i)

This lemma explicits the functional dynamics of the three variables of interest, the numbers of susceptible, infected and recovered people from the SIR model and establishes some properties of these dynamics. *(i)* details the interdependence between the dynamics of these variables. Eq.(8) shows that the proportion of recovered is a decreasing function of the proportion of susceptible; Eq.(9) shows that the proportion of infected is a non-monotone function of the proportion of susceptible. Eq.(10) shows that the proportion of susceptible varies with time depending on β_0 and γ. As expected, the proportion of susceptible decreases with time and the proportion of recovered increases with time.

(ii) proves that the relation between the proportion of infected and the proportion of susceptible is non-monotone and reaches a maximum when $s(t)$ is equal to $\frac{1}{\mathcal{R}_0}$. The higher the natural reproduction number, the higher the peak of the pandemic. As we

shall see later, the reproduction number plays a critical role in the dynamics of the pandemic once a lockdown policy is put in place. Once having recovered, one cannot be infected again. Thus the evolution of the proportion of infected depends on both the reproduction number and the evolution of the pool of susceptible. At the beginning of the pandemic for a given $\mathcal{R}_0$, the pool of susceptible is large and the number of infections increases since newly infected agents easily spread the virus into a large population of susceptible. But the pool (hence the proportion) of susceptible necessarily declines once the pandemic has started. Over time newly infected spread the virus into a smaller and smaller population even with the same reproduction number. This negative effect curbs down the rate of new infections and the proportion of infected declines. Thus the proportion of infected is a non-monotonous function of the proportion of susceptible. The maximum rate is achieved when the proportion of susceptible is just equal to the inverse of the natural reproduction number. The higher this number, the lower the susceptible proportion after which the number of infected people starts declining. At $s(t) = 1/\mathcal{R}_0$, there is a "herd immunity" threshold: the expected number of people that a newly infected person will directly infect is equal to 1.

Finally *(iii)* characterizes the final herd immunity level (end-of-time proportion of susceptible) denoted by s_∞, where s_∞ is a decreasing function of $\mathcal{R}_0$: the higher is $\mathcal{R}_0$, the more violent and deadly is the pandemic. All this is consistent with intuition and observations of actual pandemics.

Figure 1 gives a graphical representation of the dynamics of the pandemic in the plane (s, i). At the very beginning (when $t \to -\infty$) the point $(s(t), i(t))$ is close to $(1; 0)$. When t increases $s(t)$ decreases, and $i(t)$ first increases then decreases. At the end of the pandemic (when $t \to +\infty$) the point $(s(t), i(t))$ is close to $(s_\infty; 0)$.

3 Shortsighted lockdown policies.

We now assume that the "government" ruling this society (*aka* a public authority able to impose some lockdown policy) is able to act at a given period t_0.[9] Social inter-

[9] This is in line with Schlickeiser & Kröger (2021). They assume that the pandemic, while in infancy, is taken into account from some given date denoted by "0", equivalent to our t_0: at this date there is a

actions, namely “social distancing”, can be modified by law and public punishments (fines, etc.) decided by the government. We define a shortsighted lockdown policy as the capacity by the government to replace the parameter β_0 by β lower than β_0 over a limited and definite interval of time: it is supposed to be put in place in t_0 and ends in period T. It negatively affects the reproduction number which becomes $R = \frac{\beta}{\gamma}$ instead of $R_0 = \frac{\beta_0}{\gamma}$ during the interval (t_0, T). This interval is the policy duration. After T, the reproduction number is back to its “natural” value. The stricter the lockdown policy, the lower β and the lower the reproduction number. It is thus controlled by the public authority and represents its lockdown policy instrument. We do not consider a reproduction number $\mathcal{R}$ that varies continuously over time (as in optimal control), as this is not a realistic policy. As a lockdown policy is imposed on individuals, communication and persuasion are critical. It imposes simple messages and imposing social distancing rules that vary every day would not be collectively understood and followed. True, the parameters of a lockdown policy can vary infrequently. Since we reason in a deterministic framework, modelling such a strategy of succeeding different lockdown parameters would complexify the formal analysis without leading to qualitatively different policy lessons. A “shortsighted” policy is such that the consequences of this policy after T until the end of the pandemic are not taken into account by the policymaker.

3.1 Short-term dynamics of the pandemic with lockdown policy.

Assume that a decision of lockdown or general social distancing is taken at date $t_0 \in \mathbb{R}$, before the epidemic attains its maximum, i.e. $s(t_0) > \frac{1}{\mathcal{R}_0}$. A lockdown policy decided in instant t_0 consists in setting a new reproduction number $\mathcal{R}$ which applies to the period from t_0 onwards up to T. There are two periods to be distinguished in the evolution of the pandemic until the end of the lockdown:

– Before t_0, the reproduction number is $\mathcal{R}_0$ and the dynamics is governed by Eqs. (5)-(7).

small but positive number of infected people. This amounts to consider the pandemic at “its infancy”. The susceptible proportion is close to 1 and the infection number is close to 0.

– In the interval between t_0 and T, the reproduction number is $\mathcal{R} = \beta/\gamma$, with $\mathcal{R} < \mathcal{R}_0$, the dynamic system capturing the dynamics of the pandemic after t_0 becomes

$$\frac{ds}{dt} = -\beta i(t) s(t) \tag{12}$$

$$\frac{di}{dt} = \beta i(t) s(t) - \gamma i(t) \tag{13}$$

$$\frac{dr}{dt} = \gamma i(t) \tag{14}$$

with $\beta < \beta_0$.

In short, the dynamics of the pandemic starts as in Figure 1, being governed by $\mathcal{R}_0$. When $s(t_0)$ is reached, it bifurcates as it is then governed by $\mathcal{R}$, up to T. $s(t)$, $i(t)$, $r(t)$ will be denoted by $s_{\mathcal{R}}(t)$, $i_{\mathcal{R}}(t)$, $r_{\mathcal{R}}(t)$ for $t \in]t_0, T]$ when it will be necessary to stress their dependency to $\mathcal{R}$. The laws of motion of the three variables $s(t), i(t)$ and $r(t)$ under a temporary lockdown policy are specified in the following

Lemma 2. *Assume that the reproduction number is $\mathcal{R}_0 = \frac{\beta_0}{\gamma}$ on $t < t_0$ and $\mathcal{R} = \frac{\beta}{\gamma}$ on $t \in [t_0; T]$, with $\mathcal{R} < \mathcal{R}_0$, and $s(t_0) > \frac{1}{\mathcal{R}_0}$, i.e. t_0 is before the natural peak of the epidemic is reached. Assume also that $s(T) < \frac{1}{\mathcal{R}}$, i.e. T is after the lockdown-related peak.*

(i) The dynamics of the pandemic until the end of the lockdown is given by the following sets of equations:

For $t < t_0$, the dynamics is given by eqs. (8)-(10).

For $t \in [t_0, T]$,

$$r(t) = r(t_0) + \frac{1}{\mathcal{R}} \ln s(t_0) - \frac{1}{\mathcal{R}} \ln s(t) \tag{15}$$

$$i(t) = i(t_0) + s(t_0) - s(t) + \frac{1}{\mathcal{R}} \ln s(t) - \frac{1}{\mathcal{R}} \ln s(t_0) \tag{16}$$

$$\int_{s(t)}^{s(t_0)} \frac{1}{\beta s \left[i(t_0) + s(t_0) - s + \frac{1}{\mathcal{R}} \ln s - \frac{1}{\mathcal{R}} \ln s(t_0)\right]} ds = t - t_0. \tag{17}$$

$t \mapsto s(t)$ is a decreasing function, and $t \mapsto r(t)$ is an increasing function.

(ii) If $\mathcal{R} \geq \frac{1}{s(t_0)}$*, then on* $t \leq T$ *the proportion of infected individuals* $i(t)$ *is first increasing, then decreasing. It is maximal when* $s(t) = \frac{1}{\mathcal{R}}$*.*
If $\mathcal{R} < \frac{1}{s(t_0)}$*, then* $i(t)$ *is maximal at* $t = t_0$ *. It is decreasing on* $t \in [t_0, T]$*.*

(iii) For $\mathcal{R}' > \mathcal{R}$*, the curve* $(s_{\mathcal{R}'}(t), i_{\mathcal{R}'}(t))_{t \geq t_0}$ *is strictly above* $(s_{\mathcal{R}}(t), i_{\mathcal{R}}(t))_{t \geq t_0}$ *in the plane* (s, i)*, except a unique common point at* $(s(t_0), i(t_0))$*.*
$\mathcal{R} \mapsto s(t) = s_{\mathcal{R}}(t)$ *is a decreasing function of* $\mathcal{R}$*.*

Proof. See Appendix A.2. □

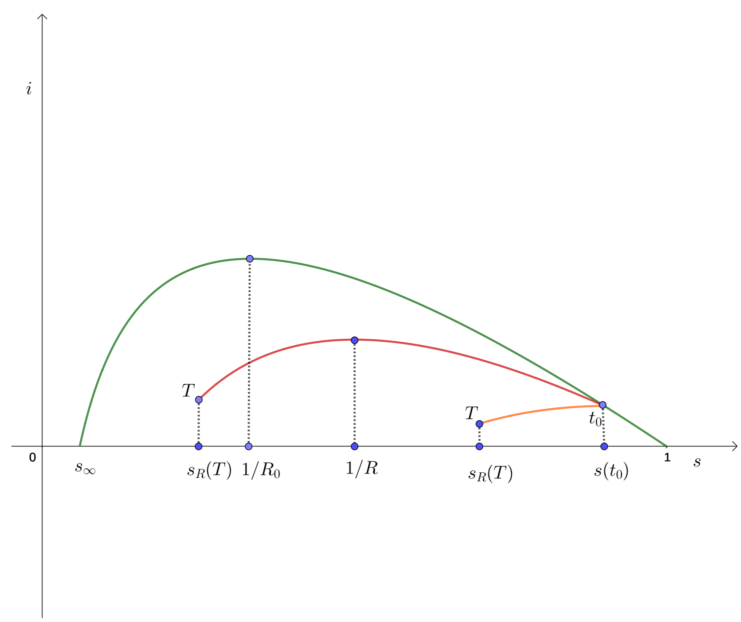

Figure 2: Variations of i with respect to s in the plane (s, i) with decision at date t_0

(i) shows the dynamic impact of the lockdown policy. It obviously has no impact over the period before t_0. Given the number of susceptible $s(t_0)$, the dynamics of the pandemic during the policy period is governed by a similar set of equations than in the previous period but depending on $\mathcal{R}$. Figure 2 gives a graphical representation of the dynamics of the pandemic with a lockdown of reproduction number $\mathcal{R}$, beginning at t_0 and ending at T.

(ii) tackles the impact of the policy reproduction number $\mathcal{R}$ over the interval $[t_0, T]$ on the dynamics of $i(t)$. It is either non-monotone or decreasing, depending on $\mathcal{R}$.

If, given the initial number of susceptible in t_0, the reproduction number $\mathcal{R}$ is not too low, $i(t)$ is first increasing, then decreasing. The explanation is similar as the one given above for the dynamics driven by $\mathcal{R}_0$. If $\mathcal{R}$ is sufficiently high, higher than $1/s(t_0)$, the contamination policy is too lenient to allow for an immediate decrease in the number of infected. This latter number first continues to grow and reaches a maximum when the susceptible proportion attains $1/\mathcal{R}$. It is important to note that this value does not depend on the other parameters of the model, in particular depends neither on t_0 nor on T. If, on the other hand, the policy number $\mathcal{R}$ is low enough (lower than $1/s(t_0)$), $i(t)$ is immediately decreasing. The social distancing is strong enough, $\mathcal{R}$ being sufficient small, to overcome the existence of a relatively large pool of agents susceptible of becoming infected and thus the easy spreading of the pandemic, so as to trigger an immediate decrease in the number of infected. The lower $\mathcal{R}$, the steeper the slope of the increasing function relating the proportion of susceptible and the proportion of infected. Such a strong lockdown policy can be dubbed a "(almost-)zero Covid policy": the policymaker wants to see the pandemic decreasing immediately and forcefully, despite the immediate negative economic consequences of this policy with the hope of getting rid of the pandemic and be able to resume a "normal" life with no lockdown and a low number of deaths. Notice that the higher is t_0, the lower is $s(t_0)$, the initial pool of susceptible, making the spreading of the pandemic more difficult.

According to *(iii)*, the number of susceptible in a given instant is a decreasing function of $\mathcal{R}$. This is consistent with intuition: the pool of susceptible agents decreases more rapidly when the reproduction number is high as the virus spreads more rapidly: the proportion of susceptible $s_{\mathcal{R}'}(t)$ related to $\mathcal{R}'$ is consistently below $s_{\mathcal{R}}(t)$ when $\mathcal{R}' > \mathcal{R}$. Consequently $s_{\mathcal{R}}(T) > s_{\mathcal{R}'}(T)$: a higher policy-chosen reproduction number leads to a lower end-of-lockdown proportion of susceptible and a higher mortality record. Furthermore there is no "catching-up" effect: a lenient lockdown policy (corresponding to $\mathcal{R}'$) cannot reach a combination (s, i) reached by a stricter policy (corresponding to $\mathcal{R}$). Since the curves $(s_{\mathcal{R}'}(T), i_{\mathcal{R}'}(T))_{t \geq t_0}$ and $(s_{\mathcal{R}}(T), i_{\mathcal{R}}(T))_{t \geq t_0}$ with $\mathcal{R} \neq \mathcal{R}'$ have a unique common point at $(s(t_0), i(t_0))$, a given pair (s, i) can be attained by one lockdown policy only and at one instant T only.

3.2 Shortsighted optimal lockdown policies.

The previous subsection highlights how a lockdown policy aiming at reducing the reproduction number of the pandemic affects its dynamics and the proportion of recovered individuals at any period. Turning to the normative aspect of a lockdown policy, we now tackle the dilemma facing the policymaker: a lockdown policy mitigates the health consequences of the pandemic but negatively affects the economy by restricting the social interactions between economic agents. In other words, there is a trade-off between economic and health objectives and a responsible government must solve this trade-off: how to optimally live with a pandemic?

In order to investigate the issue of defining the optimal lockdown policy in the presence of economic costs, we take for given the starting instant t_0 and ending instant T. Fixing the duration of a lockdown policy makes sense. The policymaker may be constrained by political conditions such as the term of her mandate or subject to economic pressures to end the policy before a certain date. She may be unable to act after a certain period for constitutional reasons or she may be unable or unwilling to anticipate the entire future of the pandemic. Above all, it corresponds to the very plausible case when the policymaker is myopic and does not foresee beyond a given date. The dynamics of a pandemic is such that the most of the contamination happens in a very short period and not much is lost by fixing this period. The marginal gains of increasing the policy period beyond a plausible duration may be limited. Lastly, addressing the optimal lockdown problem with two instruments in a non-linear system such as the one governing a pandemic is quite complex and likely to obscure the picture for little analytical gains. We will address this issue in Section 4.3, once important results, easy to understand and empirically relevant, have been obtained. In brief, it is reasonable to first reason with T given and focus on the lockdown policy-determined reproduction number: it is the parameter which attracts the most attention and is critical in the dynamics of the pandemic as proven above.

A shortsighted government limits its time horizon to the end of its lockdown policy T. In the following section we will consider a farsighted government which considers the entire future and therefore the after-lockdown dynamics of the pandemic, knowing that her lockdown policy affects this dynamics. This distinction is relevant. When observing the behavior of governments during the Covid-19 pandemic, one notices that policymakers regularly adopted a position of advocating before public opinion that the decided lockdown policy put in place would be sufficient to return to "normal life" and the economy would soon "pass the corner". Public opinion too seems to be oblivious of the long-term duration of a pandemic and such an attitude affects the decision process of the policymaker.

We consider the simple case of a pandemic developing without meeting any other barrier than the social distancing measure adopted by the government. In particular, there is no health constraint such as the availability of properly equipped hospital beds and there is no change in the therapy against it. The health system is structurally able to "properly" deal with patients. Formally, we assume that the infection fatality rate $\delta > 0$ is constant, independent of circumstances and exogenously given.

The cumulated mortality rate in the population is $m_T = \delta(r(T)+i(T)) = \delta(1-s(T))$ where $1 - s(T)$ denotes the fraction of the population which eventually recovers (or dies) from the pandemic after having being infected before T. Denoting by N the total population, the final number of deaths due to contaminations before T is $M_T = Nm_T$. A lockdown policy consists in adopting various compulsory social distancing measures so as to affect the reproduction number of the pandemic at any given period of time. Assuming that the inverse link between social distancing and this number is deterministic amounts to say that the government controls the reproduction number $\mathcal{R}$. We assume that a decision of lockdown or general social distancing is taken at date t_0, before the epidemic attains its maximum, i.e., $s(t_0) > \frac{1}{\mathcal{R}_0}$ and consists in setting a single reproduction rate which applies to any period from t_0 onwards: $\mathcal{R}(t) = \mathcal{R}, \forall t \in [t_0; T]$. As we abstract from any shock, including on the biological characteristics of the virus, and any change in government, this assumption is reasonable. M_T depends on $\mathcal{R}$, so we write $M_T = M_T(\mathcal{R})$.

A benevolent policymaker is willing to limit the amount of casualties of the pandemic by means of an active control of the reproduction number. Given her shortsightedness, the policymaker takes into account the cumulated mortality $M_T(\mathcal{R})$ due to contaminations occurring up to T. This mortality record is affected by her choice of $\mathcal{R}$. It is defined as

$$M_T(\mathcal{R}) \equiv \delta N \left(i_{\mathcal{R}}(T) + r_{\mathcal{R}}(T)\right) = \delta N \left(1 - s_{\mathcal{R}}(T)\right)$$

A lockdown policy also incurs economic losses: social distancing affects both the supply and the demand sides of the economy. On the one hand, some firms cannot open, some workers cannot work as efficiently as in "normal" times, or are out of work. On the whole the capacity to produce goods and services is impaired. On the other hand, some goods are not demanded because the social distancing prevents their consumption. Consumption and investment are depressed and the well-being of individuals is negatively affected by the desire to control the pandemic by means of social distancing measures. The goverment trades off the economic losses and the sanitary adverse consequences of the pandemic. Formally the welfare function of the decision-maker is assumed to be

$$V_T(\mathcal{R}) = (T - t_0)\, y(\mathcal{R}) - \lambda M_T(\mathcal{R}) = (T - t_0)\, y(\mathcal{R}) - \lambda \delta N \left(1 - s_{\mathcal{R}}(T)\right) \tag{18}$$

where $y(\mathcal{R})$ is an aggregate output index such as GDP per unit of time, depending on the lockdown parameter $\mathcal{R}$, and $\lambda \in [0; \infty)$ is the weight put on mortality relative to economic activity (as measured by y). It captures the "value of life" as assessed by the policymaker relative to the economic target of boosting the economy.[10] The function $y(\mathcal{R})$, in addition to the reduction of activity directly due to the reduction in mobility, may capture the change in the production process decided when the lockdown is imposed. Here we abstract from distinguishing the various channels which shape the relationship between y and $\mathcal{R}$ and directly reason on the reduced form given by $y(\mathcal{R})$.

During the pandemic, the economic index y is an increasing concave function of $\mathcal{R}$: the more lenient the lockdown policy, the higher the aggregate output index. We assume that the marginal gain of relaxing the lockdown policy (increasing the reproduction number) is diminishing with this number and an immediate impact of $\mathcal{R}$ on y without

[10] This welfare function, displaying an economic argument and a "loss of life" one, is similar to the functions used by Acemoglu et al. (2021) and Rowthorn and Maciejowski (2020).

lagged effects. The lowest value of y is obtained when the social distancing is at its maximum, that is, when $\mathcal{R}$ is equal to 0: economic losses are at their maximum. We consider that $y(0)$ is equal to 0 since then all activities, including productive ones, are frozen.[11] Thus $y(\mathcal{R})$ can be seen as the gain from relaxing the lockdown parameter from 0 to $\mathcal{R}$. When $\mathcal{R}$ increases, social distancing is relaxed, the economy partially recovers and losses are reduced. We assume y is a concave function of $\mathcal{R}$, with $y'(0) = +\infty$. When $\mathcal{R}$ equals $\mathcal{R}_0$, this corresponds to the "hands-off" regime characterized by the natural reproduction number and there are no economic losses.

The first term in (18) corresponds to the cumulative economic effect of the decision $\mathcal{R}$ from t_0 up to T. The second term corresponds to the health cost of the pandemic, measured in the total of deaths due to the pandemic up to date T. The optimal policy consists in choosing $\mathcal{R}^{opt}$ maximizing the welfare function, that is

$$\mathcal{R}^{opt} = \arg \max_{\mathcal{R} \leq \mathcal{R}_0} V_T(\mathcal{R}) \tag{19}$$

This optimal value generates a mortality record, an economic loss and thus a given level of welfare. On the whole, the global configuration with optimal policy is characterized by $(\mathcal{R}^{opt}, M_T(\mathcal{R}^{opt}), y(\mathcal{R}^{opt}), V_T(\mathcal{R}^{opt}))$. We could equivalently say that the decision-maker wants to minimize the short-term loss due to the pandemic and the lockdown

$$L_T(\mathcal{R}) = (T - t_0)(y(\mathcal{R}_0) - y(\mathcal{R})) + \lambda M_T(\mathcal{R}). \tag{20}$$

We are able to offer the following

Proposition 1.
There exists $\lambda_0 \geq 0$, with $\lambda_0 \leq \frac{(T-t_0)y'(\mathcal{R}_0)}{-\delta N \left(\frac{\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}}\right)_{\mathcal{R}=\mathcal{R}_0}}$, such that:

(i) $\mathcal{R}^{opt}$ is equal to $\mathcal{R}_0$ for $\lambda \in [0; \lambda_0]$ and $\mathcal{R}^{opt} < \mathcal{R}_0$ for $\lambda > \lambda_0$.

(ii) $\mathcal{R}^{opt}$ is a non increasing function of λ for $\lambda > \lambda_0$, with $\lim_{\lambda \to \infty} \mathcal{R}^{opt} = 0$.

Proof. See Appendix A.3. □

[11] We neglect household production.

This proposition states that the optimal lockdown policy consists in having a low reproduction number when the "value of life" weight in the welfare function is sufficiently high. When the value of life weight is close to 0, the lockdown optimal policy is the laissez-faire policy, corresponding to no imposition of social distancing measures. This comes from the fact that the marginal loss of imposing a stricter policy is high while the marginal welfare gain in terms of saved life is negligible, given that λ is low. When the weight is sufficiently high, the optimal value of $\mathcal{R}^{opt}$ starts decreasing. The more the policymaker cares about the mortality record of the pandemic relative to the adverse economic consequences of social distancing (the higher λ), the stricter is the chosen lockdown. When λ tends to infinity, $\mathcal{R}^{opt}$ tends to 0. When the value of life is arbitrarily large, the best policy is to neglect the economic costs for the sake of saving lifes by imposing an extremely severe lockdown. By a similar reasoning, notice that $\mathcal{R}^{opt}$ is a decreasing function of the fatality rate[12] δ.[13]

4 Farsighted policies.

We now investigate the pandemic and its relation wich lockdown policy in a long-term perspective. Compared to a short-term perspective, three differences can be introduced. Firstly, the delayed consequences of a lockdown policy are taken into account: Given the dynamic nature of the problem, anything happening in a given time interval impacts on the subsequent evolution of the pandemic. Secondly, we may relax the assumption made before that solely one lockdown with one reproduction number is fixed by the policymaker. Lastly, it may happen that the post-policy reproduction number differs from the initial one, being equal to $\mathcal{R}_0'$ instead of $\mathcal{R}_0$. In the sequel, we focus on the first difference: the policymaker deciding in t_0 takes into account the delayed impact after the end-of-policy instant T of her lockdown policy. Said in other words, she may be qualified as "far-sighted". We maintain the assumption that the policymaker chooses a unique reproduction number and we assume that the post-policy reproduction number

[12]The Ebola epidemics which is highly deadly leads to the most extreme lockdown measures.

[13]Rowthorn and Maciejovski (2020), based on simulation of the model, find that a 10-week lockdown is optimal if the value of life for Covid-19 victims exceeds 10 million pounds.

is the "natural" one: $\mathcal{R}_0' = \mathcal{R}_0$.

4.1 Long-term dynamics of the pandemic with lockdown policy.

Given a lockdown policy with a constant R over the duration period: $R = \frac{\beta}{\gamma}$ on $t \in [t_0; T]$, for $t < t_0$, the dynamics is given by eqs. (8)-(10). For $t \in [t_0,T]$, it is given by eqs. (15)-(17). For $t > T$, it is given by the following equations

$$r(t) = r(T) + \frac{1}{\mathcal{R}_0} \ln s(T) - \frac{1}{\mathcal{R}_0} \ln s(t) \tag{21}$$

$$i(t) = i(T) + s(T) - s(t) + \frac{1}{\mathcal{R}_0} \ln s(t) - \frac{1}{\mathcal{R}_0} \ln s(T) \tag{22}$$

$$\int_{s(t)}^{s(T)} \frac{1}{\beta_0 s \left[i(T) + s(T) - s + \frac{1}{\mathcal{R}_0} \ln s - \frac{1}{\mathcal{R}_0} \ln s(T) \right]} ds = t - T. \tag{23}$$

$s(t)$, $i(t)$, $r(t)$ will be denoted by $s_{\mathcal{R},T}(t)$, $i_{\mathcal{R},T}(t)$, $r_{\mathcal{R},T}(t)$ for $t > T$ when it will be necessary to stress their dependency to $\mathcal{R}$ and T. Eqs. (21)-(23) are similar to (15)-(17) with the crucial difference that the notations $r(T)$, $i(T)$ and $s(T)$ are introduced.[14] (21)-(23) depend on $s(T)$ and $i(T)$, that is on the outcome of the policy fixing $\mathcal{R}$ over the interval $t \in [t_0, T]$. This proves the delayed consequences of a lockdown policy after it has stopped. In the two following propositions, we investigate the dynamics of the pandemic after T.

Once a lockdown policy has ended, either the proportion of infected pursues its decline (at a different pace) or it reverts to increasing again. The latter case refers to a "rebound" which is a very common feature in actual pandemics. For example, in many countries the Covid-19 pandemic was characterized by several rebounds, not all due to the advent of variants of the original virus. Thus it is important to understand under which circumstances such a reversal happens in the absence of a renewed source of infection such as a new virus or a variant of the current one. This is answered in the

[14] They replace $r(t_0)$, $i(t_0)$ and $s(t_0)$ respectively, and $\mathcal{R}$ is replaced by $\mathcal{R}_0$.

following

Proposition 2.

(i) If $s(T) > \frac{1}{\mathcal{R}_0}$*, there is a rebound of the epidemic after* T *and* $i(t)$ *is maximal on* $t \geq T$ *when* $s(t) = \frac{1}{\mathcal{R}_0}$.

If $s(T) \leq \frac{1}{\mathcal{R}_0}$*, there is no rebound of the epidemic after* T *and* $i(t)$ *is maximal on* $t \geq T$ *when* $t = T$.

(ii) Setting $\widetilde{\mathcal{R}}_{t_0} = \frac{\ln(s(t_0)) - \ln\left(\frac{1}{\mathcal{R}_0}\right)}{i(t_0)+s(t_0)-\frac{1}{\mathcal{R}_0}}$*, we have:*

1. *For* $\mathcal{R} \leq \widetilde{\mathcal{R}}_{t_0}$*, there is necessarily a rebound after* T *for any value of* T.

2. *For* $\mathcal{R} > \widetilde{\mathcal{R}}_{t_0}$*, there exists a critical value* $T_{\min}(\mathcal{R})$*, such that:*

   - *There is a rebound after* T *if* $T < T_{\min}(\mathcal{R})$.

   - *There is no rebound after* T *if* $T \geq T_{\min}(\mathcal{R})$.

Proof. See Appendix A.4. □

Figure 3 illustrates the case of dynamics including a rebound, Figure 4 the case without a rebound.

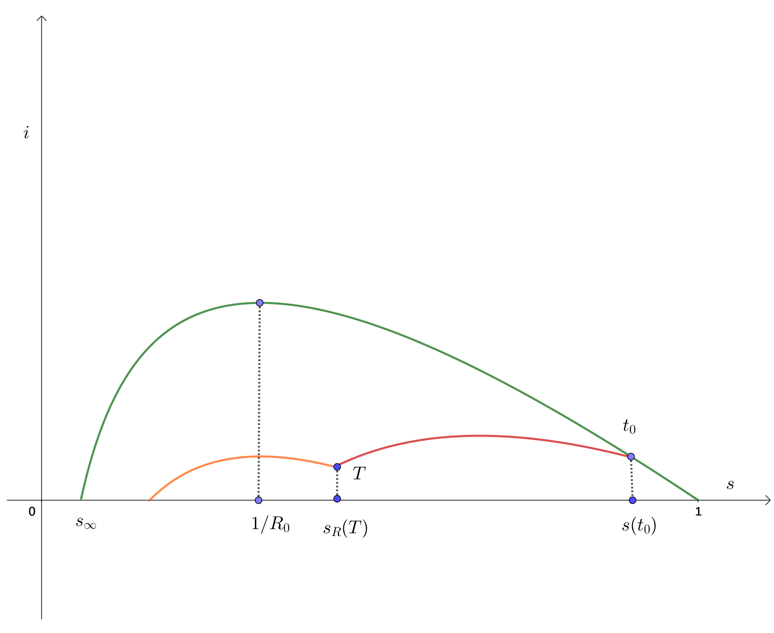

Figure 3: Rebound

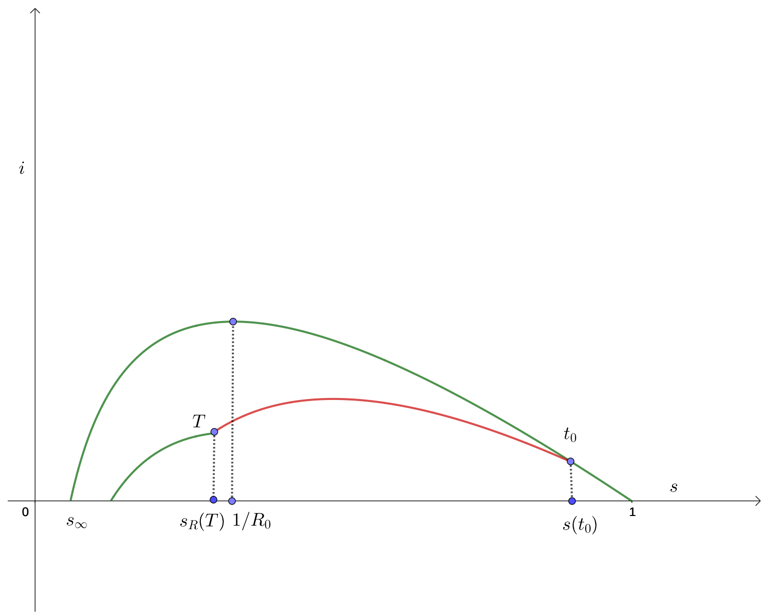

Figure 4: No rebound

(i) focuses on the impact of $s\left(T\right) = s_{\mathcal{R}}\left(T\right)$ and therefore implicitly of $\mathcal{R}$. It proves that if the susceptible proportion $s(T)$ at the end of the lockdown is above the "natural" peak of the pandemic, there will be a rebound when the reproduction number switches back to $\mathcal{R}_0$. The pair $(\mathcal{R}, T)$ is such that the control over the pandemic is not sufficient to pass this peak. The pool of people susceptible to be infected after the lockdown is too large and the contamination process starts increasing again after T: a rebound occurs. Notice that the post-T function between $s\left(t\right)$ and $i\left(t\right)$ shares the same property as in the case without lockdown: its peak is at $\frac{1}{\mathcal{R}_0}$. Therefore if $s\left(T\right)$ is higher than $\frac{1}{\mathcal{R}_0}$, it implies that the number of infected people increases after T when the susceptible proportion pursues its decline. It is solely if $s\left(T\right)$ is smaller than $\frac{1}{\mathcal{R}_0}$ that the two numbers decline together: no rebound occurs.

(ii) focuses on the impact of $\mathcal{R}$. If the reproduction number is higher than a critical value denoted by $\widetilde{\mathcal{R}}_{t_0}$, a rebound is avoided for T given and sufficiently high: both parameters conjugate to avoid a rebound after the lockdown policy. However, if $\mathcal{R}$ is too low, the control of the pandemic whatever the duration length is insufficient and it rebounds. This is due to the fact that the end-of-policy susceptible proportion (the pool of people available for infection) is high enough so as to let the natural

reproduction number have a huge impact on the number of infected and lead to a rebound. This counter-intuitive result casts doubt on a policy the severity of which is meant to efficiently control the pandemic. This is true in the short-term but eventually it will undo itself. This is particularly true to a "zero-Covid policy" which cannot last for ever given its economic costs. Sooner or later, an extremely severe lockdown policy (implying a very low reproduction number) will stop before the complete eradication of the virus (which can only happen at the end-of-time, that is, when t is arbitrarily large). At the end of the severe lockdown period, the pool of susceptible will be close to $s(t_0)$, and the dynamics of the pandemic governed by the natural reproduction number $\mathcal{R}_0$ and given by Proposition 1.[15] On the whole, society suffers from a large economic cost due to the severity of the lockdown policy without much impact on long-term mortality. This is unknown (or neglected) by the short-sighted policymaker who solely looks at the outcome at the end-of-policy instant T but this unpleasant conclusion appears clearly when a long-term perspective is adopted.

Turning to the eventual impact of a lockdown policy on the end-of-pandemic susceptible proportion we offer the following

Proposition 3.

(i) At the end of the pandemic, $(s, i, r) = (s_\infty(\mathcal{R}, T), 0, r_\infty(\mathcal{R}, T))$*, with* $r_\infty(\mathcal{R}, T) = 1 - s_\infty(\mathcal{R}, T)$ *and* $s_\infty(\mathcal{R}, T)$ *given by*

$$\mathcal{R}_0 = \frac{\ln(s(T)) - \ln(s_\infty(\mathcal{R}, T))}{i(T) + s(T) - s_\infty(\mathcal{R}, T)}, \qquad 0 < s_\infty(\mathcal{R}, T) < 1. \tag{24}$$

$s_\infty(\mathcal{R}, T) < \frac{1}{\mathcal{R}_0}$*, and* $s_\infty(\mathcal{R}, T)$ *is an increasing function of* T *and a decreasing function of* $\mathcal{R}_0$*.*

(ii) The end-of-pandemic susceptible proportion $s_\infty(\mathcal{R}, T)$ *is always higher than* $s_\infty(\mathcal{R}_0)$*. If* T *is sufficiently large,* $s_\infty(\mathcal{R}, T)$ *is a non-monotonic function of* $\mathcal{R}$*, it is increasing on* $\mathcal{R} < \widetilde{\mathcal{R}}_{t_0}$*, and decreasing on* $\mathcal{R} > \widetilde{\mathcal{R}}_{t_0}$*.*

[15] The case of Australia which pursued a zero-Covid strategy against Covid-19 is exemplary. Its prime minister Scott Morrison has declared on August 23rd 2021: "This is not a sustainable way to live in this country". See https://www.economist.com/asia/2021/08/28/australia-is-ending-its-zero-covid-strategy?utm_campaign=coronavirus-special-edition&utm_medium=newsletter&utm_source=salesforce-marketing-cloud&utm_term=2021-08-28&utm_content=article-link-1&etear=nl_special_1

Proof. See Appendix A.5. □

(i) makes clear that a lockdown policy always has an impact on herd immunity, that is, the end-of-pandemic susceptible proportion. It suffices to compare (24) with (11): in the former equation $i(T) = i_{\mathcal{R}}(T)$ and $s(T) = s_{\mathcal{R}}(T)$ which depend on the policy stance $(\mathcal{R}, T)$ now appear. The longer the lockdown policy lasts, the better it is in terms of herd immunity, which is consistent with intuition. This is due to the fact that both $i(T)$ and $s(T)$ decline with T, when T is large enough. It is logically a decreasing function of $\mathcal{R}_0$ as this number governs the post-policy dynamics: a higher reproduction number applying after T leads to a worsening of the pandemic in the post-lockdown period and eventually a higher fatality record.

(ii) proves that any lockdown policy, however light (a high $\mathcal{R}$) and/or short (a small T), leads to an improvement in the herd immunity, measured by the terminal susceptible proportion. There is never a perverse long-term effect of an active policy. Yet it does not mean that the terminal susceptible proportion is a monotone function of $\mathcal{R}$. This is due to the possible presence of rebounds. As we have seen above, a tight lockdown policy ($\mathcal{R}$ low) may lead to a large rebound whereas a not so tight policy leads to a small rebound. The ending of the large rebound may thus be at the left of the ending of the small rebound. The relationship of $s_\infty(\mathcal{R}, T)$ is a non-monotone function of $\mathcal{R}$ peaking at $\widetilde{\mathcal{R}}_{t_0}$ if T is large. In the absence of rebound ($\mathcal{R} > \widetilde{\mathcal{R}}_{t_0}$), the relationship is decreasing: a stricter lockdown policy improves the terminal susceptible proportion. With rebound ($\mathcal{R} < \widetilde{\mathcal{R}}_{t_0}$), the relationship is increasing: a stricter lockdown generates a higher rebound.

Notice that these two propositions can easily be adapted for the case $\mathcal{R}_0' \neq \mathcal{R}_0$.

4.2 Farsighted optimal lockdown policy.

In this subsection, we again adopt a normative point of view. Given the delayed consequences of a policy stance $(\mathcal{R}, T)$, a far-sighted policymaker (adopting a long-term perspective on the pandemic) should take them into account. The total economic cost of a lockdown policy $(\mathcal{R}, T)$ is equal to $(T - t_0)(y(\mathcal{R}_0) - y(\mathcal{R}))$. This cost is subject to two opposite forces: a longer duration increases economic losses whereas a higher reproduction number $\mathcal{R}$ decreases them. After T, since the reproduction number returns to $\mathcal{R}_0$ and no lagged economic effect of a policy fixed lockdown is assumed, there is no economic loss due to lockdowns. On the opposite, the health impact of the pandemic still goes on, based on $s(T)$ and $\mathcal{R}_0$, as shown in (21)-(23).

Extending (20), the decision-maker's objective is now to minimize losses over the entire future (for T given):

$$L_\infty(\mathcal{R}) = (T - t_0)\left(y(\mathcal{R}_0) - y(\mathcal{R})\right) + \lambda M_\infty(\mathcal{R}) \tag{25}$$

with $M_\infty(\mathcal{R}) = N\delta r_\infty(\mathcal{R}) = N\delta\left(1 - s_\infty(\mathcal{R}, T)\right)$. The properties of the optimal decision of a farsighted policymaker are given in

Proposition 4.

Let $\mathcal{R}_\infty^{opt}$ be the value of $\mathcal{R}$ minimizing the long-term loss $L_\infty(\mathcal{R})$ on $0 \leq \mathcal{R} \leq \mathcal{R}_0$, for T and λ given.

(i) For T given, there exists $\lambda_0' \geq 0$, with $\lambda_0' \leq \frac{(T-t_0)y'(\mathcal{R}_0)}{-\delta N s_\infty'(\mathcal{R}_0)}$, such that $\mathcal{R}_\infty^{opt}$ is equal to $\mathcal{R}_0$ for $\lambda \in [0; \lambda_0']$, and $\mathcal{R}_\infty^{opt} < \mathcal{R}_0$ for $\lambda > \lambda_0'$, where $\mathcal{R}_\infty^{opt}$ is a non increasing function of λ for $\lambda > \lambda_0'$.

(ii) For λ given, $\lambda > \lambda_0'$, we have $\mathcal{R}_\infty^{opt} \geq \widetilde{\mathcal{R}}_{t_0}$ if T is sufficiently high.

Proof. See Appendix A.6. □

(i) is a long term version of proposition 1 and is similarly explained.

(ii) assuming T sufficiently high, the long-term optimal reproduction number is above a positive value $\widetilde{\mathcal{R}}_{t_0}$ for any value of λ. This results from the desire of the farsighted policymaker to avoid a rebound after T, an event which is not anticipated by a shortsighted policymaker. A rebound leads to an increase in mortality and a lower herd immunity level. Avoiding a rebound makes sense especially when the life argument in the loss function is given a higher weight. According to Proposition 2 (ii), if T is sufficiently high, then there is a rebound if $\mathcal{R} \leq \widetilde{\mathcal{R}}_{t_0}$, and there is no rebound if $\mathcal{R} > \widetilde{\mathcal{R}}_{t_0}$. This implies that the far-sighted policymaker will choose a reproduction

number $\mathcal{R}_{\infty}^{opt}$ higher than $\widetilde{\mathcal{R}}_{t_0}$. In other words, an optimal lockdown policy is such that it rules out rebounds.

4.3 Farsighted optimal pandemic policy.

In this subsection we extend our normative approach to lockdowns and define an optimal pandemic policy as the joint choice of duration and lockdown severity parameters. Choosing a pandemic policy consists in choosing a value $R \in (0, R_0)$ and a value $T \in]t_0, \infty[$. Up to now, we considered the policy duration $T - t_0$ as given (and we refered to a lockdown policy). It is interesting to relax this assumption as interrogations about the duration of lockdowns were rife during the Covid-19 pandemic. To shed some light on the role of policy duration, we consider the following problem. Supposing the policymaker wishes to attain collective immunity consistent with a given mortality M_{∞}, i.e. equivalently with a certain final susceptible proportion $s_{\infty} = \frac{1}{\mathcal{R}_0} - \varepsilon$, which pair $(\mathcal{R}, T)$ does she choose? By considering an objective in terms of reaching a given collective immunity ratio we do not oppose here the health objective to the economic one. Instead we focus on a possible trade-off between the duration of a lockdown policy and its stringency. It may be argued that the collective immunity level should be reached in the minimal time thanks to a "tough" lockdown policy, given the impatience of the people to get rid of the pandemic as soon as possible, rather than applying a more lenient lockdown policy (a higher $\mathcal{R}$) on a longer period. Is it true? Is the shortest duration policy optimal? We answer this problem in the following proposition. We denote by $(\mathcal{C}_{\varepsilon})$ the curve in the plane (s, i) representing the end of lockdowns $(s_{\mathcal{R}}(T), i_{\mathcal{R}}(T))$ which lead after release of lockdown to $s_{\infty}(\mathcal{R}, T) = \frac{1}{\mathcal{R}_0} - \varepsilon$.

Proposition 5. *Let* $\varepsilon \in \left(0; \frac{1}{\mathcal{R}_0}\right)$ *be given.*

(i) There exist an infinity of couples $(\mathcal{R}, T)$ *such that* $s_{\infty}(\mathcal{R}, T) = \frac{1}{\mathcal{R}_0} - \varepsilon$. *More precisely, there exist* $\mathcal{R}_1$ *and* $\mathcal{R}_2$, $\mathcal{R}_2 > \mathcal{R}_1$ *and a function* T_{ε} *such that*

$$s_{\infty}(\mathcal{R}, T) = \frac{1}{\mathcal{R}_0} - \varepsilon \Leftrightarrow \mathcal{R} \in (\mathcal{R}_1, \mathcal{R}_2) \text{ and } T = T_{\varepsilon}(\mathcal{R}).$$

Moreover, $\lim_{\mathcal{R} \to \mathcal{R}_1 +} T_{\varepsilon}(\mathcal{R}) = \lim_{\mathcal{R} \to \mathcal{R}_2 -} T_{\varepsilon}(\mathcal{R}) = +\infty$.

(ii) We denote by $\left(\widehat{\mathcal{R}}_\varepsilon, \widehat{T}_\varepsilon\right)$ *the policy pair which generates the minimal economic cost and by* $\left(\mathcal{R}_\varepsilon^\circ, T_\varepsilon^\circ\right)$ *the policy which allows to reach* $s_\infty = \frac{1}{\mathcal{R}_0} - \varepsilon$ *in the minimal time. Then* $\widehat{\mathcal{R}}_\varepsilon > \mathcal{R}_\varepsilon^\circ$ *and* $\widehat{T}_\varepsilon > T_\varepsilon^\circ$.

Proof. See Appendix A.7. □

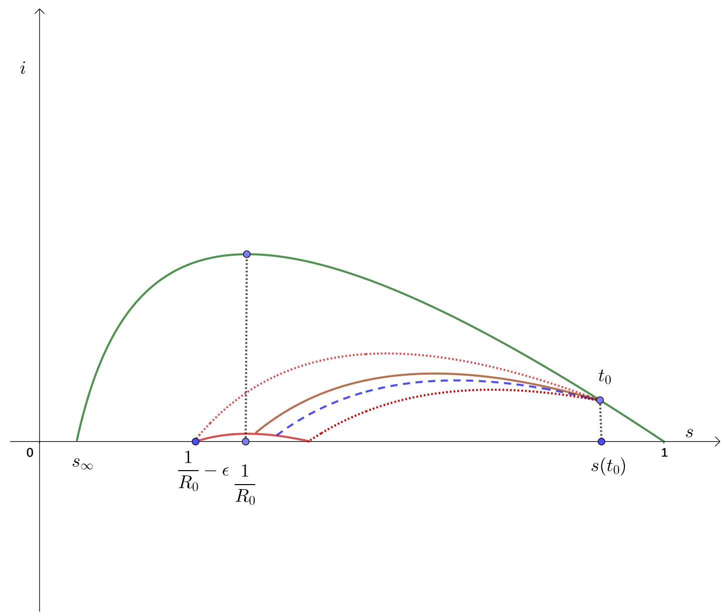

Figure 5: Minimal economic cost to reach $s_\infty = \frac{1}{R_0} - \varepsilon$

(i) states that the objective can be attained by an infinite number of combinations of duration and reproduction number (but not all reproduction numbers $\mathcal{R}$ are admissible). This is due to the adverse consequences of lengthening the policy interval (increasing T) and lightening the lockdown intensity (increasing $\mathcal{R}$). These different lockdown policies cannot be determined according to health considerations only, since they do not have the same economic impact. This raises the question of which policy is best from an economic point of view. *(ii)* shows that the shortest lockdown policy consistent with a health objective generates a higher economic cost than is necessary as it is linked to a relatively severe lockdown policy, i.e. a small reproduction number. Despite the fact that such a policy does not last long it harms too much economic activity. The cost-minimizing policy implies a higher reproduction number imposed over a longer

duration: some patience is rewarding. The conclusion is that tackling a pandemic consists in balancing policy duration and severity of a lockdown.

Figure 5 illustrates this proposition. The two dotted curves represent the curves associated with $\mathcal{R}_1$ and $\mathcal{R}_2$. The lower curve reaching $\frac{1}{\mathcal{R}_0} - \varepsilon$ represents $(\mathcal{C}_\varepsilon)$. Any trajectory corresponding to a lockdown policy with $s_\infty(\mathcal{R}, T) = \frac{1}{\mathcal{R}_0} - \varepsilon$ must terminate on $(\mathcal{C}_\varepsilon)$ at the end of the lockdown. Afterwards, the pandemic is governed by $\mathcal{R}_0$ and follows $(\mathcal{C}_\varepsilon)$. The dashed curve corresponds to the time-minimizing trajectory and the continuous curve to the cost minimizing one. Since it is above the dashed curve, it corresponds to a higher reproduction number (from Proposition 2(iii)). It reaches the same mortality record, but it lasts longer.

Proposition 5 tells us that indeed there is an optimal pandemic policy to reach a given herd immunity level. We can use this result to prove the following

Proposition 6. *There is an optimal pandemic policy* $(\mathcal{R}^*, T^*)$ *such that* $L_\infty(\mathcal{R}, T)$ *is minimized.*

Proof. See Appendix A.8. ☐

Proposition 6 claims that the policymaker, acting at t_0, is able to play on both parameters of the policy stance so as to minimize the long-term consequences of the pandemic, consistent with herd immunity. Here we do not impose the level of herd immunity to be reached. This level is endogenously derived: it is not necessarily the highest possible one but it is optimal when the adverse economic consequences of a pandemic policy are taken into account.

5 Conclusion.

This paper offers a theoretical analysis of the optimal decision about social distancing and possibly duration taken by a policymaker confronted with a pandemic and facing a dilemma between reducing the economic costs of lockdown and minimizing the mortality rate through social distancing.

Using the workhorse model of epidemiology, namely a deterministic version of the SIRD model, we first look at the dynamics of the pandemic in the absence of any policy action aiming at controlling the pandemic. We obtain a non-monotone relation between the infected proportion and the susceptible proportion which peaks at a susceptible proportion equal to the inverse of the reproduction number. This very simple expression plays a critical role in the understanding of the pandemic dynamics when a lockdown policy is put in place.

A pandemic policy is defined by three parameters: the instant of decision, the extent of lockdown which affects the reproduction number and the duration of the lockdown. We show that the dynamics of the pandemic is strongly affected by these variables. Such a policy is designed so as to trade off the health benefits and the economic losses that it generates. Solving this trade-off amounts to search for the optimal lockdown policy to be followed. This issue is the core of this paper. We distinguish between a short-term perspective, when the policymaker is short-sighted and limits her time horizon to the ending of the lockdown, and a long-term perspective, when she is far-sighted and takes into account the posterior consequences on the dynamics of the pandemic after this ending, up to infinity.

We show that there can be rebounds in the pandemic happening either when the policy duration is too short and/or when the lockdown policy reproduction number is too low. A far-sighted policy takes into account these rebounds. The existence of rebounds explains why a "zero-Covid" policy, whatever its duration, is unsustainable as it eventually leads to large rebounds with a very low terminal susceptible proportion as well as huge economic costs. Yet, even with rebounds, any policy stance leads to a higher collective immunity relative to a non-interventionist position and thus a weak improvement. The policymaker may choose not to intervene by means of imposing some social distancing measure if the relative value of life is sufficiently low. When the parameter capturing this relative value in the welfare function characterizing the policymaker is above a certain threshold, the magnitude of the lockdown depends positively on this parameter: the chosen reproduction number is a decreasing function of this parameter. This is true in both perspectives. The optimal long-term optimal reproduction number taking into account the possibility of rebounds mitigates the strictness of lockdown in order to avoid these rebounds or limit their amplitude. If the duration period is sufficiently long, the optimal reproduction number is large enough so as to avoid rebounds. It may therefore be that the long-term optimal lockdown is more lenient than the short-term optimal one, for a given lockdown duration.

In the same vein, it is not true that the shortest duration combined with a strict lockdown measure consistent with reaching a given herd immunity target is optimal. A more lenient policy (a chosen higher reproduction number) enforced over a longer duration period dominates such a policy as it rules out rebounds in the course of the pandemic, eventually reaching the herd immunity target and generates a lower total economic cost. Living with the pandemic may mean enduring a milder lockdown over a longer period. There exists an optimal pandemic policy, that is, a pair of reproduction number and duration. It leads to a herd immunity level which is not the highest possible one. This is due to the public necessity to trade-off the economic losses and the mortality gains attached to a pandemic policy. On the whole, this paper makes clear the dire consequences of policy errors made policymakers either because they are myopic or willing to eradicate a pandemic once it has been identified as such.

There exist many different instruments to tackle an expanding pandemic, in particular tracking, testing, appropriate individual equipements as masks and finally, isolation. A mix of measures is likely to be what defines an adequate policy toward the control of a pandemic. Any such measures are likely to have opposite health (positive) and economic (negative, if only because of direct costs) impacts and therefore meet our assumptions. Our policy instrument can thus be understood as a “composite” public health instrument (a combination of measures) for tackling a pandemic.

The model assumes that the population is homogeneous. It does not take into account the reaction of the population to the lockdown decision and assumes a simple framing of economic and sanitary losses. As it is, it proves an adequate basis for understanding the basic policy issues related to the control of a pandemic, in particular in relation with economic consequences of a lockdown policy when stylized laws of the dissemination of a pandemic are explicitly taken into accunt. The model can be complexified so as to take into consideration different assumptions. Relaxing these assumptions as well as analyzing lockdown policy in variants of the SIR model which have been offered in the epidemiology literature is left for further research.[16]

Finally, the model rules out uncertainty. Epidemiologists have developed a stochastic approach to capture the randomness in the matching process between infected and susceptible people. This is when the number of infected is very low and the law of large numbers does not apply (see Britton (2010)). We do not claim that the results obtained here (in particular the self-defeating nature of an extremely severe policy) are transferable to a stochastic approach. The strength of a deterministic approach is to obtain analytical results which clarify the impact of a lockdown policy, without resorting to simulation techniques. It appears that these results are useful to understand the pitfalls of a pandemic policy, in particular the adverse consequence of a short-sighted lockdown policy.

A Appendix

A.1 Proof of Lemma 1

We know that:

$\frac{dr}{ds} = \frac{\gamma i}{-\beta_0 is} = -\frac{1}{\mathcal{R}_0 s}$, i.e., $\frac{ds}{\mathcal{R}_0 s} = -dr$

$\forall t \in \mathbb{R}$, $\int_{s(-\infty)}^{s(t)} \frac{ds}{\mathcal{R}_0 s} = -\int_{r(-\infty)}^{r(t)} dr$, thus $\frac{1}{\mathcal{R}_0}\left(\ln s(t) - \ln s(-\infty)\right) = -\left(r(t) - r(-\infty)\right)$

i.e., $\forall t \in \mathbb{R}$, $\frac{1}{\mathcal{R}_0}\ln s(t) + r(t) = \frac{1}{\mathcal{R}_0}\ln s(-\infty) + r(-\infty)$.

The boundary conditions $s(-\infty) = 1$, $r(-\infty) = 0$ give $\frac{1}{\mathcal{R}_0}\ln s(t) + r(t) = 0$, $\forall t \in \mathbb{R}$, which gives (8).

Similarly, we know that

$\frac{di}{ds} = \frac{\beta_0 is - \gamma i}{-\beta_0 is} = -1 + \frac{1}{\mathcal{R}_0 s}$, i.e., $di = \left(-1 + \frac{1}{\mathcal{R}_0 s}\right) ds$

$\forall t \in \mathbb{R}$, $\int_{i(-\infty)}^{i(t)} di = \int_{s(-\infty)}^{s(t)} \left(-1 + \frac{1}{\mathcal{R}_0 s}\right) ds$, thus $i(t) - i(-\infty) = s(-\infty) - s(t) +$

[16] On the need to combine epidemiology and economics, Murray (2020, p.106) writes: "As an epidemiologist, I ask economists interested in Covid-19 to build on their expertise and ours. Indeed, the efforts of economists in tackling the economic sequelae of this pandemic are vitally needed, as are the development of tools for tracking, predicting, and preventing future pandemics based on understanding the flow of people, goods, and other economic activity around the globe."

$\frac{1}{\mathcal{R}_0}(\ln s(t) - \ln s(-\infty))$, i.e., $i(t) + s(t) - \frac{1}{\mathcal{R}_0}\ln s(t) = i(-\infty) + s(-\infty) - \frac{1}{\mathcal{R}_0}\ln s(-\infty)$. The boundary conditions $s(-\infty) = 1$, $i(-\infty) = 0$ give $i(t) + s(t) - \frac{1}{\mathcal{R}_0}\ln s(t) = 1$, $\forall t \in \mathbb{R}$, which gives (9).

$\frac{ds}{dt} = -\beta_0 i(t)s(t) = -\beta_0 s(t)\left[1 - s(t) + \frac{1}{\mathcal{R}_0}\ln s(t)\right]$

thus $\frac{ds}{-\beta_0 s\left[1-s+\frac{1}{\mathcal{R}_0}\ln s\right]} = dt$

i.e., $t = \int_{s(0)}^{s(t)} \frac{ds}{-\beta_0 s\left[1-s+\frac{1}{\mathcal{R}_0}\ln s\right]} = \int_{s(t)}^{s(0)} \frac{ds}{\beta_0 s\left[1-s+\frac{1}{\mathcal{R}_0}\ln s\right]}$.

$t \mapsto s(t)$ is a decreasing function since $\frac{ds}{dt} = -\beta_0 is < 0$

$t \mapsto r(t)$ is an increasing function since $\frac{dr}{dt} = \gamma i > 0$,

so we have proven (i) of Lemma 1.

From $i(t) = 1 - s(t) + \frac{1}{\mathcal{R}_0}\ln(s(t))$, we get:

$i'(t) = -s'(t) + \frac{s'(t)}{s(t)\mathcal{R}_0} = -s'(t)\left[1 - \frac{1}{s(t)\mathcal{R}_0}\right]$ which is of the sign of $1 - \frac{1}{s(t)\mathcal{R}_0}$ since $s'(t) < 0$. Then $i(t)$ is increasing for $s(t) \geq \frac{1}{\mathcal{R}_0}$ (i.e. for t small), and $i(t)$ is decreasing for $s(t) \leq \frac{1}{\mathcal{R}_0}$ (i.e. for t larger).

Thus $i(t)$ is maximal when $s(t) = \frac{1}{\mathcal{R}_0}$, which gives

$i_{\max} = 1 - \frac{1}{\mathcal{R}_0} + \frac{1}{\mathcal{R}_0}\ln(\frac{1}{\mathcal{R}_0}) = 1 - \frac{1}{\mathcal{R}_0}[1 + \ln(\mathcal{R}_0)]$, which proves (ii).

At the end of the epidemic, $i = 0$ thus $1 - s + \frac{1}{\mathcal{R}_0}\ln s = 0$, i.e., $\frac{\ln(s)}{\mathcal{R}_0} = s - 1$ then $\mathcal{R}_0 = \frac{\ln(s)}{s-1}$. □

A.2 Proof of Lemma 2

Lemma 1 gives the dynamics for $t < t_0$.

For $t \in [t_0, T]$,

$\frac{dr}{ds} = \frac{\gamma i}{-\beta is} = -\frac{1}{\mathcal{R}s}$, i.e., $\frac{ds}{\mathcal{R}s} = -dr$

$\forall t \in [t_0, T]$, $\int_{s(t_0)}^{s(t)} \frac{ds}{\mathcal{R}s} = -\int_{r(t_0)}^{r(t)} dr$, i.e., $\frac{1}{\mathcal{R}}(\ln s(t) - \ln s(t_0)) = r(t_0) - r(t)$

$\frac{di}{ds} = \frac{\beta is - \gamma i}{-\beta is} = -1 + \frac{1}{\mathcal{R}s}$, i.e., $di = \left(-1 + \frac{1}{\mathcal{R}s}\right)ds$

$\forall t \in [t_0, T]$, $\int_{i(t_0)}^{i(t)} di = \int_{s(t_0)}^{s(t)} \left(-1 + \frac{1}{\mathcal{R}s}\right)ds$

thus $i(t) - i(t_0) = -(s(t) - s(t_0)) + \frac{1}{\mathcal{R}}(\ln(s(t) - \ln(s(t_0)))$.

$\frac{ds}{dt} = -\beta i(t)s(t) = -\beta s(t)\left[i(t_0) - (s(t) - s(t_0)) + \frac{1}{\mathcal{R}}(\ln s(t) - \ln s(t_0))\right]$

thus $\frac{ds}{-\beta s\left[i(t_0)-(s-s(t_0))+\frac{1}{\mathcal{R}}(\ln s - \ln s(t_0))\right]} = dt$

i.e., $\int_{s(t_0)}^{s(t)} \frac{ds}{-\beta s\left[i(t_0)-(s-s(t_0))+\frac{1}{\mathcal{R}}(\ln s - \ln s(t_0))\right]} = t - t_0$

To sum up, for $t \in [t_0, T]$:

$r(t) = r(t_0) + \frac{1}{\mathcal{R}}\ln s(t_0) - \frac{1}{\mathcal{R}}\ln s(t)$

$i(t) = -s(t) + \frac{1}{\mathcal{R}}\ln s(t) + i(t_0) + s(t_0) - \frac{1}{\mathcal{R}}\ln s(t_0)$

$\int_{s(t)}^{s(t_0)} \frac{ds}{\beta s\left[i(t_0)+s(t_0)-\frac{1}{\mathcal{R}}\ln s(t_0) - s + \frac{1}{\mathcal{R}}\ln s\right]} = t - t_0$

This proves Eq. (15), (16) and (17). Moreover, $t \mapsto s(t)$ is a decreasing function and $t \mapsto r(t)$ is an increasing function according to Eq. (12) and Eq. (14). This gives the second part of Lemma 2 (i).

It is assumed that $s(t_0) > \frac{1}{\mathcal{R}_0}$, thus the maximum of $i(t)$ for $t \in]-\infty; t_0]$ is attained at t_0, thus $\max_{t\in]-\infty;T]} i(t) = \max_{t\in[t_0,T]} i(t)$.

Moreover, on $t \in [t_0, T]$, $i'(t) = -s'(t) + \frac{s'(t)}{\mathcal{R}s(t)} = -s'(t)\left[1 - \frac{1}{\mathcal{R}s(t)}\right]$ with $s'(t) < 0$, thus the sign of $i'(t)$ is that of $1 - \frac{1}{\mathcal{R}s(t)}$ on $t \in [t_0, T]$.

- If $s(t_0) < \frac{1}{\mathcal{R}}$, then $s(t) < \frac{1}{\mathcal{R}}$ $\forall t \in [t_0, T]$, thus $i'(t) < 0$ $\forall t \in [t_0, T]$. The maximum of $i(t)$ is here attained at $t = t_0$.

- If $s(t_0) \geq \frac{1}{\mathcal{R}}$, then since $s(T) < \frac{1}{\mathcal{R}}$, there exists $t \in [t_0, T]$, such that $s(t) = \frac{1}{\mathcal{R}}$. This value of t gives the maximum of $i(t)$.

This gives Lemma 2 (ii).

Now, we prove that if $\mathcal{R}' > \mathcal{R}$, the curve $(s_{\mathcal{R}'}(t), i_{\mathcal{R}'}(t))_{t\geq t_0}$ is strictly above $(s_{\mathcal{R}}(t), i_{\mathcal{R}}(t))_{t\geq t_0}$ in the plane (s, i), except a unique common point at $(s(t_0), i(t_0))$.

We need to study i as a function of s. Since

$i_{\mathcal{R}}(t) = i(t_0) + s(t_0) - s_{\mathcal{R}}(t) + \frac{1}{\mathcal{R}}\ln(s_{\mathcal{R}}(t)/s(t_0))$ according to Eq. (16).

We set:

$$\mathcal{I}_{\mathcal{R}}(s) = i(t_0) + s(t_0) - s + \frac{1}{\mathcal{R}}\ln(s/s(t_0)). \tag{26}$$

It is clear that $s = s_{\mathcal{R}}(t) \Rightarrow \mathcal{I}_{\mathcal{R}}(s) = i_{\mathcal{R}}(t)$. It means that in the plane (s, i), the curve $(s_{\mathcal{R}}(t), i_{\mathcal{R}}(t))_{t\geq t_0}$ is the curve $(s, \mathcal{I}_{\mathcal{R}}(s))_{s\leq s(t_0)}$.

We just have to prove that $(s, \mathcal{I}_{\mathcal{R}'}(s))_{s\leq s(t_0)}$ is strictly above $(s, \mathcal{I}_{\mathcal{R}}(s))_{s\leq s(t_0)}$ if $\mathcal{R}' > \mathcal{R}$, except at $s = s(t_0)$.

$\mathcal{I}_{\mathcal{R}'}(s) - \mathcal{I}_{\mathcal{R}}(s) = [i(t_0) + s(t_0) - s + \frac{1}{\mathcal{R}'}\ln(s/s(t_0))] - [i(t_0) + s(t_0) - s + \frac{1}{\mathcal{R}}\ln(s/s(t_0))]$
$= \frac{1}{\mathcal{R}'}\ln(s/s(t_0)) - \frac{1}{\mathcal{R}}\ln(s/s(t_0)) = (\frac{1}{\mathcal{R}} - \frac{1}{\mathcal{R}'})\ln(s(t_0)/s) > 0$ if $s < s(t_0)$, since $\frac{1}{\mathcal{R}} > \frac{1}{\mathcal{R}'}$.

Now we prove that $s_{\mathcal{R}'}(t) < s_{\mathcal{R}}(t)$ if $\mathcal{R}' > \mathcal{R}$.

According to Eq. (17), $\gamma(t - t_0) = \int_{s_{\mathcal{R}}(t)}^{s(t_0)} \frac{ds}{\mathcal{R} s \mathcal{I}_{\mathcal{R}}(s)}$ and $\gamma(t - t_0) = \int_{s_{\mathcal{R}'}(t)}^{s(t_0)} \frac{ds}{\mathcal{R}' s \mathcal{I}_{\mathcal{R}'}(s)}$

$\forall s < s(t_0)$, $\mathcal{I}_{\mathcal{R}'}(s) > \mathcal{I}_{\mathcal{R}}(s)$, thus $\forall s < s(t_0)$, $\mathcal{R}'\mathcal{I}_{\mathcal{R}'}(s) > \mathcal{R}\mathcal{I}_{\mathcal{R}}(s)$,

$\forall s < s(t_0)$, $\frac{1}{\mathcal{R}'\mathcal{I}_{\mathcal{R}'}(s)} < \frac{1}{\mathcal{R}\mathcal{I}_{\mathcal{R}}(s)}$ with $\int_{s_{\mathcal{R}}(t)}^{s(t_0)} \frac{ds}{\mathcal{R} s \mathcal{I}_{\mathcal{R}}(s)} = \int_{s_{\mathcal{R}'}(t)}^{s(t_0)} \frac{ds}{\mathcal{R}' s \mathcal{I}_{\mathcal{R}'}(s)}$

thus the interval $[s_{\mathcal{R}'}(t), s(t_0)]$ must be larger than $[s_{\mathcal{R}}(t), s(t_0)]$, i.e. $s_{\mathcal{R}'}(t) < s_{\mathcal{R}}(t)$.

This gives Lemma 2 (iii). □

A.3 Proof of Proposition 1

V_T is a continuous real function on the compact interval $[0; \mathcal{R}_0]$, thus it has a global maximum on this interval, attained at $\mathcal{R}^{opt}$.

$V_T(\mathcal{R}) = (T - t_0)y(\mathcal{R}) - \lambda N \delta(1 - s_{\mathcal{R}}(T))$ and $V_T'(\mathcal{R}) = (T - t_0)y'(\mathcal{R}) + \lambda N \delta \frac{\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}}$ (where $y' > 0$ and $\frac{\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}} < 0$ according to Lemma 2 (iii)).

We can distinguish 3 cases:

- First corner solution: $\mathcal{R}^{opt} = 0$ which implies $V_T'(0) \leq 0$. But $V_T'(0) \leq 0$ is not possible since we have assumed that $y'(0) = +\infty$. Thus $V_T'(0) > 0$ and $\mathcal{R}^{opt} > 0$ in the sequel.

- Second corner solution: $\mathcal{R}^{opt} = \mathcal{R}_0$ which implies $V_T'(\mathcal{R}_0) \geq 0$, which means $(T - t_0)y'(\mathcal{R}_0) \geq -\lambda \delta N \left(\frac{\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}}\right)_{\mathcal{R}=\mathcal{R}_0}$, i.e. $\lambda \leq \widehat{\lambda}_0$, setting $\widehat{\lambda}_0 = \frac{(T-t_0)y'(\mathcal{R}_0)}{-\delta N \left(\frac{\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}}\right)_{\mathcal{R}=\mathcal{R}_0}} \geq 0$.

- Interior solution: $0 < \mathcal{R}^{opt} < \mathcal{R}_0$. Here $\mathcal{R}^{opt}$ satisfies $V_T'(\mathcal{R}^{opt}) = 0$.

Since corner solution implies $\lambda \leq \widehat{\lambda}_0$, then $\lambda > \widehat{\lambda}_0$ implies interior solution.

In the interior solution case, applying the implicit function theorem at the curve $V_T'(\mathcal{R}^{opt}) = 0$ in the plane $(\lambda, \mathcal{R}^{opt})$:

$\frac{d\mathcal{R}^{opt}}{d\lambda} = -\frac{\frac{\partial V_T'}{\partial \lambda}}{\frac{\partial V_T'}{\partial \mathcal{R}}} = -\frac{\delta N \frac{\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}}}{V_T\text{"}(\mathcal{R}^{opt})} \leq 0$ since $V_T\text{"}(\mathcal{R}^{opt}) \leq 0$ (as V_T attains its maximum at $\mathcal{R}^{opt}$) and $\frac{\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}} < 0$.

We have found that $\lambda \mapsto \mathcal{R}^{opt}(\lambda)$ is a non increasing function for $\mathcal{R}^{opt}(\lambda) < \mathcal{R}_0$, with $\mathcal{R}^{opt}(0) = \mathcal{R}_0$ and $\mathcal{R}^{opt}(\lambda) < \mathcal{R}_0$ for $\lambda > \widehat{\lambda}_0$, thus there exists $\lambda_0 \in [0; \widehat{\lambda}_0]$ such that $\mathcal{R}^{opt}(\lambda) = \mathcal{R}_0 \Leftrightarrow \lambda \in [0; \lambda_0]$.

This gives Prop 1 (i) and (ii), except $\lim_{\lambda \to +\infty} \mathcal{R}^{opt} = 0$ proven below.

Now let us prove that $\lim_{\lambda \to +\infty} \mathcal{R}^{opt} = 0$.

According to Eq. (17), $\gamma(T - t_0) = \int_{s_{\mathcal{R}}(T)}^{s(t_0)} \frac{ds}{\mathcal{R} s \mathcal{I}_{\mathcal{R}}(s)} = \int_{s_{\mathcal{R}'}(T)}^{s(t_0)} \frac{ds}{\mathcal{R}' s \mathcal{I}_{\mathcal{R}'}(s)}$.

If $\mathcal{R} < \mathcal{R}'$, $s_{\mathcal{R}'}(T) < s_{\mathcal{R}}(T)$ and $\mathcal{I}_{\mathcal{R}'}(s) \geq \mathcal{I}_{\mathcal{R}}(s)$ for all $s < s(t_0)$.

$\int_{s_{\mathcal{R}'}(T)}^{s_{\mathcal{R}}(T)} \frac{ds}{\mathcal{R}' s \mathcal{I}_{\mathcal{R}}(s)} + \int_{s_{\mathcal{R}}(T)}^{s(t_0)} \frac{ds}{\mathcal{R}' s \mathcal{I}_{\mathcal{R}}(s)} = \int_{s_{\mathcal{R}'}(T)}^{s(t_0)} \frac{ds}{\mathcal{R}' s \mathcal{I}_{\mathcal{R}}(s)} \geq \int_{s_{\mathcal{R}'}(T)}^{s(t_0)} \frac{ds}{\mathcal{R}' s \mathcal{I}_{\mathcal{R}'}(s)} = \gamma(T - t_0)$

thus

$\int_{s_{\mathcal{R}'}(T)}^{s_{\mathcal{R}}(T)} \frac{ds}{\mathcal{R}' s \mathcal{I}_{\mathcal{R}}(s)} \geq \gamma(T - t_0) - \int_{s_{\mathcal{R}}(T)}^{s(t_0)} \frac{ds}{\mathcal{R}' s \mathcal{I}_{\mathcal{R}}(s)} = \gamma(T - t_0) - \frac{\mathcal{R}}{\mathcal{R}'} \int_{s_{\mathcal{R}}(T)}^{s(t_0)} \frac{ds}{\mathcal{R} s \mathcal{I}_{\mathcal{R}}(s)}$

$= \gamma(T - t_0) - \frac{\mathcal{R}}{\mathcal{R}'} \gamma(T - t_0)$

i.e.

$\frac{1}{\mathcal{R}'} \int_{s_{\mathcal{R}'}(T)}^{s_{\mathcal{R}}(T)} \frac{ds}{s \mathcal{I}_{\mathcal{R}}(s)} \geq \gamma(T - t_0) \left[1 - \frac{\mathcal{R}}{\mathcal{R}'}\right]$

$\int_{s_{\mathcal{R}'}(T)}^{s_{\mathcal{R}}(T)} \frac{ds}{s \mathcal{I}_{\mathcal{R}}(s)} \geq \gamma(T - t_0)(\mathcal{R}' - \mathcal{R})$

$\frac{1}{(\mathcal{R}' - \mathcal{R})} \int_{s_{\mathcal{R}'}(T)}^{s_{\mathcal{R}}(T)} \frac{ds}{s \mathcal{I}_{\mathcal{R}}(s)} \geq \gamma(T - t_0)$

with $\mathcal{R}' \to \mathcal{R}$, we have

$\lim_{\mathcal{R}' \to \mathcal{R}^+} \frac{1}{(\mathcal{R}' - \mathcal{R})} \int_{s_{\mathcal{R}'}(T)}^{s_{\mathcal{R}}(T)} \frac{ds}{s \mathcal{I}_{\mathcal{R}}(s)} = \frac{1}{s_{\mathcal{R}}(T) \mathcal{I}_{\mathcal{R}}(s_{\mathcal{R}}(T))} \times \lim_{\mathcal{R}' \to \mathcal{R}^+} \frac{s_{\mathcal{R}}(T) - s_{\mathcal{R}'}(T)}{(\mathcal{R}' - \mathcal{R})}$

$= \frac{1}{s_{\mathcal{R}}(T) i_{\mathcal{R}}(T)} \times \frac{-\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}}$

thus

$$\frac{-\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}} \geq \gamma(T - t_0) s_{\mathcal{R}}(T)\, i_{\mathcal{R}}(T). \tag{27}$$

Now we go back to the proof of $\lim_{\lambda \to +\infty} \mathcal{R}^{opt} = 0$.

In the case of an interior solution $V_T'(\mathcal{R}^{opt}) = 0$, i.e.

$(T - t_0) y'(\mathcal{R}^{opt}) + \lambda N \delta \frac{\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}} = 0$

$(T - t_0) y'(\mathcal{R}^{opt}) = -\lambda N \delta \frac{\partial s_{\mathcal{R}}(T)}{\partial \mathcal{R}} \geq \lambda N \delta \gamma (T - t_0) s_{\mathcal{R}}(T)\, i_{\mathcal{R}}(T)$ according to (27).

$y'(\mathcal{R}^{opt}) \geq \lambda N \delta \gamma s_{\mathcal{R}}(T)\, i_{\mathcal{R}}(T)$

Since $T < +\infty$, and $\mathcal{R} \mapsto s_{\mathcal{R}}(T)\, i_{\mathcal{R}}(T)$ is a positive and continous function on the compact set $[0; \mathcal{R}_0]$, then there exists $\eta > 0$ such that:

$s_{\mathcal{R}}(T)\, i_{\mathcal{R}}(T) \geq \eta > 0$ for all $\mathcal{R}$ (T given).

$y'(\mathcal{R}^{opt}) \geq \lambda N \delta \gamma \eta$

When $\lambda \to +\infty$, then $y'(\mathcal{R}^{opt}) \to +\infty$, i.e. $\mathcal{R}^{opt} \to 0$. $\square$

A.4 Proof of Proposition 2

(i) According to (22), for $t \geq T$, we have

$$i(t) = i(T) + s(T) - s(t) + \frac{1}{\mathcal{R}_0}\ln(s(t)) - \frac{1}{\mathcal{R}_0}\ln(s(T))$$

thus $i'(t) = -s'(t) + \frac{s'(t)}{\mathcal{R}_0 s(t)} = -s'(t)\left[1 - \frac{1}{\mathcal{R}_0 s(t)}\right]$, and since $s'(t) < 0$, thus $i'(t)$ is of the sign of $1 - \frac{1}{\mathcal{R}_0 s(t)}$.

There is a rebound after T if and only if $i'(T) > 0$, i.e. if $1 > \frac{1}{\mathcal{R}_0 s(T)}$, which means that there is a rebound after T if and only if $s(T) > \frac{1}{\mathcal{R}_0}$.

If there is a rebound, $i(t)$ is maximal on $t \geq T$ when $i'(t) = 0$, i.e. for $s(t) = \frac{1}{\mathcal{R}_0}$.

If there is no rebound, $i(t)$ is a decreasing function on $t \geq T$, thus $i(t)$ is maximal on $t \geq T$ when $t = T$.

(ii) There is a rebound after $T \Longleftrightarrow s(T) > \frac{1}{\mathcal{R}_0}$.

We must study under which conditions on $\mathcal{R}$ and T do we have $s(T) > \frac{1}{\mathcal{R}_0}$.

$s(T)$ is a decreasing function of T; we set $\widetilde{s}_\infty = \lim_{T\to+\infty} s(T)$, and $\widetilde{s}_\infty$ is the value of $s(t)$ such that $i(t) = 0$ in (16).

Thus $i(t_0) + s(t_0) - \widetilde{s}_\infty + \frac{1}{\mathcal{R}}\ln(\widetilde{s}_\infty) - \frac{1}{\mathcal{R}}\ln(s(t_0)) = 0$ and

$$\mathcal{R} = \frac{\ln(s(t_0)) - \ln(\widetilde{s}_\infty)}{i(t_0) + s(t_0) - \widetilde{s}_\infty}.$$

We claim that $\widetilde{s}_\infty$ is a decreasing function of $\mathcal{R}$.

Indeed, $\frac{d\mathcal{R}}{d\widetilde{s}_\infty} = \frac{-\frac{1}{\widetilde{s}_\infty}\left(i(t_0)+s(t_0)-\widetilde{s}_\infty\right) + \left(\ln(s(t_0)) - \ln(\widetilde{s}_\infty)\right)}{\left(i(t_0)+s(t_0)-\widetilde{s}_\infty\right)^2} = \frac{\left(\mathcal{R} - \frac{1}{\widetilde{s}_\infty}\right)\left(i(t_0)+s(t_0)-\widetilde{s}_\infty\right)}{\left(i(t_0)+s(t_0)-\widetilde{s}_\infty\right)^2} < 0$, since $\widetilde{s}_\infty \leq s(T) < \frac{1}{\mathcal{R}}$.

- We have: $s(T) > \frac{1}{\mathcal{R}_0}$ for all $T \Longleftrightarrow \widetilde{s}_\infty \geq \frac{1}{\mathcal{R}_0}$,

i.e.

$$s(T) > \frac{1}{\mathcal{R}_0} \text{ for all } T \Longleftrightarrow \mathcal{R} \leq \widetilde{\mathcal{R}}_{t_0}, \text{ setting } \widetilde{\mathcal{R}}_{t_0} = \frac{\ln(s(t_0)) - \ln\left(\frac{1}{\mathcal{R}_0}\right)}{i(t_0) + s(t_0) - \frac{1}{\mathcal{R}_0}}.$$

This proves the first point of Proposition 2(ii).

- If $\widetilde{s}_\infty < \frac{1}{\mathcal{R}_0}$, i.e. if $\mathcal{R} > \widetilde{\mathcal{R}}_{t_0}$, then since $s(T)$ is a decreasing function of T, there exists a value $T_{\min}$ such that $s(T) \leq \frac{1}{\mathcal{R}_0}$ (i.e. no rebound) for $T \geq T_{\min}$, and $s(T) > \frac{1}{\mathcal{R}_0}$ (i.e. rebound) for $T < T_{\min}$.

A.5 Proof of Proposition 3

(i) s_∞ is the value of $s(t)$ such that $i(t) = 0$ in (22).

Thus $i(T) + s(T) - s_\infty + \frac{1}{\mathcal{R}_0}\ln(s_\infty) - \frac{1}{\mathcal{R}_0}\ln(s(T)) = 0$

which gives $\mathcal{R}_0 = \frac{\ln(s(T))-\ln(s_\infty)}{i(T)+s(T)-s_\infty}$.

At the end of the pandemic we have of course $i = i_\infty = 0$, thus $r = r_\infty = 1 - s_\infty$.

- Let us show that $s_\infty = s_\infty(\mathcal{R}, T) < \frac{1}{\mathcal{R}_0}$.

If $s(T) \leq \frac{1}{\mathcal{R}_0}$, then $s_\infty(\mathcal{R}) < s(T) \leq \frac{1}{\mathcal{R}_0}$.

If $s(T) > \frac{1}{\mathcal{R}_0}$, there is a rebound after T with an epidemic peak when $s(t) = \frac{1}{\mathcal{R}_0}$, thus $s_\infty(\mathcal{R}) < s(t) = \frac{1}{\mathcal{R}_0}$.

In both cases we have $s_\infty(\mathcal{R}) < \frac{1}{\mathcal{R}_0}$.

- Let us show that $s_\infty(\mathcal{R}, T)$ is an increasing function of T (for a given $\mathcal{R}_0$).

As $\mathcal{R}_0 = \frac{\ln(s(T))-\ln(s_\infty)}{i(T)+s(T)-s_\infty} = \frac{\ln(s(T))-\ln(s_\infty)}{1-r(T)-s_\infty}$, we get: $\mathcal{R}_0\left(1 - r(T) - s_\infty\right) = \ln(s(T)) - \ln(s_\infty)$.

Derivating with respect to T :

$$\mathcal{R}_0\left(-\dot{r}(T) - \dot{s}_\infty\right) = \frac{\dot{s}(T)}{s(T)} - \frac{\dot{s}_\infty}{s_\infty}.$$

As $\dot{r}(T) = \frac{dr}{dt}(T) = \gamma i(T)$ and $\dot{s}(T) = \frac{ds}{dt}(T) = -\beta i(T)s(T)$, we get:

$$-\mathcal{R}_0\left(\gamma i(T) + \dot{s}_\infty\right) = -\beta i(T) - \frac{\dot{s}_\infty}{s_\infty}.$$

Thus $\dot{s}_\infty\left(\frac{1}{s_\infty} - \mathcal{R}_0\right) = (\beta_0 - \beta)i(T)$, we write

$\dot{s}_\infty = \frac{(\beta_0-\beta)i(T)}{\frac{1}{s_\infty}-\mathcal{R}_0} > 0$ which is positive since the numerator and the denominator are both positive.

- Let us show that $s_\infty(\mathcal{R})$ is a decreasing function of $\mathcal{R}_0$ (for a given T).

From (24) we get $\frac{d\mathcal{R}_0}{ds_\infty} = \frac{-\frac{1}{s_\infty}(i(T)+s(T)-s_\infty)+\ln(s(T))-\ln(s_\infty)}{(i(T)+s(T)-s_\infty)^2} = \frac{\left(\mathcal{R}_0-\frac{1}{s_\infty}\right)(i(T)+s(T)-s_\infty)}{(i(T)+s(T)-s_\infty)^2} < 0$

since $s_\infty < \frac{1}{\mathcal{R}_0}$. This proves Prop 3(i).

(ii) The representative curve of the function $s \mapsto \mathcal{I}_{\mathcal{R}_0}(s)$ is above the one representing $s \mapsto \mathcal{I}_{\mathcal{R}}(s)$ from Lemma 2(iii), thus $s_\infty(\mathcal{R}) \geq s_\infty(\mathcal{R}_0)$.

- Let us show that $s_\infty(\mathcal{R})$ is a non-monotonic function of $\mathcal{R}$ if T is high.

$\mathcal{R}_0 = \frac{\ln(s(T))-\ln(s_\infty)}{i(T)+s(T)-s_\infty}$, thus $\mathcal{R}_0\left(i(T) + s(T) - s_\infty\right) = \ln(s(T)) - \ln(s_\infty)$

i.e., $i(T) + s(T) - s_\infty = \frac{1}{\mathcal{R}_0}\ln(s(T)) - \frac{1}{\mathcal{R}_0}\ln(s_\infty)$

$$-s_\infty + \frac{1}{\mathcal{R}_0}\ln(s_\infty) = -i(T) - s(T) + \frac{1}{\mathcal{R}_0}\ln(s(T))$$

and since $i(T) + s(T) = 1 - r(T)$:

$$1 - s_\infty + \frac{1}{\mathcal{R}_0}\ln(s_\infty) = r(T) + \frac{1}{\mathcal{R}_0}\ln(s(T)).$$

If T is sufficiently high, then $s(T) \simeq \widetilde{s}_\infty$ (since $\widetilde{s}_\infty = \lim_{T\to\infty} s(T)$), and $r(T) \simeq 1-\widetilde{s}_\infty$. This leads to:

$$1 - s_\infty + \frac{1}{\mathcal{R}_0}\ln(s_\infty) \simeq 1 - \widetilde{s}_\infty + \frac{1}{\mathcal{R}_0}\ln(\widetilde{s}_\infty)$$

i.e.

$$h_{\mathcal{R}_0}(s_\infty) \simeq h_{\mathcal{R}_0}(\widetilde{s}_\infty)$$

where $h_{\mathcal{R}_0}$ is defined by $h_{\mathcal{R}_0}(s) = 1 - s + \frac{1}{\mathcal{R}_0}\ln(s)$.

According to the proof of Prop 2 (ii), $\widetilde{s}_\infty$ is a decreasing function of $\mathcal{R}$ on $\mathcal{R} \in (0;\mathcal{R}_0]$, with $\widetilde{s}_\infty \geq \frac{1}{\mathcal{R}_0}$ if $\mathcal{R} \leq \widetilde{\mathcal{R}}_{t_0}$, and $\widetilde{s}_\infty \leq \frac{1}{\mathcal{R}_0}$ if $\mathcal{R} \geq \widetilde{\mathcal{R}}_{t_0}$.

For a given s_∞, $s_\infty < \frac{1}{\mathcal{R}_0}$, the equation $h_{\mathcal{R}_0}(s_\infty) = h_{\mathcal{R}_0}(\widetilde{s}_\infty)$ of unknown $\widetilde{s}_\infty$ has two different roots: the first one is higher than $\frac{1}{\mathcal{R}_0}$, the second one is lower than $\frac{1}{\mathcal{R}_0}$.

For $\mathcal{R} \in [\widetilde{\mathcal{R}}_{t_0}, \mathcal{R}_0]$, we have $s_\infty = \widetilde{s}_\infty \leq \frac{1}{\mathcal{R}_0}$ and s_∞ is a decreasing function of $\mathcal{R}$.

For $\mathcal{R} \in (0, \widetilde{\mathcal{R}}_{t_0})$, we have $s_\infty < \frac{1}{\mathcal{R}_0} < \widetilde{s}_\infty$ and s_∞ is here an increasing function of $\mathcal{R}$.

Summing up, for T high enough, s_∞ is a non-monotonic function of $\mathcal{R}$, it is first increasing, then decreasing.

A.6 Proof of Proposition 4

(i) $L_\infty(\mathcal{R}) = (T-t_0)\,[y(\mathcal{R}_0) - y(\mathcal{R})] + \lambda M_\infty(\mathcal{R})$.

L_∞ is a continuous function on the compact interval $[0;\mathcal{R}_0]$, thus it has a global minimum on this interval, attained at $\mathcal{R}_\infty^{opt}$.

$L'_\infty(\mathcal{R}) = -(T-t_0)y'(\mathcal{R}) - \lambda N\delta s'_\infty(\mathcal{R})$ with $y' > 0$.

We can distinguish 3 cases:

- First corner solution:

$\mathcal{R}_\infty^{opt} = 0$ which implies $L'_\infty(0) \geq 0$. But $L'_\infty(0) \geq 0$ is not possible since $y'(0) = +\infty$ by assumption. Thus $L'_\infty(0) < 0$ and $\mathcal{R}_\infty^{opt} > 0$ in the sequel.

- Second corner solution:

$\mathcal{R}_\infty^{opt} = \mathcal{R}_0$ which implies $L'_\infty(\mathcal{R}_0) \leq 0$ which means $-\lambda\delta N s'_\infty(\mathcal{R}_0) \leq (T-t_0)y'(\mathcal{R}_0)$, i.e. $\lambda \leq \widehat{\lambda}'_0$, setting $\widehat{\lambda}'_0 = \frac{(T-t_0)y'(\mathcal{R}_0)}{-\delta N s'_\infty(\mathcal{R}_0)} \geq 0$.

- Interior solution: $0 < \mathcal{R}_\infty^{opt} < \mathcal{R}_0$. Here $\mathcal{R}_\infty^{opt}$ satisfies $L'_\infty(\mathcal{R}_\infty^{opt}) = 0$.

$L'_\infty(\mathcal{R}_\infty^{opt}) = 0$ then $\lambda N\delta s'_\infty(\mathcal{R}_\infty^{opt}) = -(T-t_0)y'(\mathcal{R}_\infty^{opt}) < 0$.

Since corner solution implies $\lambda \leq \widehat{\lambda}'_0$, then $\lambda > \widehat{\lambda}'_0$ implies interior solution.

In the interior solution case, applying the implicit function theorem at the curve $L'_\infty(\mathcal{R}_\infty^{opt}) = 0$ in the plane $(\lambda, \mathcal{R}_\infty^{opt})$:

$\frac{d\mathcal{R}_\infty^{opt}}{d\lambda} = -\frac{\frac{\partial L'_\infty}{\partial \lambda}}{\frac{\partial L'_\infty}{\partial \mathcal{R}}(\mathcal{R}_\infty^{opt})} = \frac{\delta N s'_\infty(\mathcal{R}_\infty^{opt})}{L_\infty"(\mathcal{R}_\infty^{opt})} \leq 0$ since $L_\infty"(\mathcal{R}_\infty^{opt}) \geq 0$ (because there is a minimum at $\mathcal{R}_\infty^{opt}$), and $s'_\infty(\mathcal{R}_\infty^{opt}) < 0$ by the F.O.C.

We have found that $\lambda \mapsto \mathcal{R}_\infty^{opt}(\lambda)$ is a non increasing function for $\mathcal{R}_\infty^{opt}(\lambda) < \mathcal{R}_0$, with $\mathcal{R}_\infty^{opt}(0) = \mathcal{R}_0$ and $\mathcal{R}_\infty^{opt}(\lambda) < \mathcal{R}_0$ for $\lambda > \widehat{\lambda}'_0$, thus there exists $\lambda'_0 \in [0; \widehat{\lambda}'_0]$ such that $\mathcal{R}_\infty^{opt}(\lambda) = \mathcal{R}_0 \Leftrightarrow \lambda \in [0; \lambda'_0]$.

This gives Prop 4 (i).

(ii) For λ given, $\lambda > \lambda'_0$, the interior solution leads to $s'_\infty(\mathcal{R}_\infty^{opt}) = \frac{-(T-t_0)y'(\mathcal{R}_\infty^{opt})}{\lambda N \delta} < 0$. According to Proposition 3(ii), for T high enough $s'_\infty(\mathcal{R}) < 0$ only on $\mathcal{R} > \widetilde{\mathcal{R}}_{t_0}$, thus $\mathcal{R}_\infty^{opt} > \widetilde{\mathcal{R}}_{t_0}$. □

A.7 Proof of Proposition 5

(i) Let us denote by $(\mathcal{C}_\varepsilon)$ the curve in the plane (s, i) representing the end of lockdowns $(s_\mathcal{R}(T), i_\mathcal{R}(T))$ which lead after release of lockdown to $s_\infty = s_\infty(\mathcal{R}, T) = \frac{1}{\mathcal{R}_0} - \varepsilon$.

According to Eq. (22) with $t \to +\infty$, and $i = i(T)$, $s = s(T)$, since $\lim_{t\to\infty} i(t) = 0$ and $\lim_{t\to\infty} s(t) = s_\infty$, the equation of $(\mathcal{C}_\varepsilon)$ in the plane (s, i) is $0 = i + s - s_\infty + \frac{1}{\mathcal{R}_0}\ln(s_\infty) - \frac{1}{\mathcal{R}_0}\ln(s)$,

i.e.

$$i = s_\infty - \frac{1}{\mathcal{R}_0}\ln(s_\infty) - s + \frac{1}{\mathcal{R}_0}\ln(s)$$

Under the assumption $s_\infty = \frac{1}{\mathcal{R}_0} - \varepsilon$, with $\varepsilon > 0$ given, $(\mathcal{C}_\varepsilon)$ meets the x-axis at two points: $s = s_\infty$ and $s = \check{s}_\infty$, with $s_\infty < \frac{1}{\mathcal{R}_0} < \check{s}_\infty$,

thus $-s_\infty + \frac{1}{\mathcal{R}_0}\ln(s_\infty) = -\check{s}_\infty + \frac{1}{\mathcal{R}_0}\ln(\check{s}_\infty)$, i.e. $h_{\mathcal{R}_0}(s_\infty) = h_{\mathcal{R}_0}(\check{s}_\infty)$.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be defined by $\lim_{T\to\infty} s_{\mathcal{R}_1}(T) = \check{s}_\infty$ and $\lim_{T\to\infty} s_{\mathcal{R}_2}(T) = s_\infty$.

Then for $\mathcal{R} \leq \mathcal{R}_1$ or $\mathcal{R} \geq \mathcal{R}_2$, we have $s_\infty(\mathcal{R}) < \frac{1}{\mathcal{R}_0} - \varepsilon$, i.e. $s_\infty(\mathcal{R}) = \frac{1}{\mathcal{R}_0} - \varepsilon$ is impossible.

If $\mathcal{R} \in (\mathcal{R}_1, \mathcal{R}_2)$, there exists a value of T (denoted by $T_\varepsilon(\mathcal{R})$), such that $s_\infty(\mathcal{R}) = s_\infty(\mathcal{R}, T) = \frac{1}{\mathcal{R}_0} - \varepsilon$, and $\lim_{\mathcal{R}\to\mathcal{R}_1} T_\varepsilon(\mathcal{R}) = \lim_{\mathcal{R}\to\mathcal{R}_2} T_\varepsilon(\mathcal{R}) = +\infty$, since $\lim_{T\to\infty} s_{\mathcal{R}_1}(T) =$

$\check{s}_\infty$ and $\lim_{T\to\infty} s_{\mathcal{R}_2}(T) = s_\infty$.

(ii) For a given mortality $M_\infty(\mathcal{R},T) = \delta N s_\infty(\mathcal{R},T) = \delta N\left(\frac{1}{\mathcal{R}_0} - \varepsilon\right)$, the economic cost of the lockdown is $C_\varepsilon(\mathcal{R}) = \left(T_\varepsilon(\mathcal{R}) - t_0\right)\left(y(\mathcal{R}_0) - y(\mathcal{R})\right)$.

$\mathcal{R} \mapsto C_\varepsilon(\mathcal{R})$ is a continuous function on $(\mathcal{R}_1, \mathcal{R}_2)$, with $\lim_{\mathcal{R}\to\mathcal{R}_1} C_\varepsilon(\mathcal{R}) = \lim_{\mathcal{R}\to\mathcal{R}_2} C_\varepsilon(\mathcal{R}) = +\infty$.

$\mathcal{R} \mapsto C_\varepsilon(\mathcal{R})$ is continuous on any closed (compact) interval included in $(\mathcal{R}_1, \mathcal{R}_2)$, thus there exists a minimum in $(\mathcal{R}_1, \mathcal{R}_2)$.

The first order equation is $C'_\varepsilon(\mathcal{R}) = 0$, with $C'_\varepsilon(\mathcal{R}) = T'_\varepsilon(\mathcal{R})\left(y(\mathcal{R}_0) - y(\mathcal{R})\right) - \left(T_\varepsilon(\mathcal{R}) - t_0\right) y'(\mathcal{R})$

The minimal time is obtained with $\mathcal{R} = \mathcal{R}^\circ_\varepsilon$ satisfying $T'_\varepsilon(\mathcal{R}) = 0$.

The minimal economic cost is obtained with $\mathcal{R} = \widehat{\mathcal{R}}_\varepsilon$ satisfying $C'_\varepsilon(\mathcal{R}) = 0$.

Since $T'_\varepsilon(\mathcal{R}) = 0$ and $C'_\varepsilon(\mathcal{R}) = 0$ cannot be simultaneously obtained, thus $\mathcal{R}^\circ_\varepsilon \neq \widehat{\mathcal{R}}_\varepsilon$.

If $\mathcal{R}^\circ_\varepsilon > \widehat{\mathcal{R}}_\varepsilon$, then $\mathcal{R}^\circ_\varepsilon$ is not only the fastest, but also the less costly. It is impossible since the less costly is $\widehat{\mathcal{R}}_\varepsilon$. Thus $\mathcal{R}^\circ_\varepsilon < \widehat{\mathcal{R}}_\varepsilon$.

We have $\widehat{T}_\varepsilon > T^\circ_\varepsilon$ by definition. □

A.8 Proof of Proposition 6

$L_\infty(\mathcal{R},T) = (T - t_0)\left(y(\mathcal{R}_0) - y(\mathcal{R})\right) + \lambda M_\infty(\mathcal{R})$ is a continuous function on $(\mathcal{R},T) \in [0;\mathcal{R}_0] \times [t_0;+\infty)$.

Let $a = \inf_{\substack{0\leq\mathcal{R}\leq\mathcal{R}_0 \\ T\geq t_0}} L_\infty(\mathcal{R},T)$. Is this infimum a minimum?

We can write $a = \inf_{0\leq\varepsilon\leq\frac{1}{\mathcal{R}_0}}\left[\inf_{\left\{(\mathcal{R},T)\text{ with } s_\infty=\frac{1}{\mathcal{R}_0}-\varepsilon\right\}} L_\infty(\mathcal{R},T)\right] = \inf_{0\leq\varepsilon\leq\frac{1}{\mathcal{R}_0}} a(\varepsilon)$,

where $a(\varepsilon) = \inf_{(\mathcal{R},T)\text{ with } s_\infty=\frac{1}{\mathcal{R}_0}-\varepsilon} L_\infty(\mathcal{R},T)$ has been studied in Proposition 5.

$a(0) = L_\infty\left(\widetilde{\mathcal{R}}_{t_0}, +\infty\right) = +\infty$, thus $a = \inf_{0<\varepsilon\leq\frac{1}{\mathcal{R}_0}} a(\varepsilon)$ and for ε very close to 0, $a(\varepsilon)$ is arbitrarily high, thus if $\varepsilon_0 > 0$ is sufficiently small, we have $a = \inf_{\varepsilon_0\leq\varepsilon\leq\frac{1}{\mathcal{R}_0}} a(\varepsilon)$.

The function $\varepsilon \mapsto a(\varepsilon)$ is continuous on the compact interval $[\varepsilon_0, \frac{1}{\mathcal{R}_0}]$, thus the infimum is a minimum according to the extreme value theorem, i.e. there exists a couple $(\mathcal{R}^*, T^*) \in [0;\mathcal{R}_0] \times [t_0;+\infty)$ such that $a = L_\infty(\mathcal{R}^*, T^*)$. □

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