Job Matching and Affirmative Action: The Impact of Transfer Policies

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Abstract

Subsidies and taxes are widely used in labor markets to influence employment outcomes. This paper uses a Kelso-Crawford framework to assess how transfers affect the welfare of minority workers. We show that affirmative action policies, though well-intentioned, can unintentionally harm this group. We establish that only uniform transfers—where all workers in a group receive the same subsidy or tax across all firms—guarantee no welfare loss. Building on this insight, we explore how transfers can be designed to achieve representation goals for minority workers without reducing their welfare.

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1 Introduction

Subsidies and taxes are widely used policy instruments in labor markets to promote employment and reduce inequalities. In the United States, companies receive tax abatements for employing at least a certain number of minority workers (Byrnes et al. 1999, Slattery and Zidar 2020). In the United Kingdom, minimum wage laws aim to reduce wage disparities (*Living Wage Laws*, Neumark et al. 2007, Derenoncourt and Montialoux 2021). In many OECD countries, employment subsidies support the integration of marginalized workers (Snower 1994). These interventions aim to improve worker welfare and reduce labor market segregation. Building on the framework of Kelso and Crawford (1982), this paper studies how such transfer policies affect market allocation, focusing jointly on assignment and salary.

Affirmative action policies aim to improve outcomes for minority workers but can induce complex responses in firms' hiring and wage-setting behavior. For example, subsidies increase a worker's attractiveness, which may prompt firms to adjust wage offers in competitive equilibrium

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(Shapley and Shubik 1971; Crawford and Knoer 1981; Kelso and Crawford 1982; Gul and Stacchetti 1999; Hatfield et al. 2019; Kojima et al. 2020, 2024). Although both assignment and salary are key determinants of worker welfare, most of the existing literature focuses on the former, leaving the latter underexplored. This paper addresses this gap by demonstrating that certain affirmative action policies, such as subsidizing minority workers or taxing majority workers, may paradoxically reduce minority workers' welfare.[1]

To formally analyze these effects, we develop a competitive equilibrium model in which worker *utility* depends on both *assignment* and *salary*. Each firm is endowed with a *revenue function* mapping any subset of workers to a real number. *Transfer policies* are represented by a matrix assigning a *subsidy* or *tax* to each worker-firm pair. Firms maximize *profits*, defined as the sum of revenue and transfers minus total salaries associated with the workers they employ. For each firm, *demand* corresponds to the sets of workers that maximize profit given prevailing salaries and transfers. Within this framework, the *gross substitutes condition* is sufficient for the existence of competitive equilibria (Kelso and Crawford 1982, Gul and Stacchetti 1999).[2] It also ensures that the set of equilibria forms a *lattice*, with the *firm-optimal* and *worker-optimal stable allocations* at its extremities, where firms maximize profits and workers maximize utility, respectively. We analyze two *mechanisms*: one yielding the firm-optimal and the other the worker-optimal stable allocation.

This paper first examines which *transfer policies do not reduce the welfare of minority workers* at a given mechanism. A transfer *reduces the welfare of minority workers* if, under the allocation induced by the mechanism, at least one minority worker experiences a decrease in utility. We show that only *uniform* transfers—those that provide the same subsidy or tax to every member of a group (minority or majority) across all firms—guarantee no welfare reduction for minority workers under the optimal stable mechanisms. The intuition is that granting a larger subsidy to certain workers makes them more attractive to firms, thereby intensifying competition among workers. Those receiving smaller or no subsidies lose bargaining power, which may result in lower wages or job loss, ultimately reducing their welfare. As a consequence, policies that target only a subset of the population—such as those aimed at promoting the employment of unemployed minority workers or supporting specific sectors—may inadvertently reduce the welfare of minority workers.

While uniform transfers are the only ones that never reduce the welfare of minority workers, they do not necessarily improve it. In practice, many policies—such as minimum wage laws or

[1]Kojima (2012) shows that affirmative action policies harm minority students in a model without transfers for any stable mechanisms. We discuss in greater detail the distinction between this model and the one presented in the current paper in Appendix A.

[2]The gross substitutes condition roughly requires that a set of demanded workers remains demanded after an increase in other workers' salaries. This condition is widely adopted in the literature for both technical and practical reasons. Cramton et al. (2006), Bichler and Goeree (2017), and Milgrom (2017) discuss how market design incorporating the gross substitutes condition is effectively addressed, whereas its absence leads to unavoidable trade-offs, such as between efficiency, incentives, and complexity properties. Moreover, this condition is particularly relevant for specialized markets, such as the labor market for gastroenterologists (Niederle and Roth 2003), in situations where resources are relatively homogeneous (e.g., unskilled labor), when resources are prepackaged to reduce complementarity (Loertscher et al. 2015), and when complementary resources are fixed in the short term (e.g., rental markets for agricultural land that require specific machinery investments).

hiring subsidies—aim to ensure that targeted workers achieve a sufficient level of well-being. We provide a sufficient condition under which a subsidy strictly increases the utility of its targeted workers by at least the subsidy amount, without changing the assignment or harming other workers. This condition requires that the (weakly) highest-value subsidy for minority workers be directed to the firm that employs them. Intuitively, if the most favorable subsidy is granted to the current employer, the worker has no incentive to switch firms, competition among workers remains unchanged, and the worker retains the entire surplus generated by the transfer under the worker-optimal stable allocation.

We then examine which transfers support common policy goals, such as *promoting employment for unemployed minority workers* and *increasing the representation of minority workers in firms*. We restrict attention to the subclass of *additively separable revenue functions* and identify conditions under which these goals can be achieved without reducing the welfare of minority workers. Roughly speaking, additive separability means that, for each firm, workers are assigned real numbers, and that the revenue generated by any set of employed workers equals the sum of these numbers. This framework allows us to design transfers that induce a desired representation of minority groups in the labor market. Moreover, to limit the cost of such policies, we analyze the substitutability between subsidies and taxes. We show that, within this subclass, if a transfer uniformly subsidizes minority workers, then taxing majority workers by the same amount leads to an identical market allocation.

In practice, groups are formed based on various criteria, such as gender, location, or ethnicity. Since different minority groups may be considered, affirmative action policies may not benefit them uniformly. We show that this situation is similar to implementing a non-uniform transfer, which may reduce the welfare of a minority group. In this context, uniform taxes prevent the rest of the population from experiencing a welfare reduction. However, taxes do not allow for targeting the effects of the policy to favor specific groups.

Related Literature

A large literature in market design examines the existence of equilibria—or stable allocations—under affirmative action policies. This literature primarily focuses on non-transferable utility (NTU) frameworks and identifies the conditions under which equilibrium exists despite representation constraints such as minimum guarantees (or lower quotas), upper quotas, and regional quotas.(Abdulkadiroğlu and Sönmez 2003; Kamada and Kojima 2012, 2015, 2017, 2024, Ehlers et al. 2014; Kominers and Sönmez 2016; Sönmez et al. 2019, among others). In transferable utility (TU) settings, Gul et al. (2019), Kojima et al. (2020), 2024, and Echenique et al. (2021) identify conditions under which market equilibrium is preserved when such constraints are imposed. While these studies ensure minority representation in firms, they often overlook how such policies affect the welfare of minority workers. In contrast, our paper investigates the impact of affirmative action policies on minority workers' welfare. In addition, we show that, without imposing quotas, it is possible to incentivize firms to achieve targeted minority representation through the use of taxes and subsidies. Thus, our paper offers insights into the compensations required to incentivize firms to achieve a desired level of minority representation in equilibrium.

Most existing studies examining the welfare effects of affirmative action for minority agents are conducted within NTU frameworks, which differ fundamentally from our TU approach. While Ehlers et al. (2014) demonstrate that imposing floor constraints can result in an empty set of stable matchings in NTU settings, Kojima et al. (2020) show that, under *generalized interval constraints*, the substitutes condition is preserved in TU markets, thereby ensuring equilibrium existence. Similarly, Kojima (2012) establishes the impossibility of designing affirmative action policies in NTU environments that guarantee no harm to minority agents under any stable mechanism. By contrast, we show that in TU settings, uniform transfers within groups never reduce the welfare of minority workers. To the best of our knowledge, this is the first paper to demonstrate that affirmative action policies can reduce minority welfare in a TU environment, and to identify the exact conditions under which this effect can be prevented. A key contribution of our paper is the characterization, i.e., the identification of necessary and sufficient conditions, for transfer-based affirmative action policies that do not harm minority workers.

In recent years, new objectives for affirmative action policies have emerged, such as reducing unemployment among minority workers (Snower 1994) and ensuring a minimum wage (Neumark et al. 2007). Surprisingly, these objectives have received limited attention in the market design literature. We address this gap by exploring how subsidies can be designed to reduce unemployment and increase the wages of minority workers.

Finally, our theoretical analysis complements a growing empirical literature on the effects of affirmative action on both representation and welfare. Bleemer (2022) provides evidence that affirmative action in education may reduce labor market earnings for minorities. Other studies, including those by Leonard (1990) and Coate and Loury (1993), examine the implementation of affirmative action in employment contexts, often finding that firms fail to comply with the regulations, resulting in continued disadvantages for minority workers. Our paper complements these empirical findings by identifying how employment and desegregation policies may lead to reduced welfare for minority workers. We then propose transfers—implemented through subsidies and taxes—that align firm incentives with affirmative action objectives. In our framework, firms act as profit maximizers, yet the proposed instruments ensure both successful recruitment and proper representation of minority workers.

This paper proceeds as follows. Section 2 introduces the model. Section 3 investigates transfers that never reduce the welfare of minority workers under optimal stable allocations. Section 4 explores common affirmative action policies, their associated costs, and how surplus is distributed. Section 5 examines the effects of transfers in markets with multiple minority groups. Section 6 discusses policy implications and highlights potential sources of welfare loss. Appendix A provides additional insights. Appendix B details the algorithms underlying the mechanisms discussed. Auxiliary results are presented in Appendix C, while all proofs are provided in Appendix D.

2 The Model

There are finite sets W and F of *workers* and *firms*. Agents on both sides of the market derive value from being matched to agents on the other side. Let $\sigma_{w,f} \in \mathbb{R}$ denote the value that worker w obtains when employed by firm f. For each firm $f \in F$, we define a *revenue function* $R_f : 2^W \to \mathbb{R}$, which maps a subset of workers to the revenue of firm f if it hires them.

We assume that for each $W' \subset W$, worker $w \in W \setminus W'$ and firm f,

$$R_f(W' \cup \{w\}) - R_f(W') - \sigma_{w,f} \geq 0.$$[3]

Without loss of generality, we normalize the value of agents that are unassigned to zero, $\sigma_{w,\emptyset} = 0$ and for any subset $W' \subseteq W, R_\emptyset(W') = 0$, where $\emptyset$ denotes the null firm to which the corresponding agent is assigned. All workers have the option of remaining unemployed.

In markets, we allow for utility transfers between firms and workers, which we refer to as *salaries*. Let $s_{w,f} \in \mathbb{R}$ denote the salary received by worker $w \in W$ when working for firm $f \in F \cup \{\emptyset\}$. We denote $\mathbf{s}$ as a *vector of salaries*. We assume that workers are indifferent about which other workers firms hire, and that $s_{w,\emptyset} = 0$.

Each worker's *utility function* is strictly increasing and continuous in the salaries. Let $u_w(f, s_{w,f}) \in \mathbb{R}$ be the utility function of worker w when working for firm f at salary $s_{w,f}$, such that $u_w(f, s_{w,f}) = \sigma_{w,f} + s_{w,f}$. Consequently, $u_w(f, -\sigma_{w,f}) = 0$ and $-\sigma_{w,f}$ is the lowest salary at which worker w would consider working for firm f.[4]

A policymaker regulates the market using *transfers*, which represent *subsidies* or *taxes*. A transfer is a matrix $\mathbf{T} = (t_{w,f})_{w \in W, f \in F}$, where each element $t_{w,f} \in \mathbb{R}$ represents the transfer for worker w when employed by firm f. If $t_{w,f} < 0$, we say that $\mathbf{T}$ *taxes* worker w at f by $t_{w,f}$, and if $t_{w,f} > 0$ we say that $\mathbf{T}$ *subsidizes* worker w at f by $t_{w,f}$. A *null transfer* is denoted by $\mathbf{T}_0$ such that for each worker w and each firm f, $t_{w,f} = 0$. Let $\mathbb{T}$ be the set of all possible transfers.

If a firm f hires a subset of workers $W' \subset W$ while facing a vector of salaries $\mathbf{s}$, a transfer $\mathbf{T}$, and a revenue function R_f, its *profit* is given by

$$V_f(W'; \mathbf{s}, R_f, \mathbf{T}) = R_f(W') - \sum_{w \in W'} s_{w,f} + \sum_{w \in W'} t_{w,f},$$

that is, the firm's revenue minus salaries of workers it hires and the transfers received by the firm.[5] We define the *maximal profit function* as $\Pi_f(\,\cdot\,; R_f, \mathbf{T})$ and the *demand* by $D_f(\,\cdot\,; R_f, \mathbf{T})$

[3]This restriction, ensuring that workers' *marginal product* is non-negative, is natural and is discussed by Kelso and Crawford (1982).

[4]In our model, wages can be negative. This occurs when σ is positive, meaning that the worker derives positive utility from working. In practice, this situation arises in the context of on-the-job training, where workers pay (or accept negative wages) in exchange for acquiring skills that will increase their future employability.

[5]Subsidies and taxes are typically determined based on the firm's workforce. By adding the transfer to the profit, we assume the transfer is an *additively separable function*. Kojima et al. (2024) provides further discussion on transfers that preserve the substitutability condition.

such that for each vector of salaries $\mathbf{s}$,

$$\Pi_f(\mathbf{s}; R_f, \mathbf{T}) = \max\{V_f(W'; \mathbf{s}, R_f, \mathbf{T}) : W' \subset W\};$$
$$D_f(\mathbf{s}; R_f, \mathbf{T}) = \{W' \subset W : V_f(W'; \mathbf{s}, R_f, \mathbf{T}) = \Pi_f(\mathbf{s}; R_f, \mathbf{T})\}.$$

Each element of $D_f(\mathbf{s}; R_f, \mathbf{T})$ is referred to as a *demand set.*

The *gross substitutes condition*, introduced by Kelso and Crawford (1982), ensures that if a worker's salary increases, firms retain their demand for workers whose salaries remain unchanged.

Definition 1. (Gross Substitutes). A demand $D_f(\cdot\,; R_f, \mathbf{T})$ satisfies the gross substitutes condition if, for any two vectors of salaries $\mathbf{s}$ and $\mathbf{s}'$, and for all $W' \in D_f(\cdot\,; R_f, \mathbf{T})$, if $\mathbf{s}' \geq \mathbf{s}$ and $s_w = s'_w$, then $w \in W'$, there exists $W'' \in D_f(\mathbf{s}'; R_f, \mathbf{T})$ such that $w \in W''$.

A market with salaries is denoted by $G = (W, F, R, \mathbf{T}, \mathbf{s})$. Let $\mathbb{G}$ be the set of all possible markets. As W, F, and R are fixed throughout, and $\mathbf{s}$ is determined by R, we write $G = (\mathbf{T})$ for simplicity. We assume that in all markets, firms' demands satisfy the gross substitutes condition.

An *assignment* is a mapping (or correspondence) $\mu : W \times F \rightrightarrows W \cup \{\emptyset\} \times F \cup \{\emptyset\}$ satisfying:

- $\mu(w) \in F \cup \{\emptyset\}$,
- For any $w \in W$ and $f \in F$, we have $\mu(w) = f$ if and only if $w \in \mu(f)$, and
- $\mu(f) \subseteq W \cup \{\varnothing\}$.

We use the notation $\mu(w) = \emptyset$ to indicate that worker w is unemployed at μ. Given an assignment μ and a vector of salaries $\mathbf{s}$ we define an *allocation* as a pair $(\mu, \mathbf{s})$. With a slight abuse of notation, we note $u_w(\mu, \mathbf{s})$ for $u_w(\mu(w), s_{w,\mu(w)})$.
An allocation $(\mu, \mathbf{s})$ is *individually rational* if for each firm $f \in F$, $V_f(\mu(f); \mathbf{s}, R_f, \mathbf{T}) \geq 0$ and for each worker $w \in W, u_w(\mu, \mathbf{s}) \geq 0$. We now introduce the definition of *stability.*

Definition 2. (Stable Allocation). An allocation $(\mu, \mathbf{s})$ is stable if it is individually rational, and there is no firm-worker set $W' \cup \{f\}$, with $W' \subseteq W$, and vector of salaries $\mathbf{s}'$, that *block* $(\mu, \mathbf{s})$ such that:

- $u_w(f, s'_{w,f}) \geq u_w(\mu, \mathbf{s})$, for each $w \in W'$, and
- $V_f(W'; \mathbf{s}', R_f, \mathbf{T}) \geq V_f(\mu(f); \mathbf{s}, R_f, \mathbf{T})$

with strict inequality holding for at least one member of $W' \cup \{f\}$.

We define *stable allocation optimality* in markets where firms' demands satisfy the gross substitutes condition. In such markets, there exists a stable allocation unanimously preferred by all firms, and another unanimously preferred by all workers. Without transfers, we denote the *worker-optimal stable allocation* as $(\mu_W, \mathbf{s}_W)$ and the *firm-optimal stable allocation* as $(\mu_F, \mathbf{s}_F)$.[6] With a non-zero transfer $\mathbf{T}$, the corresponding allocations are $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$ for workers and $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ for firms.

A *mechanism* φ maps each market $G \in \mathbb{G}$ to an allocation. The *worker-optimal stable mechanism* φ_W assigns to each market its worker-optimal stable allocation, $\varphi_W(\mathbf{T}) = (\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$,

[6]The existence of these allocations in the context of job markets has been extensively studied by Kelso and Crawford (1982), Roth (1984), Roth (1985), Demange and Gale (1985), Gul and Stacchetti (1999), Hatfield and Milgrom (2005) among others.

while the *firm-optimal stable mechanism* φ_F assigns the firm-optimal stable allocation, $\varphi_F(\mathbf{T}) = (\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$.

In this paper, we study the impact of affirmative action on groups of the population.[7] Mathematically, we fix a *partition* of W, denoted by $\mathcal{P} \subset 2^W \setminus \{\varnothing\}$, where $\bigcup_{P \in \mathcal{P}} P = W$, and $P \cap P' = \{\varnothing\}$ for $P \neq P'$. Each element of $\mathcal{P}$ is a *group*. Throughout this paper, we restrict the partition $\mathcal{P}$ to two groups: m, representing *minority workers*, and M, representing *majority workers*. We extend the analysis to more than two groups in Section 5. A transfer $\mathbf{T}$ *taxes majority workers* if $t_{w,f} \leq 0$ for all (w, f), and $t_{w,f} < 0$ only if $w \in M$. Similarly, a transfer $\mathbf{T}$ *subsidizes minority workers* if $t_{w,f} \geq 0$ for all (w, f), and $t_{w,f} > 0$ only if $w \in m$.[8] A transfer *taxes majority and subsidizes minority workers* if $t_{w,f} < 0$ only if $w \in M$, and $t_{w,f} > 0$ only if $w \in m$.

Affirmative action policies aim to favor certain groups. A natural requirement is that a transfer should not reduce the welfare of minority workers. A transfer $\mathbf{T}$ *does not reduce the welfare of minority workers* under φ if, for a market $G = (\mathbf{T}_0)$ where $\varphi(\mathbf{T}_0) = (\mu, \mathbf{s})$ and $\varphi(\mathbf{T}) = (\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}})$, the following condition holds for each $w \in m$:

- $u_w(\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}}) \geq u_w(\mu, \mathbf{s})$.

A transfer $\mathbf{T}$ *reduces the welfare of minority workers* under φ if this condition is violated.[9] A transfer $\mathbf{T}$ *never reduces the welfare of minority workers* under φ if this condition holds for all markets $G \in \mathbb{G}$; otherwise, $\mathbf{T}$ *may reduce the welfare of minority workers*.

3 Uniform Transfers and Minority Workers Welfare

In this section, we examine the impact of commonly used targeted transfers. Using an example, we illustrate the potential adverse effects such transfers may have on minority workers.

Example 1. Consider a market with three firms, $F = \{f_1, f_2, f_3\}$ and three workers, $W = \{w_1, w_2, w_3\}$. For simplicity, assume that for each worker $w \in W$ and each firm $f \in F$, $\sigma_{w,f} = 0$, also, for each $W' \subset W$, $R_f(W') = \max_{w \in W'}(R_f(\{w\}))$.[10] Revenue functions are provided in Table 1, while stable optimal allocations are detailed in Table 2.

Firms	w_1	w_2	w_3
f_1	5	4	2
f_2	1	5	0
f_3	5	0	3

Table 1: Revenue functions.

[7] A group of workers may be defined based on factors such as gender, ethnicity, qualifications, location, educational background, or any other combination of individual characteristics.

[8] Throughout the paper, we implicitly assume that minority (majority) workers are never taxed (subsidized).

[9] This can be seen as a weak Pareto improvement for minority workers. More specific objectives are discussed in Section 4.

[10] These revenue functions belong to a special class known as *unit-demand revenue functions*, which satisfy the gross substitutes condition.

f_1	f_2	f_3
w_1	w_2	w_3
$s_{w_1} = 2$	$s_{w_2} = 1$	$s_{w_3} = 0$

f_1	f_2	f_3
w_1	w_2	w_3
$s_{w_1} = 5$	$s_{w_2} = 5$	$s_{w_3} = 3$

Table 2: Firm-optimal and worker-optimal stable allocations.

Let the partition of workers $\mathcal{P}$ be defined as $m = \{w_2, w_3\}$ and $M = \{w_1\}$. Assume the policymaker intends to incentivize firm f_1 to recruit minority workers.[11] To that end, a subsidy of 2 is provided to all minority workers at f_1, i.e., $\mathbf{T}$ is introduced such that $t_{w_2,f_1} = 2$, $t_{w_3,f_1} = 2$, and zero for all other elements. The resulting worker-optimal stable allocation is given in Table 3.

f_1	f_2	f_3
w_1	w_2	w_3
$s_{w_1} = 4$	$s_{w_2} = 5$	$s_{w_3} = 2$

Table 3: Worker-optimal stable allocation with a subsidy of 2 at f_1.

The assignment is not modified, and w_3's salary is reduced. The intuition is that, by subsidizing w_2 at firm f_1, w_2 can expect a higher salary and thus compete with worker w_1. Because of the competition, worker w_1, in turn, considers switching to firm f_3, creating competition for worker w_3. Competition leads to w_3's salary reduction. Note that the same reasoning holds if the subsidy only concerns worker w_2, and leads to the same allocation.

Similarly, consider that the subsidy now concerns firm f_3, that is $t_{w_2,f_3} = 2, t_{w_3,f_3} = 2$ and 0 for the other elements. The resulting firm-optimal stable allocation is given in Table 4.

f_1	f_2	f_3
w_1	w_2	w_3
$s_{w_1} = 0$	$s_{w_2} = 0$	$s_{w_3} = 0$

Table 4: Firm-optimal stable allocation with a subsidy of 2 at f_3.

Before the introduction of the subsidy, firms f_1 and f_3 were competing for worker w_1. This resulted in a higher salary for w_2, which is the best alternative for f_1. By subsidizing the worker w_3 at f_3, firm f_3 becomes indifferent between w_1 and w_3. As a result, competition between f_1 and f_3 for worker w_1 is reduced, and w_2's salary is lowered. Note that the same reasoning holds if the subsidy applies only to worker w_3. Therefore, when a transfer is directed toward subsidizing minority workers across a subset of firms, it may have adverse effects on minority workers at the worker-optimal and firm-optimal stable allocations.

Another common instrument in affirmative action policies is the implementation of taxes. These taxes are designed to penalize firms that fail to meet government-imposed representation requirements. Firms may receive tax abatements for employing a minimum number of minority

[11] Such transfers can be implemented by policymakers to support specific market sectors with the aim of increasing minority worker representation.

workers. In this context, imposing a tax is analogous to reducing the abatement. The key distinction between taxes and subsidies lies in the range of possible amounts. While there may be no mathematical distinction, from an economic perspective, it is important to limit the amount of the tax. If the tax exceeds the surplus generated by a worker at a firm, the firm will never employ that worker. Therefore, throughout this paper, we assume that for each worker w and each firm f, we have $R_f(\{w\}) + \sigma_{w,f} + t_{w,f} \geq 0$.

We now establish the difference between taxes and subsidies. In Example 1, consider a transfer $\mathbf{T}$ where $t_{w_1,f_1} = -1$, and 0 for all other elements. The resulting optimal stable allocations are given in Table 5.

f_1	f_2	f_3
w_1	w_2	w_3
$s_{w_1} = 2$	$s_{w_2} = 2$	$s_{w_3} = 0$

f_1	f_2	f_3
w_1	w_2	w_3
$s_{w_1} = 4$	$s_{w_2} = 5$	$s_{w_3} = 2$

Table 5: Firm-optimal and worker-optimal stable allocations with a tax of 1 at f_1.

For the worker-optimal stable allocation, the argument is the same as presented in Table 3: by taxing the worker w_1 at firm f_1,[12] worker w_1, aiming to obtain the highest salary, competes for firm f_3, thereby intensifying the competition with worker w_3. However, at the firm-optimal allocation, salaries are not reduced. The intuition is that in the worker-optimal stable allocation, workers compete to achieve the highest possible salary, while, when the firm-optimal stable allocation is considered, firms compete to attain the highest possible profit. By taxing majority workers, minority workers become relatively more attractive, leading firms to offer them higher salaries. This illustrates how allocations can be affected even when transfers do not directly involve the agents concerned.

The following proposition establishes that taxing majority workers may reduce the welfare of minority workers under, but never under φ_F.

Proposition 1. A transfer $\mathbf{T}$ that taxes majority workers may reduce the welfare of minority workers under φ_W and never reduce the welfare of minority workers under φ_F.

Proof. See Appendix D.1. ■

Although taxes do not reduce the welfare of minority workers under the firm-optimal stable allocation, they may do so under the worker-optimal stable allocation. Our first main theorem establishes that uniform transfers form the entire class of transfers that never reduce the welfare of minority workers under φ_F and φ_W. A transfer $\mathbf{T}$ is *uniform* if there exist constants $t \geq 0$ and $t' \leq 0$ such that for each $w \in m$ and each $f \in F$, $t_{w,f} = t$, and for each $w' \in M$ and each $f \in F$, $t_{w',f} = t'$.

Theorem 1. A transfer $\mathbf{T}$ never reduces the welfare of minority workers under φ_F and φ_W if and only if $\mathbf{T}$ is uniform.

[12]This transfer may correspond to real-life situations where firms are taxed because they do not employ minority workers.

Proof. See Section D.3. ■

The policy implication of Theorem 1 is that if a transfer—be it a tax or a subsidy—is implemented to target a specific worker, subset of workers, firm, or sector, then it must be introduced uniformly across all minority workers and all firms in order to prevent a reduction in welfare.

The following corollary of Theorem 1 provides the characterization of transfers that never reduce the welfare of minority workers under φ_W.

Corollary 1. A transfer $\mathbf{T}$ never reduces the welfare of minority workers under φ_W if and only if $\mathbf{T}$ is uniform.

Example 1 (Continued). Consider a uniform transfer $\mathbf{T}$ that subsidizes each minority worker by 1 and taxes the majority worker by 1. The resulting stable optimal allocations for this market are as follows:

f_1	f_2	f_3
w_1 $s_{w_1} = 0$	w_2 $s_{w_2} = 1$	w_3 $s_{w_3} = 0$

f_1	f_2	f_3
w_1 $s_{w_1} = 4$	w_2 $s_{w_2} = 6$	w_3 $s_{w_3} = 4$

Table 6: Firm-optimal and worker-optimal stable allocations with a uniform transfer.

Table 6 shows that uniform transfers prevent any reduction in the welfare of minority workers at both the firm-optimal and worker-optimal stable allocations. While these transfers may intensify competition among workers, their uniform structure across firms offsets potential negative effects by ensuring identical support. However, they do not guarantee any improvement. Table 6 shows that, under the firm-optimal stable allocation, the welfare of minority workers remains unchanged. Our next theorem establishes a sufficient condition on transfers that ensures a welfare gain for minority workers at the worker-optimal stable allocation.

Theorem 2. Consider a market $G \in \mathbb{G}$. If $\mathbf{T}$ subsidizes minority workers and satisfies $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f})$ for each $w \in m$, then $\mu_W = \mu_W^{\mathbf{T}}$, and for each $w \in W$,

$$u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W, \mathbf{s}_W) + t_{w,\mu_W(w)}.$$

Proof. See Appendix D.2. ■

Theorem 2 demonstrates that, given knowledge of the worker-optimal stable allocation, it is possible to design transfers that raise minority workers' salaries without altering the assignment. The key insight is that, because each minority worker receives the highest subsidy at their current employer, no worker has an incentive to switch firms. As a result, competition is not intensified, and the entire surplus generated by the subsidy is captured by the workers.[13] More-

[13]Theorem 2 degenerates into Theorem 1 when transfers uniformly subsidize all minority workers. In this case, the condition in Theorem 2 is trivially satisfied, and the result reduces to that of Theorem 1.

over, the welfare of all workers—both minority and majority—is weakly improved.[14] However, the transfers described in Theorem 2 may reduce the welfare of minority workers under φ_F, as shown in Table 4, where only w_3 receives a subsidy at f_3.

The design of transfer policies critically depends on the information available to the policymaker. When the mechanism implementing the equilibrium is unknown, uniform transfers are the only class of transfers that guarantee minority workers are not adversely affected under both the worker-optimal and firm-optimal stable allocations. If the policymaker observes that the market operates under the firm-optimal stable allocation, then any tax on majority workers—regardless of its structure—preserves the welfare of minority workers. However, such taxes may fail to improve minority outcomes. In contrast, if the equilibrium corresponds to the worker-optimal stable allocation, Theorem 2 provides a sufficient condition under which targeted subsidies raise minority wages without reducing the welfare of majority workers. This result is particularly relevant in practice, as minority workers frequently earn lower wages.[15] By allowing subsidies to vary across workers, this approach enables targeted interventions—such as sector-specific wage policies—that can improve welfare without changing the assignment.[16]

4 Affirmative Action Policies

This section examines two common objectives of affirmative action policies: decreasing the segregation of minority workers across firms or sectors and reducing unemployment. Under certain revenue functions associated with demand satisfying the gross substitutes condition, a key feature of the model is the absence of unemployment.[17] To introduce unemployment, we impose firm capacity constraints and assume an excess supply of workers: each firm $f \in F$ has a *capacity* $q_f \in \mathbb{N}$, and $|W| > \sum_{f \in F} q_f$. Let $\mathbf{q} \equiv (q_f)_{f \in F}$ denote the *capacity vector*. The null firm has capacity $q_\emptyset = |W|$, so all workers can remain unemployed.

To guarantee existence of worker-optimal stable allocations under capacity constraints, we assume, in this section, that firm revenue functions are *additively separable*, that is, for any subset of workers $W' \subset W$, the revenue function satisfies $R_f(W') = \sum_{w \in W'} R_f(\{w\})$.[18] Since firms' revenue functions are additively separable, we simplify notation by writing $R_f(w)$ instead of $R_f(\{w\})$.

We define a market with salaries and capacities as $\overline{G} = (W, F, R, \mathbf{T}, \mathbf{s}, \mathbf{q})$. Let $\overline{\mathbb{G}}$ be the class

[14] The increase in workers' utility may strictly exceed the amount of the transfer due to the potential reduction in competition resulting from the subsidy.

[15] The empirical literature documenting wage disparities faced by minority workers is too extensive to cite exhaustively. See, for instance, Derenoncourt and Montialoux (2021), who document these disparities and analyze the impact of minimum wage policies on minority workers' welfare.

[16] A sector-specific minimum wage can be interpreted as a direct application of Theorem 2.

[17] The assumption of positive marginal product, which is necessary for the existence of stable allocations, rules out unemployment. Thus, unemployment only arises when a tax is introduced that renders a worker's marginal product negative.

[18] Without additive separability, the structure of the market may become asymmetric, and the set of stable allocations may fail to form a complete lattice (see Roth, 1984). This assumption has been widely discussed in the literature; see, Crawford and Knoer (1981), Demange and Gale (1985), Sotomayor (1999), Kojima et al. (2024), among others. This assumption is naturally satisfied in many practical settings, such as unskilled or specialized labor markets.

of such markets where all R_f are additively separable. Since W, F, R, and $\mathbf{q}$ are fixed, and $\mathbf{s}$ is determined by R, we write $\overline{G} = (\mathbf{T})$. Firm capacities modify profit and demand functions as follows:

$$\Pi_f(\mathbf{s}; R_f, \mathbf{T}, q_f) = \max\{V_f(W'; \mathbf{s}, R_f, \mathbf{T}) : W' \subset W, |W'| \leq q_f\},$$
$$D_f(\mathbf{s}; R_f, \mathbf{T}, q_f) = \{W' \subset W : |W'| \leq q_f \text{ and } V_f(W'; \mathbf{s}, R_f, \mathbf{T}) = \Pi_f(\mathbf{s}; R_f, \mathbf{T}, q_f)\}.$$

An *assignment with capacities* is a mapping (or correspondence) $\mu : W \times F \rightrightarrows W \cup \{\emptyset\} \times F \cup \{\emptyset\}$ satisfying:

- $\mu(w) \in F \cup \{\emptyset\}$,
- For any $w \in W$ and $f \in F$, we have $\mu(w) = f$ if and only if $w \in \mu(f)$,
- $\mu(f) \subseteq W \cup \{\varnothing\}$ and $|\mu(f)| \leq q_f$.

The definition of stability extends naturally to assignments with capacities by requiring that any blocking coalition respects the firm's capacity constraint.

Assuming additively separable revenue functions, firm demand satisfies the gross substitutes condition, ensuring that the results from the previous section continue to apply.[19]

When unemployment is possible, competition among workers intensifies. Unemployed workers may be willing to accept wages just sufficient to offset the disutility of working. As a result, subsidizing these workers can inadvertently lower the welfare of employed workers—without necessarily improving employment outcomes for the targeted group. The example below illustrates this effect.

Example 2. Consider a market with two firms, $F = \{f_1, f_2\}$ and five workers, $W = \{w_1, w_2, w_3, w_4, w_5\}$. Each firm $f \in F$ has capacity $q_f = 2$, and for each worker $w \in W, \sigma_{w,f} = 0$. Table 7 presents the revenue functions, and Table 8 shows the firm-optimal and worker-optimal stable allocations.

Firms	w_1	w_2	w_3	w_4	w_5
f_1	8	7	5	4	2
f_2	3	6	8	5	3

Table 7: Revenue functions.

f_1	f_2
w_1, w_2 $s_{w_1} = 0, s_{w_2} = 1$	w_3, w_4 $s_{w_3} = 0, s_{w_4} = 0$

f_1	f_2
w_1, w_2 $s_{w_1} = 6, s_{w_2} = 5$	w_3, w_4 $s_{w_3} = 5, s_{w_4} = 2$

Table 8: Firm-optimal and worker-optimal stable allocations.

Suppose that the partition of workers $\mathcal{P}$ is defined as $m = \{w_1, w_2, w_5\}$ and $M = \{w_3, w_4\}$, where w_5 is an unemployed minority worker. To reduce minority unemployment, the policymaker

[19] The only structural change is the introduction of capacity constraints. The examples in Section 3 can be seen as special cases where each firm has capacity one. To meet the requirement $|W'| \leq q_f$, one can always add a worker with zero productivity.

implements a subsidy: $t_{w_5,f_1} = t_{w_5,f_2} = 1$, with all other components of $\mathbf{T}$ set to zero.[20] The resulting stable optimal allocations are shown in Table 9.

f_1	f_2
w_1, w_2 $s_{w_1} = 0, s_{w_2} = 1$	w_3, w_4 $s_{w_3} = 0, s_{w_4} = 0$

f_1	f_2
w_1, w_2 $s_{w_1} = 5, s_{w_2} = 4$	w_3, w_4 $s_{w_3} = 4, s_{w_4} = 1$

Table 9: Firm-optimal and worker-optimal stable allocations.

Table 9 illustrates that subsidizing unemployed minority workers can intensify market competition, thereby reducing the welfare of employed minority workers. To design transfers that do not reduce the welfare of minority workers, we distinguish between employed and unemployed workers within the minority group.

Given a market $\overline{G} \in \overline{\mathbb{G}}$ and a stable allocation $(\mu, \mathbf{s})$, let

$$m_E \equiv \{w \in m : \mu(w) \neq \emptyset\} \quad \text{and} \quad m_U \equiv \{w \in m : \mu(w) = \emptyset\},$$

denote the sets of *employed* and *unemployed* minority workers, respectively.[21]

When a transfer $\mathbf{T}$ is introduced, we define

$$m_E^{\mathbf{T}} \equiv \{w \in m : \mu^{\mathbf{T}}(w) \neq \emptyset\} \quad \text{and} \quad m_U^{\mathbf{T}} \equiv \{w \in m : \mu^{\mathbf{T}}(w) = \emptyset\},$$

as the sets of employed and unemployed minority workers under the resulting stable allocation.

The following proposition provides a sufficient condition on transfers that guarantees no reduction in the welfare of minority workers under the worker-optimal stable allocation.

Proposition 2. Consider a market $\overline{G} \in \overline{\mathbb{G}}$. If $\mathbf{T}$ subsidizes minority workers and satisfies for each $w \in m_E$,

- $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f})$, and
- $t_{w,\mu_W(w)} \geq \max_{w' \in m_U, f' \in F}(t_{w',f'})$,

then $\mathbf{T}$ does not reduce the welfare of minority workers under φ_W.

Proof. See Section D.5. ■

Proposition 2 states that when subsidies are granted to unemployed workers, employed minority workers must receive at least as much—through the firm that employs them—to avoid a reduction in welfare under the worker-optimal stable allocation. The intuition follows from Example 2: subsidizing unemployed workers increases their attractiveness to firms, which can intensify competition and result in lower salaries—or even job loss—for employed minority workers. By providing an equivalent subsidy to the firm employing a given worker, the policy offsets this competitive pressure and preserves the worker's utility, as formalized in Theorem 2.

[20] Such targeted subsidies are commonly used to support the employment of disadvantaged groups.

[21] When firms face capacity constraints and revenue functions are additively separable, Sotomayor (1999) shows that the worker-optimal and firm-optimal stable allocations involve the same assignment. The only difference lies in the salary vector. The intuition is that each worker is assigned to the firm where they can generate the highest surplus, which also maximizes the firm's profit at the firm-optimal stable allocation.

The policy implication of Proposition 2 is that subsidies targeting unemployed minority workers must be complemented by equivalent transfers for employed minority workers. In practice, however, many affirmative action policies focus exclusively on the unemployed, without accounting for potential welfare losses among employed workers. We discuss these policies in more detail in Section 6.

4.1 Increasing the Representation of Minority Workers in a Firm

In this section, we explore transfers that increase the representation of minority workers in a firm. A transfer $\mathbf{T}$ *increases the representation of minority workers in firm* f under φ if, given a market $\overline{G} = (\mathbf{T}_0)$ with $\varphi(\mathbf{T}_0) = (\mu, \mathbf{s})$ and $\varphi(\mathbf{T}) = (\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}})$, the following two conditions hold:

(i) $q_f \geq |\mu^{\mathbf{T}}(f) \cap m| > |\mu(f) \cap m|$, and

(ii) $(\mu(f) \cap m) \subset (\mu^{\mathbf{T}}(f) \cap m)$.

Condition (i) ensures that at least one additional minority worker is hired by firm f.[22] Condition (ii) guarantees that no previously employed minority workers lose their job. Taken together, these imply that the increased representation of minority workers necessarily comes at the expense of majority workers employed by f prior to the transfer.

For simplicity and without loss of generality, we consider a single minority worker targeted for employment at a firm, replacing a majority worker. The replaced majority worker is the one who yields the lowest profit to the firm under the firm-optimal stable allocation.[23] Let w_m and w_M denote the minority and majority workers, respectively, where $\mu_F(w_m) \neq f$ and $\mu_F(w_M) = f$, and define $w_M = \arg\min_{w \in \mu_F(f) \cap M}(R_f(w) - s_{Fw,f})$. For w_m to be hired by firm f instead of w_M, the minority worker w_m must generate a higher profit at f than the majority worker w_M does. We define $t_{w_m \to f}$ as the *subsidy required for* w_m *to be hired by firm* f. It is given by:

$$t_{w_m \to f} \equiv \overbrace{(R_f(w_M) - s_{Fw_M,f}) - (R_f(w_m) + \sigma_{w_m,f} - u_{w_m}(\mu_F, \mathbf{s}_F))}^{\text{Compensation to firm } f \text{ for replacing } w_M} +$$
$$\underbrace{R_{\mu_F(w_m)}(w_m) + \sigma_{w_m,\mu_F(w_m)} - u_{w_m}(\mu_F, \mathbf{s}_F) - \max_{w' \in (W_U \cup \{w_M\}) \setminus \{w_m\}} (R_{\mu_F(w_m)}(w') + \sigma_{w',\mu_F(w_m)} - u_{w'}(\mu_F, \mathbf{s}_F))}_{\text{Compensation to the current employer of } w_m}.$$

The required subsidy depends on the revenue functions of both firm f and $\mu_F(w_m)$.[24] The first term ensures that firm f is willing to replace w_M with w_m by covering the difference in profits. The second term compensates w_m's current employer, ensuring it cannot outbid firm f by offering a higher salary. If w_m is unemployed, this second term is zero, as no compensation is needed to retain the worker in the current position.

[22] The reference to firm capacity in condition (i) ensures that firm f must initially employ majority workers who can be replaced.

[23] Under the worker-optimal stable allocation, all workers employed at a given firm generate the same profit (see Lemma 4). We, therefore, rely on the firm-optimal stable allocation to differentiate among workers.

[24] Claim 3 shows that $t_{w_m \to f}$ enables w_m to generate (weakly) higher profit at firm f than w_M at the firm-optimal stable allocation.

Example 2 (Continued). Suppose the set of minority workers is given by $m = \{w_1, w_4\}$, and the policymaker aims to have w_4 employed by firm f_1. The required subsidy is $t_{w_4 \to f_1} = 2$. Let $\mathbf{T}$ be a transfer such that $t_{w_4, f_1} = 2$, and all other components of $\mathbf{T}$ are zero. The resulting stable optimal allocations are:

f_1	f_2
w_1, w_4 $s_{w_1} = 0, s_{w_4} = 0$	w_2, w_3 $s_{w_2} = 1, s_{w_3} = 0$

f_1	f_2
w_1, w_4 $s_{w_1} = 4, s_{w_4} = 2$	w_2, w_3 $s_{w_2} = 3, s_{w_3} = 5$

Table 10: Firm-optimal worker-optimal stable allocations with a subsidy of 2 for w_4 at f_1.

This example illustrates that the transfer can reduce the welfare of minority workers already employed by firm f under the worker-optimal stable allocation.

The following theorem provides a sufficient condition on transfers, for a given market, to increase the representation of minority workers in a firm and to prevent a reduction in the welfare of minority workers at the worker-optimal stable allocation.

Theorem 3. Consider a market $\overline{G} \in \overline{\mathbb{G}}$. If the transfer $\mathbf{T}$ subsidizes minority workers and satisfies the following conditions:

- For each $w \in \{w \in m_E : \mu_W(w) \neq f\} \backslash \{w_m\}$, we have $t_{w,\mu_W(w)} = \max_{f' \in F}(t_{w,f'}) \geq t_{w_m \to f}$;
- For each $w \in \{w \in m_E : \mu_W(w) = f\}$, we have $t_{w,f} \geq \max_{f' \in F \backslash \{f\}}(t_{w,f'}) + t_{w_m \to f}$;
- For each $w' \in W \setminus (m_E \cup \{w_m\})$ and each $f \in F$, we have $t_{w',f} = 0$;
- For each $f \in F \setminus \{f\}$, we have $t_{w_m,f} = 0$, and $t_{w_m,f} \geq t_{w_m \to f}$.

Then, $\mathbf{T}$ does not reduce the welfare of minority workers under φ_W and increases the representation of minority workers in firm f under both φ_W and φ_F.

Proof. See Section D.6. ∎

In contrast to the salary increases discussed in Theorem 2, modifying the assignment of minority workers necessitates a non-uniform transfer, specifically targeted at the minority workers employed by firm f. Theorem 3 establishes that these workers must receive a subsidy at firm f that is at least $t_{w_m \to f}$ higher than any subsidy they would receive at other firms. This condition ensures that the workers remain employed at f and that the transfer does not reduce their welfare under φ_W.

For minority workers who are employed but not by f, their highest subsidy must be provided by their current employer, and it must be at least $t_{w_m \to f}$, as presented in Proposition 2. While w_m is chosen arbitrarily, it is possible that other minority workers could, in principle, be hired by firm f with lower subsidies. However, by the construction of $\mathbf{T}$, even if such workers exist, they remain employed at their original firm. In the proof of Theorem 3, we apply the same argument as in Theorem 2, showing that the assignment remains unchanged. Consequently, only w_m is newly recruited by firm f.

The proof of Theorem 3 proceeds by decomposing the transfer $\mathbf{T}$ into a sum of transfers, following Lemma 1 (Appendix C). The first transfer subsidizes all employed minority workers other than w_m, according to the amounts they receive under $\mathbf{T}$. We use Theorem 2 to show that the salaries of workers increase. We then introduce a transfer that subsidizes w_m by $t_{w_m \to f}$. According to Lemma 2 (Appendix C), we know that the welfare of all workers decreases by at most $t_{w_m \to f}$. Since their previous utility gains exceeded $t_{w_m \to f}$, we can conclude that the welfare of minority workers is not reduced. The remaining subsidy for w_m at f is then introduced, allowing w_m to be recruited by f and being subsidized by $t_{w_m,f} - t_{w_m \to f}$. This, following Theorem 2, results in an increase in w_m's salary by at least that amount, without reducing the salaries of other workers.

Similar to the transfers described in Theorem 2, the transfers described in Theorem 3 may reduce the welfare of minority workers under φ_F.[25]

Proposition 3 complements Theorem 3 by identifying a sufficient condition for increasing the representation of minority workers in firm f, without imposing any restrictions on the welfare of minority workers. Specifically, only the minority workers employed at firm f must retain their positions, and some minority workers may experience a reduction in their welfare.

Proposition 3. Consider a market $\overline{G} \in \overline{\mathbb{G}}$. If the transfer $\mathbf{T}$ subsidizes minority workers and satisfies the following conditions:

- For each $w \in \{w \in m_E : \mu_W(w) = f\}$, we have $t_{w,f} \geq \max(0, R_f(w_M) - s_{Fw_M,f} - (R_f(w) - s_{Fw,f}))$;
- For each $w' \in W \setminus (m_E \cap (\mu_F(f) \cup \{w_m\}))$, we have for each $f' \in F$, $t_{w',f'} = 0$;
- For each $f \in F \setminus \{f\}$, we have $t_{w_m,f} = 0$, and $t_{w_m,f} \geq t_{w_m \to f}$.

Then, $\mathbf{T}$ increases the representation of minority workers in firm f under both φ_W and φ_F.

Proof. See Section D.7. ■

To retain their positions at firm f, minority workers already employed by f must generate a profit higher than that of w_M. Therefore, the transfer described ensures that each minority worker is subsidized by at least 0 if the profit they generate at f already exceeds that generated by w_M, and by the difference otherwise.

4.2 Reducing Unemployment Among Minority Workers

In this section, we consider a policy aimed at reducing unemployment among minority workers. A transfer $\mathbf{T}$ *promotes the employment of minority workers* under φ if, for a market $\overline{G} = (\mathbf{T}_0)$, we have $\varphi(\mathbf{T}_0) = (\mu, \mathbf{s})$ and $\varphi(\mathbf{T}) = (\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}})$, the following two conditions hold:

(i) $|m_E^{\mathbf{T}}| > |m_E|$, and

(ii) $m_E \subset m_E^{\mathbf{T}}$.

Condition (i) ensures that at least one additional minority worker is employed, while condition (ii) guarantees that employed minority workers retain their positions. Taken together, these conditions imply that the promotion of employment for minority workers comes at the expense of majority workers.

[25] In Example 2, consider a partition such that $m = \{w_2, w_5\}$, and suppose the objective is to have w_5 employed by firm f_2. Then $t_{w_5 \to f_2} = 2$. Consider a transfer $\mathbf{T}$ with $t_{w_2,f_1} = 2$, $t_{w_5,f_2} = 3$, and 0 for other elements. The salary of w_2 is 0 under $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$.

By definition, if a transfer $\mathbf{T}$ promotes the employment of minority workers, there exists $f \in F$ such that $\mathbf{T}$ increases the representation of minority workers in firm f. However, there are two key distinctions to note. First, minority workers who are already employed may switch firms, provided they remain employed after the transfer. Second, unemployed minority workers must obtain employment with a firm, without specification of which firm that may be.

As in the previous section, and without loss of generality, we focus on a single unemployed minority worker who replaces a majority worker at a firm. Let $w_m \in m_U$ denote the targeted minority worker. We select the firm f^* that minimizes the required subsidy to implement the replacement. Specifically,

$$f^* = \arg\min_{f \in F} \left(\min_{w \in \mu_F(f) \cap M} (R_f(w) - s_{Fw,f} - R_f(w_m) - \sigma_{w,f}) \right).$$

Let w_M be the majority worker at firm f^* who yields the lowest profit to the firm, i.e., $w_M = \arg\min_{w \in \mu_F(f)} (R_f(w) - s_{Fw,f})$.

We define $t_{w_m \to m_E}$ as the *subsidy required for worker w_m to be hired by firm f^**,[26] given by:

$$t_{w_m \to m_E} \equiv R_{f^*}(w_M) + \sigma_{w_M,f^*} - (R_{f^*}(w_m) + \sigma_{w_m,f^*}).$$

As discussed in the previous section, since w_m is unemployed, there is no firm to compensate for the worker's employment. Therefore, only the firm employing w_M needs to be compensated.

The following theorem provides a sufficient condition on transfers, for a given market, to promote the employment of minority workers and prevent a reduction in the welfare of minority workers at the worker-optimal stable allocation.

Theorem 4. Consider a market $\overline{G} \in \overline{\mathbb{G}}$. If the transfer $\mathbf{T}$ subsidizes minority workers and satisfies the following conditions:

- For each $w \in m_E$, we have $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f}) \geq t_{w_m \to m_E}$;
- For each $w' \in W \setminus (m_E \cup \{w_m\})$ and each $f \in F$, we have $t_{w',f} = 0$;
- For each $f \in F \setminus \{f^*\}$, we have $t_{w_m,f} = 0$, and $t_{w_m,f^*} \geq t_{w_m \to m_E}$.

Then, $\mathbf{T}$ does not reduce the welfare of minority workers under φ_W and promotes the employment of minority workers under both φ_W and φ_F.

Proof. See Section D.9. ■

Theorem 4 states that transfers, in which minority worker employed by a firm receives the highest subsidy within their respective firm (exceeding $t_{w_m \to m_E}$), while w_m receives a subsidy of at least $t_{w_m \to m_E}$ at f^*, do not reduce the welfare of minority workers under φ_W. Moreover, these transfers promote the employment of minority workers under both φ_F and φ_W.[27]

For employed minority workers, the requirement that the largest subsidy be granted by their current employer directly follows from Theorem 2. The proof proceeds similarly to that of

[26]We prove that $t_{w_m \to m_E}$ is sufficient to guarantee that w_m is employed in the proof of Proposition 4.

[27]Unlike in Theorem 3, minority workers already employed by f^* are not treated differently in the transfer. This is because the policy here is solely aimed at promoting minority employment, without imposing any constraints on the firm at which such employment occurs.

Theorem 3, where the transfer is decomposed. Specifically, the salaries of employed minority workers decrease by at most $t_{w_m \to m_E}$; thus, by offering a higher subsidy, the transfer ensures that the welfare of minority workers is not reduced. While the transfer in Theorem 3 is non-uniform—subsidizing employed minority workers at firm f differently—Theorem 4 demonstrates that the transfer can be uniform. This distinction highlights that a uniform transfer, which subsidizes minority workers, and is at least $t_{w_m \to m_E}$, does not reduce the welfare of minority workers and promotes the employment of minority workers under both φ_W and φ_F.

One might argue that a minority worker w_m, receiving a subsidy of $t_{w_m \to m_E}$, would have zero utility in all competitive equilibria with the transfer. However, as shown in Theorem 2, once employed, it is possible to increase w_m's salary without reducing the welfare of other workers. Thus, employment can be promoted while establishing a minimum wage for w_m.

The policymaker may aim to reduce unemployment without imposing restrictions on the welfare of employed minority workers. However, it is essential to ensure that these workers remain employed following the introduction of the transfer.

Proposition 4 provides a sufficient condition on transfers to guarantee the continued employment of minority workers while enabling the hiring of at least one unemployed minority worker by a firm.

Proposition 4. Consider a market $\overline{G} \in \overline{\mathbb{G}}$. If the transfer $\mathbf{T}$ subsidizes minority workers and satisfies the following conditions:

- For each $w \in m_E$, we have $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f}) \geq \max(0, t_{w_m \to m_E} - (s_{Ww,\mu_W(w)} + \sigma_{w,\mu_W(w)}))$;
- For each $w' \in W \setminus (m_E \cup \{w_m\})$ and each $f \in F$, we have $t_{w',f} = 0$;
- For each $f \in F \setminus \{f^*\}$, we have $t_{w_m,f} = 0$, and $t_{w_m,f^*} \geq t_{w_m \to m_E}$.

Then, $\mathbf{T}$ promotes the employment of minority workers under both φ_W and φ_F.

Proof. See Section D.8. ∎

Similar to Proposition 3, the utility of minority workers must remain positive after the introduction of the transfer to ensure that they are employed by a firm.[28]

4.3 Implementation of Objectives, Costs, and Surplus Sharing

For the two objectives presented, it is clear that the transfers discussed can be utilized to achieve both employment and minority worker representation goals. Specifically, it is sufficient to increase the values of $t_{w_m \to f}$ and $t_{w_m \to m_E}$. However, such policies may come with a significant cost. This section demonstrates that these costs can be mitigated when the objective is to reduce unemployment.

[28] Sotomayor (1999) shows that multiple worker-optimal and firm-optimal stable allocations can exist in a given market. However, worker utility remains identical across all worker-optimal stable allocations, and similarly for all firm-optimal stable allocations (Theorem 1 of Sotomayor 1999). Due to the construction of the transfers presented in Proposition 4, some minority workers will have a utility of 0 in the worker-optimal stable allocation. This implies that in some worker-optimal stable allocations, these workers will not be employed. To guarantee that these workers are employed in all worker-optimal stable allocations, it is sufficient to consider strict inequalities.

Interestingly, Theorem 5 establishes the substitution between subsidies and taxes in uniform transfers. It shows that introducing a subsidy of t through a uniform transfer—without taxing majority workers—yields an allocation equivalent to one in which majority workers are taxed by t, while minority workers receive no subsidy.

Theorem 5. Consider a market $\overline{G} \in \overline{\mathbb{G}}$, and two uniform transfers, $\mathbf{T}$ and $\mathbf{T}'$, where $\mathbf{T}$ subsidizes minority workers by t, and $\mathbf{T}'$ taxes majority workers by t'. If $t = |t'|$, then, there exist worker-optimal and firm-optimal stable allocations such that

$$(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = (\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'}) \text{ and } (\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) = (\mu_F^{\mathbf{T}'}, \mathbf{s}_F^{\mathbf{T}'}).$$

Proof. See Section D.4. ■

Given this equivalence, taxes and subsidies are substitutable in uniform transfers when firm revenue functions are additively separable. Corollary 2 complements Theorem 5 by demonstrating that, under uniform transfers, combining taxes and subsidies yields additive effects on optimal stable allocations.

Corollary 2. Consider a market $\overline{G} \in \overline{\mathbb{G}}$, and three uniform transfers $\mathbf{T}, \mathbf{T}', \mathbf{T}''$ with $t \geq 0$, $t' \leq 0$, such that:

- $\mathbf{T}$ subsidizes minority workers by t and taxes majority workers by t'.
- $\mathbf{T}'$ subsidizes minority workers by $t + |t'|$.
- $\mathbf{T}''$ taxes majority workers by $t + |t'|$.

Then, there exist worker-optimal and firm-optimal stable allocations such that:

$$(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = (\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'}) = (\mu_W^{\mathbf{T}''}, \mathbf{s}_W^{\mathbf{T}''}) \text{ and } (\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) = (\mu_F^{\mathbf{T}'}, \mathbf{s}_F^{\mathbf{T}'}) = (\mu_F^{\mathbf{T}''}, \mathbf{s}_F^{\mathbf{T}''}).$$

Theorem 5 and Corollary 2 have practical implications, particularly regarding the cost of affirmative action policies.[29] To lower policy costs, policymakers may prioritize taxation over subsidies within uniform transfers. For instance, a uniform transfer that taxes majority workers by $t_{w_m \to m_E}$ can promote the employment of minority workers. Moreover, the resulting tax revenue can be used to subsidize minority workers—e.g., to establish a minimum wage. However, taxes alone cannot ensure higher wages for minority workers when at least one remains unemployed. The reason is that taxing majority workers raises the relative attractiveness of all minority workers, including the unemployed, thereby increasing intra-group competition. Notably, under uniform transfers, taxes reduce firm profits.

When subsidies are used to reduce unemployment or increase minority representation within firms, the following result shows that part of the surplus generated by the transfer is captured by firms.

[29] Theorem 5 and Corollary 2 do not extend to other demand classes satisfying the gross substitutes condition. Appendix A details the condition under which these results hold.

Proposition 5. Consider a market $\overline{G} \in \overline{\mathbb{G}}$. If the transfer $\mathbf{T}$ subsidizes minority workers and, for some $f \in F$, $|\mu_W^{\mathbf{T}}(f) \cap m| > |\mu_W(f) \cap m|$, then there exists $w \in \mu_W^{\mathbf{T}}(f)$ such that,

$$t_{w,f} > u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) - u_w(\mu_W, \mathbf{s}_W),$$

and at least one of the following holds:

- $V_f(\mu_W^{\mathbf{T}}; \mathbf{s}_W^{\mathbf{T}}, R_f, \mathbf{T}) > V_f(\mu_W; \mathbf{s}_W, R_f, \mathbf{T}_0)$,
- $V_{\mu_W(w)}(\mu_W^{\mathbf{T}}; \mathbf{s}_W^{\mathbf{T}}, R_{\mu_W(w)}, \mathbf{T}) > V_{\mu_W(w)}(\mu_W; \mathbf{s}_W, R_{\mu_W(w)}, \mathbf{T}_0)$.

Proof. See Section D.10. ■

This result highlights that changes in firm composition induced by subsidies increase the profits of some firms. This applies to both policies studied previously. In contrast, in the transfers described in Theorem 2, the surplus generated by the transfer is fully absorbed by minority workers, and firm profits are not increased. In the proof of Proposition 5, we show that if $t_{w,f} = 0$, then at least the firm $\mu_W(w)$ experiences an increase in profit; otherwise, if $t_{w,f} > 0$, it is firm f that benefits.

5 Affirmative Action with more than two Groups

In this section, we relax the restrictions on the number of groups. In practice, populations may consist of more than two groups, with several of them classified as minority groups. An interpretation of the previous sections is that we considered the union of all minority groups and the union of all non-minority groups. While such aggregate treatment is analytically convenient, policymakers may seek to implement group-specific interventions. We therefore extend the framework to examine transfer policies that differentiate between groups, considering a partition $\mathcal{P}$ of the population.

We build on the concepts introduced earlier. A transfer $\mathbf{T}$ *does not reduce the welfare group* P under φ if, for a market $G = (\mathbf{T}_0)$ where $\varphi(\mathbf{T}_0) = (\mu, \mathbf{s})$ and $\varphi(\mathbf{T}) = (\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}})$, the following condition holds for each $w \in P$:

- $u_w(\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}}) \geq u_w(\mu, \mathbf{s})$.

A transfer $\mathbf{T}$ *reduces the welfare of group* P under φ if this condition is violated. A transfer $\mathbf{T}$ *never reduces the welfare of group* P under φ if this condition holds for all markets $G \in \mathbb{G}$; otherwise, $\mathbf{T}$ *may reduce the welfare of group* P. A transfer $\mathbf{T}$ *uniformly subsidizes (taxes) group* P if there exists a $t \geq 0$ ($t \leq 0$) such that, for each $w \in P$ and each $f \in F$, we have $t_{w,f} = t$, and for each $w' \in W \setminus P$, $t_{w',f} = 0$.

Our previous results highlight the importance of implementing uniform transfers to avoid a reduction in welfare. In particular, if a transfer does not subsidize each group uniformly, it may adversely affect some groups under both φ_W and φ_F.

Corollary 3. A transfer $\mathbf{T}$ that uniformly subsidizes group P may reduce the welfare of group P' under φ_W and φ_F with $P' \in \mathcal{P} \setminus \{P\}$.

Theorem 1 and Corollary 3 demonstrate that equal treatment across minority groups is essential to prevent welfare disparities. Unequal subsidies may reduce the welfare of certain groups, highlighting the importance of careful transfer design when multiple groups are involved.

The following corollary complements the characterization in Theorem 1 by showing that if a transfer uniformly taxes one group, it never reduces the welfare of any other group.

Corollary 4. A transfer $\mathbf{T}$ that uniformly taxes group P, never reduces the welfare of group P' under φ_W and φ_F with $P' \in \mathcal{P} \setminus \{P\}$.

Corollary 4 implies that, in the presence of multiple minority groups, uniformly taxing a single group—such as the majority—provides a robust policy instrument for preventing welfare losses among other groups. As established in Theorem 1, uniform transfers that tax majority workers do not reduce the welfare of minority workers. While uniform taxation constitutes a sufficient condition for avoiding welfare reduction across groups, it is not the only class of transfers with this property. For example, a transfer that uniformly subsidizes all groups in $\mathcal{P} \setminus P$ by the same amount also avoids welfare reduction. These results highlight the importance of transfer design in multi-group settings.

6 Discussion and Conclusion

In this section, we examine two policies implemented in the United States, and in Germany that are applications of our model. For each, we highlight potential challenges identified by our results and discuss their policy implications. We then discuss lump-sum taxes in the labor market and the insights provided by our paper.

6.1 United States: *Work Opportunity Tax Credit*

The *Work Opportunity Tax Credit* (WOTC) was introduced in the United States in 1996 to incentivize firms to hire workers from disadvantaged backgrounds. The program primarily targets African Americans and Latinos from low-income households, former inmates, veterans, youth from impoverished neighborhoods, and individuals with disabilities. Firms participating in the program can receive a tax credit of up to \$9,600 per eligible employee. In 2023, a total of 1,988,528 workers benefited from the program, amounting to \$4.7 billion in tax credits—an average of approximately \$2,400 per worker.[30] Notably, the tax credit varies across groups and depends both on individual and firm characteristics. In our framework, we interpret the WOTC as a worker-level subsidy.

Our results show that when subsidies vary across individuals and across firms (e.g., by sector), they may inadvertently reduce the welfare of the very groups they are intended to support—under both the firm-optimal and worker-optimal stable allocations. Moreover, differentiating subsidies across multiple minority groups may reduce their welfare (Corollary 3). To

[30] WOTC statistics for 2023 https://www.cmswotc.com/work-opportunity-tax-credit-statistics-2023/ (Accessed on 01/02/2025).

avoid this unintended consequence while still promoting minority employment, we identify two policy options:

(i) *Implementing a uniform subsidy* for all targeted workers across all firms (Theorem 1), or

(ii) *Imposing a uniform tax* on all other workers across all firms (Theorem 1 and Corollary 4).

If the magnitude of the uniform transfer is sufficiently large, these policies not only avoid harming minority workers but may also reduce their unemployment (Theorem 4).

6.2 Germany: *Eingliederungszuschuss*

The *Eingliederungszuschuss* was introduced as part of Germany's 2002 labor market reform, commonly known as *Hartz IV*. This program aims to facilitate the integration of individuals facing barriers to labor market participation, including the long-term unemployed, persons with disabilities, older workers, and those from ethnic minorities. It incentivizes firms to hire by subsidizing up to 50% of the worker's salary for a period ranging from 12 to 36 months. The amount and duration of the subsidy depend on the worker's age and disability status. In 2023, 1,017,190 workers benefited from this subsidy, totaling €25.4 billion—an average of approximately €24,970 per worker.[31] Although the program is available to all sectors, subsidies are concentrated in industrial, service, and retail sectors, particularly among firms employing low-skilled labor.[32]

As with the WOTC, subsidies under the Eingliederungszuschuss vary by worker and firm characteristics. Moreover, the program is targeted exclusively at unemployed workers. Our results suggest that this design may reduce the welfare of employed minority workers. To prevent such unintended effects, policy should extend subsidies to employed minority workers as well, with transfers at least as large as those offered to unemployed workers (Proposition 2). A further goal of the program is to increase minority representation in specific sectors. However, uniform transfers cannot achieve this. When targeting specific sectors, our results show that employed minority workers must receive a sector-specific subsidy to ensure no welfare loss and to support representation goals (Theorem 3).

6.3 Lump-Sum Taxation

In our framework, taxes are levied directly on firms. This contrasts with the model of Dupuy et al. (2020a), in which taxation applies to transfers between agents—that is, to wages. More specifically, they analyze how taxation affects the *value* of a market allocation, that is, the sum of workers' utilities and firms' profits.[33] Under such taxation, transfers between firms and workers become imperfect, as part of the surplus is captured by a central authority. As in their model, we do not explicitly represent this authority.

Uniform transfers can be interpreted as *lump-sum taxes*: rather than taxing a fixed proportion of the worker's transfer, a fixed amount is deducted per worker. Lump-sum taxes are

[31] Eingliederungszuschuss statistics for 2023 https://www.destatis.de/DE/Themen/Gesellschaft-Umwelt/Soziales/Sozialhilfe/eingliederungshilfe.html (Accessed on 01/02/2025).

[32] A similar initiative, known as the *emplois francs* program, was introduced in France in 2018, aimed at supporting workers from priority urban areas. It provides a financial subsidy of up to €5,000 per worker each year, with a maximum duration of three years.

[33] The value is invariant across stable allocations; only the division of surplus changes.

common in labor markets and may take the form of hiring costs (e.g., employee healthcare contributions) or employment entry costs (e.g., licensing requirements or mandatory training). Theorem 2 of Dupuy et al. (2020a),[34] shows that reducing a lump-sum tax on transfers increases the value of the market allocation. Our analysis complements theirs in two key respects.

First, we study how uniform transfers affect the distribution of welfare across workers. Uniform subsidies, for instance, capture exemptions from hiring costs or other entry barriers to employment.

Second, in markets where firms have additively separable revenue functions, we show that a uniform transfer—whether implemented as a tax on majority workers or as a subsidy to minority workers—of equal magnitude yields the same allocation. In this case, only firms' profits are affected, shedding light on how reductions in total value are distributed among market participants.

6.4 Concluding Remarks

This paper examines the impact of transfer policies on minority workers in the labor market. We show that transfers targeting specific segments of the population can reduce the well-being of minority workers. We characterize transfers that never reduce the well-being of minority workers at both the firm-optimal and worker-optimal stable allocations. Our results highlight the importance of information in the design of transfer policies. When the policymaker knows the market allocation, it is possible to achieve a desired distribution of minority workers and a target level of well-being.

The results are easily transferable to an object allocation setting. In many applications of object allocation, prices are specific to certain agents (Gul et al. 2019). Thus, uniform transfers represent a price adjustment based on the group to which an agent belongs.

We conclude by raising two open questions: First, which types of transfers that tax majority workers can effectively achieve the goals of affirmative action policies? Second, which transfers can minimize the negative impacts on majority workers while enhancing the positive effects on minority workers?

A Additional Insights

In this appendix, we provide further insights on questions that may arise from our main analysis. First, we examine the distinction between transferable utility (TU) and non-transferable utility (NTU) settings. Second, we explore whether the results from Section 4.3 extend to other revenue functions within the class of demand satisfying the gross substitutes condition.

[34]Theorem 2 is presented in Appendix D in the working paper version (Dupuy et al., 2020b). I am grateful to Scott Duke Kominers for highlighting how this result fits into the broader framework developed in the present paper.

A.1 Distinction Between Markets With and Without Transfers

Kojima (2012) establishes the impossibility of designing affirmative action policies that never reduce the welfare of minority agents in a NTU setting. In this section, we highlight a key distinction between NTU and TU frameworks, using the terminology of agents and institutions.

Example 3. Consider a market with two institutions, i_1 and i_2, and three agents, a_1, a_2, and a_3. Each institution has one available position. Agents are evaluated with institution-specific scores as follows:[35]

	a_1	a_2	a_3
i_1	6	4	3
i_2	6	7	5

Table 11: Scores of individuals across institutions.

Agent preferences are:

- a_1: $i_2 \succ i_1$, a_2: $i_1 \succ i_2$, a_3: $i_2 \succ i_1$

Institutions rank agents according to their scores. In the agent-optimal stable assignment, a_2 is assigned to i_1 and a_1 to i_2; in the institution-optimal stable assignment, a_1 is assigned to i_1 and a_2 to i_2.

Now suppose that a_2 and a_3 belong to a disadvantaged group and receive a +2 bonus to their scores at institution i_2. The scores at i_2 become:

	a_1	a_2	a_3
i_2	6	9	7

The agent-optimal assignment now coincides with the institution-optimal one: a_1 is assigned to i_1 and a_2 to i_2. Agent a_2, who was initially assigned to i_1, is now worse off.

Now suppose transfers are allowed between agents and institutions. Prior to the score adjustment, in the agent-optimal stable allocation, a_1 is assigned to i_1 with a compensation of 3, and a_2 to i_2 with a compensation of 2. In the institution-optimal stable allocation, the same assignment prevails, but both agents receive zero compensation.[36] After the score adjustment, both the agent-optimal and institution-optimal stable allocations remain unchanged.

In NTU models, the outside option of an institution has no impact on the welfare of the agents assigned to it. By contrast, under TU settings, each agent is affected by the outside options of other institutions, as these shape the compensation the agent receives. As a result, certain score adjustments may reduce the welfare of minority agents in NTU settings, but not in TU settings, where transfers allow for adjustments.

[35] These scores may represent entrance exam results in a college admissions context or productivities in the labor market.

[36] Throughout this example, we assume that agents have lexicographic preferences where monetary compensation is the primary criterion. When offered compensations are equal, agents rank institutions according to their ordinal preferences.

In particular, Proposition 5 establishes that taxes never reduce the welfare of minority workers under the firm-optimal stable allocation. Suppose that, initially, instead of the scores presented in Table 11, the agents' scores at institution i_2 are as follows:

	a_1	a_2	a_3
i_2	8	7	5

In this case, the agent-optimal and institution-optimal stable assignments coincide: a_2 is assigned to i_1 and a_1 to i_2. By reducing a_1's score at i_2, by 2, we return to the initial scores presented in Table 11. This change reduces the welfare of a_2 under the institution-optimal stable assignment.

In NTU markets, the stable assignment only changes if institutional rankings change. In contrast, under TU models, equilibrium allocations can shift even when the relative rankings remain constant, due to the flexibility allowed by transfers. To illustrate the second point, suppose that in Table 11, only a_3 receives a bonus of $+0.5$ at i_2, which leaves the rankings of all institutions unchanged. In the NTU market, the resulting assignments are identical. However, in the TU setting, under the agent-optimal stable allocation, a_1 is assigned to i_1 with a compensation of 3, and a_2 to i_2 with a compensation of 1.5 (down from 2). The institution-optimal allocation remains unchanged. Thus, even when rankings are unaffected, increasing competition through affirmative action policies can reduce the welfare of some agents.

A.2 Limits to the Substitutability Between Uniform Taxes and Subsidies

In Section 4.3, we show that uniform transfers allow for a substitution between taxes and subsidies when revenue functions are additively separable. In this appendix, we demonstrate that this result does not extend to more general revenue functions within the class of demand functions satisfying the gross substitutes condition. Intuitively, substitution between taxes and subsidies requires that firms' demand sets remain unchanged under either transfer. Under additively separable revenue functions, each worker can be evaluated independently, and the transfer serves as a normalization across groups. For any subset of workers, the marginal contribution of each worker is simply their individual revenue plus the transfer. By contrast, when the marginal revenue generated by workers is lower than their individual revenues, taxes and subsidies may have asymmetric effects. The following example illustrates this asymmetry.

Example 4. Consider a market with a single firm, $F = \{f\}$, and two workers, $W = \{w_1, w_2\}$. Assume that $\sigma_{w,f} = 0$ for each $w \in W$. Let the revenue function be such that $R_f(\{w_1\}) = R_f(\{w_2\}) = x$ and $R_f(\{w_1, w_2\}) = 2x - \varepsilon$, with $x > \varepsilon > 0$. Suppose that $M = \{w_1\}$ and $m = \{w_2\}$, and consider the salary vector $\mathbf{s} = (0, 0)$. It is clear that $D_f(\mathbf{s}, R_f, \mathbf{T}_0) = \{\{w_1, w_2\}\}$.

Now consider two uniform transfers $\mathbf{T}$ and $\mathbf{T}'$ such that $\mathbf{T}$ subsidizes minority workers by $t = x$, and $\mathbf{T}'$ taxes majority workers by $t' = -x$, so that $t = |t'|$. Moreover, for all $w \in W$, we have $R_f(\{w\}) + \sigma_{w,f} + t'_{w,f} \geq 0$. The corresponding demand sets are:

$$D_f(\mathbf{s}, R_f, \mathbf{T}) = \{\{w_1, w_2\}\} \text{ and } D_f(\mathbf{s}, R_f, \mathbf{T}') = \{\{w_2\}\}.$$

The two demand sets differ because the marginal revenue of w_1 under transfer $\mathbf{T}'$ is given by:

$$R_f(\{w_1, w_2\}) + t'_{w_1,f} - (R_f(\{w_2\})) = -\varepsilon.$$

Thus, although both transfers are uniform and of the same magnitude, they yield distinct demand behaviors. This illustrates that when revenue functions exhibit submodularity—i.e., the marginal contribution of a worker decreases in the presence of others—taxes and subsidies may not be interchangeable, even under uniform transfers.

B Algorithms

This section introduces two algorithms that implement the mechanisms discussed in this article. We first present the ascending salary adjustment process, which leads to a firm-optimal stable allocation. We then turn to the descending salary adjustment process with capacity constraints, used when revenue functions are additively separable, as in Section 4. Note that only firms' demand sets are adjusted during the ascending salary adjustment process in the presence of capacity constraints. For simplicity and clarity, we present the algorithms such that the utility decreases by one at each step, while the proposed wage increases by one at each step. This formulation follows the approach used by Kelso and Crawford (1982).

B.1 Ascending Salary Adjustment Processes

This section introduces the *Ascending Salary Adjustment Processes* (Kelso and Crawford, 1982).

Step 0. For each firm $f \in F$, consider an initial vector of salaries $\mathbf{s}_f(0) \equiv (-\sigma_{w,f})_{w\in W}$, and let $W'_f(0) \subset W$ be a subset of workers such that $W'_f(0) \in D_f(\mathbf{s}_f(0); R_f, \cdot)$, i.e., a subset that maximizes the profit of firm f under the current vector of salaries. If there is a tie between workers, a tie-breaking rule is imposed. For each worker $w \in W'_f(0)$, firm f offers the salary $\mathbf{s}_f(0)(w)$ to each worker $w \in W'_f(k)$ (i.e., $-\sigma_{w,f}$).

Step $k \geq 1$. If a worker receives more than one offer, she rejects all offers except her most preferred one, which she tentatively accepts. In case of a tie, a tie-breaking rule is imposed. For each firm f, if a worker w rejects the offer, the salary for w is increased by 1, i.e.,

$$\mathbf{s}_f(k)(w) = \mathbf{s}_f(k-1)(w) + 1.$$

Next, consider the updated vector of salaries $\mathbf{s}_f(k)$, and let $W'_f(k) \subset W$ be the subset of workers such that $W'_f(k) \in D_f(\mathbf{s}_f(k); R_f, \cdot)$, i.e., the subset that maximizes the profit of firm f at step t, given the current vector of salaries. If there is a tie, a tie-breaking rule is imposed. Firm f then offers the updated salary $\mathbf{s}_f(k)(w)$ to each worker $w \in W'_f(k)$.
The algorithm terminates if no worker rejects an offer.

The process continues until at least one offer is rejected. Since the number of firms is finite and each firm's profit cannot be negative, the algorithm terminates in a finite number of steps.

Workers who have not rejected the salary offers accept them and are employed by the firms making the offers. The final outcome is a firm-optimal stable allocation.

B.2 Descending Salary Adjustment Processes

This subsection introduces the *Descending Salary Adjustment Processes* with capacity, with transfer $\mathbf{T}$.

Step 0. For each worker $w \in W$, consider an initial vector of expected utility $\mathbf{u}_w(0) \equiv (R_f(w) + \sigma_{w,f} + t_{w,f})_{f \in F}$. Each worker selects a firm f such that

$$\mathbf{u}_w(0)(f) = \max_{f \in F}(\mathbf{u}_w(0)(f)) \geq 0,$$

i.e., the firm that offers the highest expected utility. If there is a tie between firms, a tie-breaking rule is imposed. Each worker w then proposes a salary of $\mathbf{u}_w(0)(f)$ to that firm.

Step $k \geq 1$. If a firm f receives more proposals than its capacity q_f, it rejects all proposals. For each firm $f \in F$ that rejects all its proposals, all workers $w \in W$ reduce their expected utility at that firm by 1, i.e.,

$$\mathbf{u}_w(k)(f) = \mathbf{u}_w(k-1)(f) - 1.$$

Each worker then selects a firm f such that

$$\mathbf{u}_w(k)(f) = \max_{f \in F}(\mathbf{u}_w(k)(f)) \geq 0.$$

If there is a tie between firms, a tie-breaking rule is imposed. Each worker w proposes a salary of $\mathbf{u}_w(k)(f)$ to that firm.

If no such firm exists, the algorithm terminates.

The algorithm continues while at least one proposal is rejected. Since the number of workers and firms is finite, and utility cannot become negative, the algorithm terminates in a finite number of steps. Firms that have not rejected the salary offers accept them and employ the workers who proposed them. The final outcome is a worker-optimal stable allocation.

C Auxiliary Results

Lemma 1 provides a decomposition of transfers, allowing us to analyze their effects on minority workers' welfare. The intuition is that for a given market $G \in \mathbb{G}$ and a transfer $\mathbf{T}$, the transfer can be decomposed into several matrices whose sum equals $\mathbf{T}$.

Lemma 1. Consider a market $G \in \mathbb{G}$, a transfer $\mathbf{T} \in \mathbb{T}$, and $\mathbb{T}' \subset \mathbb{T}$ such that $\mathbf{T} = \sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i$. Then, there exist worker-optimal and firm-optimal stable allocations in the markets $(\mathbf{T})$ and

$(\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i)$ such that:

$$(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = (\mu_W^{\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i}, \mathbf{s}_W^{\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i}) \text{ and } (\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) = (\mu_F^{\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i}, \mathbf{s}_F^{\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i}).$$

For convenience, we denote by $(\mathbf{T}_0) \xrightarrow{\mathbf{T}} (\mathbf{T})$ the *implementation of the transfer* $\mathbf{T}$ in a market $(\mathbf{T}_0)$, with $(\mathbf{T})$ as the resulting market. By decomposing $\mathbf{T}$, we have $\mathbb{T}' \subset \mathbb{T}$ such that $\mathbf{T} = \sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i$, and

$$(\mathbf{T}_0) \xrightarrow{\mathbf{T}_1} (\mathbf{T}_1) \xrightarrow{\mathbf{T}_2} (\mathbf{T}_1 + \mathbf{T}_2) \xrightarrow{\mathbf{T}_3} \cdots \xrightarrow{\mathbf{T}_l} \left(\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i \right).$$

This decomposition primarily facilitates the analysis of the impact of transfers on workers. For instance, if $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$, and both $\mathbf{T}_1$ and $\mathbf{T}_2$ do not reduce the welfare of minority workers, then $\mathbf{T}$ does not reduce the welfare of minority workers either. Similarly, it follows that Lemma 2 can capture the maximum impact of each transfer.

Lemma 2. Consider a market $G \in \mathbb{G}$, and a transfer $\mathbf{T} \in \mathbb{T}$ that subsidizes minority workers. Then, for each worker $w \in W$,
(i) $u_w(\mu_F, \mathbf{s}_F) - u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \leq \max(\mathbf{T})$, and
(ii) $u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \leq \max(\mathbf{T})$.

Proof. (i) Let $(\mu_F, \mathbf{s}_F)$ be the firm-optimal stable allocation, and consider a transfer $\mathbf{T}$ such that $t_{w,f} \geq 0$ for all (w, f), and $t_{w,f} > 0$ only if $w \in m$. Suppose, for contradiction, that there exists a worker $w \in W$ such that

$$u_w(\mu_F, \mathbf{s}_F) - u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) > \max(\mathbf{T}).$$

Since $\max(\mathbf{T}) \geq 0$ and $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ satisfies individual rationality, it follows that $u_w(\mu_F, \mathbf{s}_F) > \max(\mathbf{T})$.

Because $(\mu_F, \mathbf{s}_F)$ is the firm-optimal stable allocation, and $u_w(\mu_F, \mathbf{s}_F) > 0$, there exists a firm $f \neq \mu_F(w)$ and a salary $s_{w,f}$ such that $u_w(\mu_F, \mathbf{s}_F) = u_w(f, s_{w,f})$. Thus, w belongs to a demand set for both $\mu_F(w)$ and f at salaries $\mathbf{s}_{Fw,\mu_F(w)}$ and $s_{w,f}$, respectively.

When $\mathbf{T}$ is introduced, firm profits increase by at most $\max(\mathbf{T})$ per worker. Furthermore, by the gross substitutes condition, reducing a worker's salary by up to $\max(\mathbf{T})$ does not remove that worker from any firm's demand set.[37]

Thus, at salaries $\mathbf{s}_{Fw,\mu_F(w)} - \max(\mathbf{T})$ and $s_{w,f} - \max(\mathbf{T})$, worker w remains in a demand set for both $\mu_F(w)$ and f. Hence, w's utility can decrease by at most $\max(\mathbf{T})$ after the introduction of $\mathbf{T}$. Therefore,

$$u_w(\mu_F, \mathbf{s}_F) - u_w(\mu_F^{\mathbf{T}}, s_F^{\mathbf{T}}) \leq \max(\mathbf{T}).$$

This concludes the proof of part (i).

[37]Reducing wages weakly expands a firm's demand set (see Kelso and Crawford, 1982; Hatfield and Milgrom, 2005; Echenique, 2012).

(ii) Let $(\mu_W, \mathbf{s}_W)$ be the worker-optimal stable allocation, and consider a transfer $\mathbf{T}$ such that $t_{w,f} \geq 0$ for all (w, f), and $t_{w,f} > 0$ only if $w \in m$. Suppose, for contradiction, that there exists a worker $w \in W$ such that

$$u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) > \max(\mathbf{T}).$$

As in part (i), it follows that $u_w(\mu_W, \mathbf{s}_W) > \max(\mathbf{T})$. By definition of the worker-optimal stable allocation, no worker $w' \in W$ and no firm $\mu_W(w') \neq \mu_W(w)$ exist together with a salary vector $\mathbf{s}$ such that

$$u_{w'}(\mu_W(w), s_{w',\mu_W(w)}) > u_{w'}(\mu_W, \mathbf{s}_W),$$

and

$$V_{\mu_W(w)}(W', \mathbf{s}, R_{\mu_W(w)}, \mathbf{T}_0) \geq V_{\mu_W(w)}(\mu_W, \mathbf{s}_W, R_{\mu_W(w)}, \mathbf{T}_0),$$

for some W' containing w'.

Following similar logic to part (i), after introducing $\mathbf{T}$, each firm's profit increases by at most $\max(\mathbf{T})$ per worker. The firms' demand satisfies the gross substitutes condition. Thus, if the salary of worker w is reduced to $s_{Ww,\mu_W(w)} - \max(\mathbf{T})$, w will remain in the demand set of $\mu_W(w)$, as no worker is subsidized by more than $\max(\mathbf{T})$. Therefore,

$$u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \leq \max(\mathbf{T}),$$

which contradicts the assumption.

This completes the proof. ■

Lemma 3. [Stated as Theorem 5 of Sotomayor (1999).] Consider a market $G \in \mathbb{G}$. There exists a worker-optimal stable allocation $(\mu_W, \mathbf{s}_W)$ if and only if there exists a firm-optimal stable allocation $(\mu_F, \mathbf{s}_F)$ such that $\mu_W = \mu_F$.

Lemma 4 provides the expression for worker utility at worker-optimal stable allocation.

Lemma 4. Consider a market $\overline{G} \in \overline{\mathbb{G}}$. In any worker-optimal stable allocation $(\mu_W, \mathbf{s}_W)$, for each firm $f \in F$ and any pair of workers $w, w' \in \mu_W(f)$, it holds that:

$$R_f(w) - s_{Ww,f} = R_f(w') - s_{Ww',f}.$$

Proof. If $w = w'$, the statement is trivial. Now suppose $w \neq w'$ and, for contradiction, assume that:

$$R_f(w) - s_{Ww,f} \neq R_f(w') - s_{Ww',f}.$$

Without loss of generality, suppose that:

$$R_f(w) - s_{Ww,f} > R_f(w') - s_{Ww',f}.$$

Since $(\mu_W, \mathbf{s}_W)$ is the worker-optimal stable allocation, workers receive their highest possible utility over all stable allocations, there does not exist any worker $w'' \notin \mu_W(f)$ and salary vector $\mathbf{s}'$ such that

$$V_f((\mu_W(f) \cup \{w''\}) \setminus \{w'\}, \mathbf{s}', R_f, \mathbf{T}, \mathbf{q}) \geq V_f(\mu_W, \mathbf{s}_W, R_f, \mathbf{T}, \mathbf{q}),$$

and,

$$u_{w''}(f, s'_{w'',f}) > u_{w''}(\mu_W, \mathbf{s}_W).$$

Hence, there exists a salary vector $\mathbf{s}'$ such that $s'_{w',f} = s_{Ww',f}$ for every $w' \in W \setminus \{w\}$, and $s'_{w,f} > s_{Ww,f}$ under which $(\mu_W, \mathbf{s})$ is not blocked. This contradicts the worker-optimality of $(\mu_W, \mathbf{s}_W)$. Therefore, we must have:

$$R_f(w) - s_{Ww,f} = R_f(w') - s_{Ww',f} \quad \text{for all } w, w' \in \mu_W(f).$$

■

D Proofs

The results are not proven in the order in which they appear in the paper. Instead, we first prove certain results that will be used as building blocks for subsequent proofs.

D.1 Proof of Proposition 1

Proof. Consider a market $G \in \mathbb{G}$, and let $\mathbf{T} \in \mathbb{T}$ be a transfer such that $t_{w,f} < 0$ only if $w \in M$, and $t_{w,f} = 0$ otherwise. The proof proceeds by analyzing the ascending salary adjustment process under $\mathbf{T}$.

At initialization, define for each firm f the salary vector $\mathbf{s}_f(0) = (-(\sigma_{w',f} + t_{w',f}))_{w' \in W}$. By construction, for all $w \in m$ and all f, $\mathbf{s}_f^{\mathbf{T}}(0)(w) = \mathbf{s}_f(0)(w)$, while for some $w' \in M$, $\mathbf{s}_f^{\mathbf{T}}(0)(w') > \mathbf{s}_f(0)(w')$. We have $\mathbf{s}_f^{\mathbf{T}}(0) \geq \mathbf{s}_f(0)$.

Since firms' demand correspondences satisfy the gross substitutes condition, it follows that if $W' \in D_f(\mathbf{s}_f(0), R_f, \mathbf{T}_0)$, with $w \in W'$, then there exists $W'' \in D_f(\mathbf{s}_f^{\mathbf{T}}(0), R_f, \mathbf{T})$ such that $w \in W''$.

Moreover, salaries are normalized by adjusting for the transfers, so that the effective salary offered to each worker w at step 0 is $\mathbf{s}_f^{\mathbf{T}}(0)(w) + t_{w,f}$. Under these adjustments, minority workers receive weakly more offers at each step of the process.

By monotonicity of workers' preferences in salaries, each minority worker attains weakly higher utility under the allocation $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ than under $(\mu_F, \mathbf{s}_F)$. ■

D.2 Proof of Theorem 2

Proof. Consider a market $G \in \mathbb{G}$, and let $\mathbf{T} \in \mathbb{T}$ be a transfer such that $t_{w,f} > 0$ only if $w \in m$, $t_{w,f} \geq 0$ for all (w, f), and for each $w \in m$, $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f})$.

By Lemma 1, $\mathbf{T}$ can be decomposed as

$$\mathbf{T} = \sum_{w \in W} \mathbf{T}_w,$$

where each $\mathbf{T}^w \in \{\mathbf{T}^w\}_{w\in W} \subset \mathbb{T}$ affects only worker w, with $t_{w,f}$ as in $\mathbf{T}$, i.e., for each $f \in F$, $t^w_{w,f} = t_{w,f}$, and all other entries are zero.

Fix $w \in W$, and consider $\mathbf{T}^w$. We first show that under $\mathbf{T}^w$, worker w's utility at the worker-optimal stable allocation weakly increases by at least $t_{w,\mu_W(w)}$.

Since $(\mu_W, \mathbf{s}_W)$ is the worker-optimal stable allocation under $\mathbf{T}_0$, no allocation $(\mu', \mathbf{s}')$ satisfies

$$u_w(\mu', \mathbf{s}') > u_w(\mu_W, \mathbf{s}_W) \quad \text{and} \quad V_{\mu'(w)}(\mu'; \mathbf{s}', R_{\mu'(w)}, \mathbf{T}_0) \geq V_{\mu_W(w)}(\mu_W; \mathbf{s}_W, R_{\mu_W(w)}, \mathbf{T}_0).$$

By stability, the profit of firm $\mu_W(w)$ under $(\mu_W, \mathbf{s}_W)$ cannot be increased without decreasing the utility of some workers.

Introducing $\mathbf{T}^w$ shifts the profit of firm $\mu_W(w)$ by $t_{w,\mu_W(w)}$, yielding

$$V_{\mu_W(w)}(\mu_W; \mathbf{s}_W, R_{\mu_W(w)}, \mathbf{T}^w) = V_{\mu_W(w)}(\mu_W; \mathbf{s}_W, R_{\mu_W(w)}, \mathbf{T}_0) + t_{w,\mu_W(w)}.$$

For each worker $w' \in m$, define the salary vector $\mathbf{s}^{\mathbf{T}^w}$ by:

$$s^{\mathbf{T}^w}_{w',\mu_W(w')} = \begin{cases} s_{Ww',\mu_W(w')} + t_{w',\mu_W(w)} & \text{if } w' = w, \\ s_{Ww',\mu_W(w')} & \text{otherwise.} \end{cases}$$

By construction, we have

$$V_{\mu_W(w)}(\mu_W; \mathbf{s}^{\mathbf{T}^w}, R_{\mu_W(w)}, \mathbf{T}^w) = V_{\mu_W(w)}(\mu_W; \mathbf{s}_W, R_{\mu_W(w)}, \mathbf{T}_0),$$

ensuring that stability is preserved at $\mu_W(w)$ under $(\mu_W, \mathbf{s}^{\mathbf{T}^w})$ with the transfer $\mathbf{T}^w$.

Moreover, since the largest transfer $t_{w,\mu_W(w)}$ is received at $\mu_W(w)$, worker w has no incentive to deviate to another firm. Otherwise, this would contradict that $(\mu_W, \mathbf{s}_W)$ is the worker-optimal stable allocation. Thus, the assignment remains unchanged: $\mu^{\mathbf{T}^w}_W(w) = \mu_W(w)$. Furthermore, in the allocation $(\mu_W, \mathbf{s}^{\mathbf{T}^w})$, worker w's utility increases exactly by $t_{w,\mu_W(w)}$.

Repeating this argument for each $w \in m$, and recalling that $\mathbf{T} = \sum_{w\in W} \mathbf{T}^w$, it follows that under the transfer $\mathbf{T}$, the salary vector $\mathbf{s}^{\mathbf{T}}$ defined by

$$s^{\mathbf{T}}_{w',\mu_W(w')} = s_{Ww',\mu_W(w')} + t_{w',\mu_W(w')}, \text{ for all } w \in W,$$

yields a stable allocation $(\mu_W, \mathbf{s}^{\mathbf{T}})$. Therefore, for each $w \in m$, we have

$$u_w(\mu^{\mathbf{T}}_W, \mathbf{s}^{\mathbf{T}}_W) \geq u_w(\mu_W, \mathbf{s}^{\mathbf{T}}),$$

with an improvement of at least $t_{w,\mu_W(w)}$ and $\mu^{\mathbf{T}}_W = \mu_W$.

■

D.3 Proof of Theorem 1

We begin by showing that a uniform transfer never reduces the welfare of minority workers under both φ_F and φ_W.

Proof. Consider a market $G \in \mathbb{G}$ and a uniform transfer $\mathbf{T} \in \mathbb{T}$ such that for each $w \in m$ and each $f \in F$, $t_{w,f} = t \geq 0$, and for each $w' \in M$ and each $f \in F$, $t_{w',f} = t' \leq 0$. By Lemma 1, we decompose $\mathbf{T}$ into two transfer as follows:

- $\mathbf{T}^m$ is defined such that for each $w \in m$ and each $f \in F$, $t^m_{w,f} = t_{w,f}$, and all other entries are zero;
- $\mathbf{T}^M$ is defined such that for each $w \in M$ and each $f \in F$, $t^M_{w,f} = t_{w,f}$, and all other entries are zero.

We first examine $\mathbf{T}^M$. From Proposition 1, we know that taxing majority workers does not reduce the welfare of minority workers under the firm-optimal stable allocation. Thus, for each $w \in m$, we have

$$u_w(\mu_F^{\mathbf{T}^M}, \mathbf{s}_F^{\mathbf{T}^M}) \geq u_w(\mu_F, \mathbf{s}_F).$$

Consider now the worker-optimal stable allocation $(\mu_W, \mathbf{s}_W)$. By definition, $(\mu_W, \mathbf{s}_W)$ yields the highest utility to each worker among all stable allocations. In particular, for any $w \in W$, no firm $f \in F \setminus \mu_W(w)$ offers a strictly higher salary without decreasing firm profits, that is, there does not exist $f \neq \mu_W(w)$ and $\mathbf{s}$ such that

$$u_w(f, s_{w,f}) > u_w(\mu_W, \mathbf{s}_W),$$

and simultaneously

$$V_f(W', \mathbf{s}, R_f, \mathbf{T}_0) \geq V_f(\mu_W, \mathbf{s}_W, R_f, \mathbf{T}_0),$$

for some W' containing w.

Introducing $\mathbf{T}^M$, we know that for all firms $f \in F$ and all majority workers $w' \in M$, the transfer is $t^M_{w',f} = t' \leq 0$. Since the tax applies to all firms, it follows that for each majority worker, the highest utility w' can achieve at any stable allocation, when employed, is

$$\sigma_{w',\mu_W(w')} + s_{Ww',\mu_W(w')} + t'.$$

As in the proof of Proposition 1, consider a salary vector $\mathbf{s}'$ defined by

$$s^{\mathbf{T}^M}_{w,\mu_W(w)} = s_{Ww',\mu_W(w')} - t^M_{w,\mu_W(w)},$$

for each $w \in W$. Since $t' \leq 0$, it follows that $\mathbf{s}^{\mathbf{T}^M} \geq \mathbf{s}_W$. Because firms' demands satisfy the gross substitutes condition, we know that for each minority worker $w \in m$, w remains in a demand set in $D_{\mu_W(w)}(\mathbf{s}^{\mathbf{T}^M}, R_{\mu_W(w)}, \mathbf{T}^M)$. Therefore, at the worker-optimal stable allocation, for each minority worker $w \in m$,

$$u_w(\mu_W^{\mathbf{T}^M}, \mathbf{s}_W^{\mathbf{T}^M}) \geq u_w(\mu_W, \mathbf{s}_W).$$

We now consider $\mathbf{T}^m$ and the worker-optimal stable allocation. By the construction of $\mathbf{T}^m$, for each $w \in m$, the maximum subsidy is attained at the firm to which w is assigned under μ_W. That is,

$$t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f}).$$

From Theorem 2, it follows immediately that $\mathbf{T}^m$ does not reduce the welfare of minority workers at the worker-optimal stable allocation.

It remains to show that $\mathbf{T}^m$ does not reduce the welfare of minority workers at the firm-optimal stable allocation. Suppose, for contradiction, that there exists $w \in m$ such that

$$u_w(\mu_F, \mathbf{s}_F) > u_w(\mu_F^{\mathbf{T}^m}, \mathbf{s}_F^{\mathbf{T}^m}).$$

If $u_w(\mu_F, \mathbf{s}_F) = 0$, the contradiction is immediate, as individual rationality ensures

$$u_w(\mu_F^{\mathbf{T}^m}, \mathbf{s}_F^{\mathbf{T}^m}) \geq 0,$$

and thus

$$u_w(\mu_F^{\mathbf{T}^m}, \mathbf{s}_F^{\mathbf{T}^m}) \geq u_w(\mu_F, \mathbf{s}_F).$$

Suppose instead that $u_w(\mu_F, \mathbf{s}_F) > 0$. Then, by the optimality of $(\mu_F, \mathbf{s}_F)$, there exists a firm $f \in F$, $f \neq \mu_F(w)$, and a salary vector $\mathbf{s}$ such that

$$u_w(f, s_{w,f}) = u_w(\mu_F, \mathbf{s}_F),$$

and

$$V_f(W'; \mathbf{s}, R_f, \mathbf{T}_0) = V_f(\mu_F; \mathbf{s}_F, R_f, \mathbf{T}_0),$$

for some W' containing w.

From Lemma 2, we can further decompose $\mathbf{T}^m$ into two transfers, $\{\mathbf{T}^m_w, \mathbf{T}^m_{-w}\} \subset \mathbb{T}$, where $\mathbf{T}^m_w$ consists only of the transfers received by w, i.e., for each $f \in F$, $t^m_{w_{w,f}} = t^m_{w,f}$ and zero for other elements, and $\mathbf{T}^m_{-w}$ consists of the transfers received by all other workers, i.e., for each $w' \in W \setminus \{w\}$, for each $f \in F$, $t^m_{-w_{w',f}} = t^m_{w',f}$, and $t_{-w_{w,f}} = 0$.

Consider $(\mathbf{T}_0) \overset{\mathbf{T}^m_w}{\to} (\mathbf{T}^m_w)$. It is direct that firm f can offer a salary of $s^t_{w,f} = s_{w,f} + t$ while

$$V_f(W'; (s^t_{w,f}, \mathbf{s}_{-w}), R_f, \mathbf{T}_0) = V_f(W'; \mathbf{s}, R_f, \mathbf{T}_0).$$

Similarly, $\mu_F(w)$ can offer a salary of $s_{w,\mu_F(w)} + t$ while also preserving its profit. Therefore, the utility of w increases by t at the firm-optimal stable allocation $(\mu_F^{\mathbf{T}^m_w}, \mathbf{s}_F^{\mathbf{T}^m_w})$, that is

$$u_w(\mu_F^{\mathbf{T}^m_w}, \mathbf{s}_F^{\mathbf{T}^m_w}) = u_w(\mu_F, \mathbf{s}_F) + t.$$

We then introduce $\mathbf{T}^m_{-w}$, such that $(\mathbf{T}^m_w) \overset{\mathbf{T}^m_{-w}}{\to} (\mathbf{T}^m)$ since $\mathbf{T}^m_w + \mathbf{T}^m_{-w} = \mathbf{T}^m$. From Lemma 2 (i), we know that the utility of w decreases by at most $\max(\mathbf{T}^m_{-w}) = t$. Hence,

$$u_w(\mu_F^{\mathbf{T}^m}, \mathbf{s}_F^{\mathbf{T}^m}) \geq u_w(\mu_F, \mathbf{s}_F) + t - t = u_w(\mu_F, \mathbf{s}_F),$$

contradicting the assumption. The proof is complete.

■

It remains to show that if a transfer $\mathbf{T}$ is not uniform, then there exists a market in which $\mathbf{T}$ reduces the welfare of minority workers under either φ_F or φ_W. It suffices to demonstrate that $\mathbf{T}$ reduces the welfare of minority workers under φ_W in a single instance.

Proof. Consider a market with one firm, $F = \{f\}$, and two workers, $W = \{w_1, w_2\}$, with $\sigma_{w,f} = 0$ for all $w \in W$. Let the group partition be $\mathcal{P}$ such that all workers are minorities: $m = W$, $M = \{\varnothing\}$. The firm's revenue function is defined such that for any subset $W' \subseteq W$, $R_f(W') = \max_{w \in W'} (R_f(\{w\}))$.

Firm	w_1	w_2
f	x_1	x_2

Table 12: Firm's revenue from employing each worker.

Assume that $\mathbf{T}$ is not uniform, so $t_{w_1,f} \neq t_{w_2,f}$. Without loss of generality, suppose $t_{w_1,f} > t_{w_2,f}$, and further assume $x_2 > x_1$.

Before the implementation of $\mathbf{T}$, the worker-optimal stable allocation is straightforward:

f
w_1, w_2 $s_{w_1} = 0$, $s_{w_2} = x_2 - x_1$

Table 13: Worker-optimal stable allocation without transfer.

Now suppose transfer $\mathbf{T}$ is implemented. A worker-optimal stable allocation with transfer $\mathbf{T}$ is given by:

f
w_1, w_2 $s_{w_1} = 0$, $s_{w_2} = x_2 + t_{w_2,f} - (x_1 + t_{w_1,f})$

Table 14: Worker-optimal stable allocation with transfer $\mathbf{T}$.

Since $t_{w_1,f} > t_{w_2,f}$, it follows directly that

$$x_2 - x_1 > x_2 + t_{w_2,f} - (x_1 + t_{w_1,f}),$$

so that

$$u_{w_2}(\mu_W, \mathbf{s}_W) > u_{w_2}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}).$$

That is, the introduction of $\mathbf{T}$ reduces the welfare of minority worker w_2 under φ_W, completing the proof. ■

D.4 Proof of Theorem 5

Proof. Consider a market $\overline{G} \in \overline{\mathbb{G}}$, and let $\mathbf{T}, \mathbf{T}' \in \mathbb{T}$ be uniform transfers such that:

- $\mathbf{T}$ subsidizes minority workers by t without taxing majority workers, i.e., $t_{w,f} \geq 0$ for all (w, f), and $t_{w,f} > 0$ only if $w \in m$, and
- $\mathbf{T}'$ taxes majority workers by t' without subsidizing minority workers, i.e., $t_{w,f} \leq 0$ for all (w, f), and $t_{w,f} < 0$ only if $w \in M$

such that $t = |t'|$. We consider the construction of the ascending salary adjustment process (Appendix B.1) under both transfers.

Claim 1. At each step k of the ascending salary adjustment process, the set of workers $W_f'(k)$ selected by any firm $f \in F$ is identical under both $\mathbf{T}$ and $\mathbf{T}'$. Moreover, salaries offered to these workers also coincide.

Proof. We implicitly assume that the same tie-breaking rules apply throughout the ascending salary adjustment process. Fix a firm $f \in F$. Since the initial salary vector $\mathbf{s}_f(0)$ is identical in both markets, we compare the firm's initial choice sets under transfers $\mathbf{T}$ and $\mathbf{T}'$. Specifically, we show that the firm selects the same set of workers, that is, $W_f'(0) \in D_f(\mathbf{s}_f(0); R_f, \mathbf{T}, \mathbf{q})$ and $W_f'(0) \in D_f(\mathbf{s}_f(0); R_f, \mathbf{T}', \mathbf{q})$.

Suppose instead that these sets differ. Then there exist workers w and w' such that $w \in W_f'(0)$ for market ($\mathbf{T}$) and $w \notin W_f'(0)$ for market ($\mathbf{T}'$), while $w' \notin W_f'(0)$ for market ($\mathbf{T}$) and $w' \in W_f'(0)$ for market ($\mathbf{T}'$), since for each worker $w \in W$, $R_F(w) + \sigma_{w,f} + t_{w,f} \geq 0$. In this case, firm f's profit from employing these workers must satisfy:

$$R_f(w) + t - \mathbf{s}_f(0)(w) > R_f(w') + t - \mathbf{s}_f(0)(w') \text{ under } (\mathbf{T}),$$

and,

$$R_f(w') + t' - \mathbf{s}_f(0)(w') > R_f(w) + t' - \mathbf{s}_f(0)(w) \text{ under } (\mathbf{T}').$$

Summing both inequalities yields:

$$R_f(w) + t - \mathbf{s}_f(0)(w) + R_f(w') + t' - \mathbf{s}_f(0)(w') > \\ R_f(w') + t - \mathbf{s}_f(0)(w') + R_f(w) + t' - \mathbf{s}_f(0)(w),$$

a contradiction, since $t + t' = 0$.

It follows that $W_f'(0)$ is the same in both markets. Since the set of offers made is identical, workers make the same acceptance and rejection decisions. Therefore, salary vectors evolve identically across both settings. By induction, this argument extends to all subsequent steps.

Moreover, the algorithm continues since for all $f \in F$ and $w \in W$, we have $R_f(w)+\sigma_{w,f}+t_{w,f} \geq 0$. ■

Hence, we conclude that $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) = (\mu_F^{\mathbf{T}'}, \mathbf{s}_F^{\mathbf{T}'})$.

We now consider the worker-optimal stable allocation. By Lemma 3, we know that $\mu_F^{\mathbf{T}} = \mu_W^{\mathbf{T}}$. Since $\mu_F^{\mathbf{T}} = \mu_F^{\mathbf{T}'}$, it follows that $\mu_W^{\mathbf{T}} = \mu_W^{\mathbf{T}'}$. To complete the argument, we show that $\mathbf{s}_W^{\mathbf{T}} = \mathbf{s}_W^{\mathbf{T}'}$.

We rely on the construction of the descending salary adjustment process described in Appendix B.2. We begin by showing that, for each worker, the firm they initially propose to under transfer $\mathbf{T}$ is the same as under transfer $\mathbf{T}'$.

Claim 2. For each $w \in W$, let

- $f = \arg\max_{f\in F}(\mathbf{u}_w(0)(f))$ under transfer $\mathbf{T}$, and
- $f' = \arg\max_{f\in F}(\mathbf{u}_w(0)(f))$ under transfer $\mathbf{T}'$.

Then $f = f'$.

Proof. Suppose for contradiction that there exists $w \in W$ such that $f \neq f'$. Without loss of generality, assume that $f = \arg\max_{f\in F}(\mathbf{u}_w(0)(f))$ under $\mathbf{T}$. Then, under $\mathbf{T}'$, there exists $f^* \in F \setminus \{f\}$ such that

$$R_{f^*}(w) + \sigma_{w,f^*} + t'_{w,f^*} > R_f(w) + \sigma_{w,f} + t'_{w,f}. \tag{1}$$

At the same time, by definition of f under $\mathbf{T}$, we must have

$$R_f(w) + \sigma_{w,f} + t_{w,f} > R_{f^*}(w) + \sigma_{w,f^*} + t_{w,f^*}. \tag{2}$$

Since $t_{w,f} = t_{w,f^*}$ and $t'_{w,f} = t'_{w,f^*}$, inequalities (1) and (2) are directly contradictory. This completes the proof. ■

As a consequence, under both $\mathbf{T}$ and $\mathbf{T}'$, each worker proposes to the same firm at step 0 of the descending salary adjustment process. For notational convenience, let $\mathbf{u}'_w$ denote the expected utility of worker w in market $(\mathbf{T}')$.

It follows that the resulting expected utility difference for each worker $w \in W$, namely

$$\mathbf{u}_w(0) - \mathbf{u}_w(1) = \mathbf{u}'_w(0) - \mathbf{u}'_w(1),$$

is identical across the two markets.

It then follows that workers make proposals to the same firms at every step of the process, by the same argument, until the descending salary adjustment process terminates. Let k denote the final step of the descending salary adjustment process in market $(\mathbf{T}')$. We know that, by construction, the transfer satisfies $t_{w,f} = t$ for all f and $w \in m$ in market $(\mathbf{T})$ and $t'_{w,f} = 0$ in market ($\mathbf{T}'$), while for $w \in M$, $t_{w,f} = 0$ in market ($\mathbf{T}$) and $t'_{w,f} = -t$ in market ($\mathbf{T}'$), we have that, for every worker w and firm $f \in F$,

$$\mathbf{u}_w(0)(f) = \mathbf{u}'_w(0)(f) + t.$$

Since

$$\mathbf{u}_w(0)(f) - \mathbf{u}_w(k)(f) = \mathbf{u}'_w(0)(f) - \mathbf{u}'_w(k)(f),$$

it follows that

$$\mathbf{u}_w(k)(f) = \mathbf{u}'_w(k)(f) + t.$$

Since the assignment is the same under both $\mathbf{T}$ and $\mathbf{T}'$, i.e., $\mu_W^{\mathbf{T}} = \mu_W^{\mathbf{T}'}$, it follows that workers propose to the same firms at each step of the process until it terminates. As a result, the process converges to the same allocation under both transfers.

■

D.5 Proof of Proposition 2

Proof. Consider a market $\overline{G} \in \overline{\mathbb{G}}$, and let $\mathbf{T} \in \mathbb{T}$ be a transfer such that $t_{w,f} > 0$ only if $w \in m$, and $t_{w,f} \geq 0$ for all (w, f). Furthermore, for each $w \in m_E$, we have $t_{w,\mu_F(w)} = \max_{f\in F}(t_{w,f}) \geq \max_{w'\in m_U, f\in F}(t_{w',f})$. From Lemma 1, we decompose $\mathbf{T}$ into two transfers:

- $\mathbf{T}_1$: For each $w \in m_E$ and $f \in F$, we have $t_{1_{w,f}} = t_{w,f}$ and 0 for other elements.
- $\mathbf{T}_2$: For each $w' \in m_U$ and $f \in F$, we have $t_{2_{w',f}} = t_{w',f}$ and 0 for other elements.

It follows that $t_{1_{w,\mu_W(w)}} = \max_{f\in F}(t_{w,f})$. From Theorem 2, we know that for each $w \in m_E$, with $(\mathbf{T}_0) \overset{\mathbf{T}_1}{\to} (\mathbf{T}_1)$, we have:

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) \geq u_w(\mu_W, \mathbf{s}_W) + t_{1_{w,\mu_W(w)}}.$$

We now consider $(\mathbf{T}_1) \overset{\mathbf{T}_2}{\to} (\mathbf{T})$ since $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$. From Lemma 2, the impact of $\mathbf{T}_2$ is bounded by $\max(\mathbf{T}_2)$. Since for each $w \in m_E$, we have $t_{1_{w,\mu_W(w)}} \geq \max(\mathbf{T}_2)$, it follows that

$$u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W, \mathbf{s}_W).$$

■

D.6 Proof of Theorem 3

Proof. Consider a market $\overline{G} \in \overline{\mathbb{G}}$, and let $\mathbf{T} \in \mathbb{T}$ be a transfer such that $t_{w,f} > 0$ only if $w \in m$, and $t_{w,f} \geq 0$ for all (w, f). Furthermore:

- For each $w \in \{w \in m_E : \mu_W(w) \neq f\} \setminus \{w_m\}$, we have $t_{w,\mu_W(w)} = \max_{f'\in F}(t_{w,f'}) \geq t_{w_m \to f}$;
- For each $w \in \{w \in m_E : \mu_W(w) = f\}$, we have $t_{w,f} \geq \max_{f'\in F\setminus\{f\}}(t_{w,f'}) + t_{w_m\to f}$;
- For each $w' \in W \setminus (m_E \cup \{w_m\})$ and each $f \in F$, we have $t_{w',f} = 0$;
- For each $f \in F \setminus \{f\}$, we have $t_{w_m,f} = 0$, and $t_{w_m,f} \geq t_{w_m\to f}$.

We examine the firm-optimal stable allocations $(\mu_F, \mathbf{s}_F)$ and $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$.[38] Recall that $w_M \equiv \arg\min_{w \in \mu_F(f) \cap M} (R_f(w) - s_{Fw_M,f})$. We now show that $\mathbf{T}$ increases the representation of minority workers in firm f through the hiring of w_m.

Claim 3. $\mathbf{T}$ increases the representation of minority workers in firm f under both φ_W and φ_F.

Proof. We show that (i) $\mu_F^{\mathbf{T}}(w_m) = f$, and (ii) $(\mu(f) \cap m) \subset (\mu^{\mathbf{T}}(f) \cap m)$.

Recall that:[39]

$$t_{w_m \to f} \equiv R_f(w_M) - s_{Fw_M,f} - (R_f(w_m) + \sigma_{w_m,f}) + R_{\mu_F(w_m)}(w_m) + \sigma_{w_m,\mu_F(w_m)}$$
$$- \max_{w' \in (W_U \cup \{w_M\}) \setminus \{w_m\}} \left(R_{\mu_F(w_m)}(w') + \sigma_{w',\mu_F(w_m)} - u_{w'}(\mu_F, \mathbf{s}_F) \right).$$

(i) Firm f hires w_m under $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ if the following condition is satisfied:

$$R_f(w_m) + t_{w_m \to f} - s^{\mathbf{T}}_{Fw_m,f} \geq R_f(w_M) - s_{Fw_M,f}. \tag{3}$$

From the stability of $(\mu_F, \mathbf{s}_F)$, we know:

$$R_f(w_M) - s_{Fw_M,f} \geq R_f(w_m) + \sigma_{w_m,f} - u_{w_m}(\mu_F, \mathbf{s}_F), \tag{4}$$

which implies:

$$u_{w_m}(\mu_F, \mathbf{s}_F) \geq R_f(w_m) + \sigma_{w_m,f} - (R_f(w_M) - s_{Fw_M,f}). \tag{5}$$

Now consider the maximum utility that w_m can obtain at her current firm $\mu_F(w_m)$. The surplus generated by w_m at this firm is $R_{\mu_F(w_m)}(w_m) + \sigma_{w_m,\mu_F(w_m)}$. To replace w_m, the firm would need to attract another worker—assumed here to be either unemployed or w_M—offering that worker at least the utility they currently receive. Hence, the best alternative available to $\mu_F(w_m)$ yields:

$$\max_{w' \in (W_U \cup \{w_M\}) \setminus \{w_m\}} \left(R_{\mu_F(w_m)}(w') + \sigma_{w',\mu_F(w_m)} - u_{w'}(\mu_F, \mathbf{s}_F) \right).$$

Therefore, the maximum utility that w_m can extract at $\mu_F(w_m)$ is:

$$R_{\mu_F(w_m)}(w_m) + \sigma_{w_m,\mu_F(w_m)} - \max_{w' \in (W_U \cup \{w_M\}) \setminus \{w_m\}} \left(R_{\mu_F(w_m)}(w') + \sigma_{w',\mu_F(w_m)} - u_{w'}(\mu_F, \mathbf{s}_F) \right). \tag{6}$$

[38] We focus on the firm-optimal stable allocation for two reasons. First, under the worker-optimal stable allocation, all workers employed by a given firm yield the same profit (see Lemma 4), making it impossible to differentiate among them. Second, under the firm-optimal stable allocation, each firm secures its highest achievable profit across all stable outcomes. Consequently, any decrease in a worker's salary implies the worker has a strictly better outside option, either through employment at another firm or by remaining unemployed.

[39] For notational simplicity, we omit the term $u_{w_m}(\mu_F, \mathbf{s}_F)$, which cancels out directly.

Combining inequalities (5) and (6), we obtain:

$$R_{\mu_F(w_m)}(w_m)+\sigma_{w_m,\mu_F(w_m)}-\max_{w'\in(W_U\cup\{w_M\})\setminus\{w_m\}}\Big(R_{\mu_F(w_m)}(w')+\sigma_{w',\mu_F(w_m)}-u_{w'}(\mu_F,\mathbf{s}_F)\Big)\geq$$
$$u_{w_m}(\mu_F,\mathbf{s}_F)\geq R_f(w_m)+\sigma_{w_m,f}-(R_f(w_M)-s_{Fw_M,f})\,. \quad (7)$$

It follows that the transfer $t_{w_m\to f}$ renders firm f indifferent between hiring w_m and retaining w_M. Additionally, w_m's utility under $(\mu_F^{\mathbf{T}},\mathbf{s}_F^{\mathbf{T}})$ is weakly higher than under the original allocation:

$$u_{w_m}(\mu_F^{\mathbf{T}},\mathbf{s}_F^{\mathbf{T}})\geq u_{w_m}(\mu_F,\mathbf{s}_F).$$

Now, substituting $t_{w_m\to f}$ into inequality (3), we obtain:

$$R_{\mu_F(w_m)}(w_m)+\sigma_{w_m,\mu_F(w_m)}-\max_{w'\in(W_U\cup\{w_M\})\setminus\{w_m\}}\Big(R_{\mu_F(w_m)}(w')+\sigma_{w',\mu_F(w_m)}-u_{w'}(\mu_F,\mathbf{s}_F)\Big)\geq u_{w_m}(\mu_F^{\mathbf{T}},\mathbf{s}_F^{\mathbf{T}}). \quad (8)$$

Since $u_{w_m}(\mu_F^{\mathbf{T}},\mathbf{s}_F^{\mathbf{T}})\geq u_{w_m}(\mu_F,\mathbf{s}_F)$, inequality (3) must hold. By optimality of $(\mu_F^{\mathbf{T}},\mathbf{s}_F^{\mathbf{T}})$, we conclude that $\mu_F^{\mathbf{T}}(w_m)=f$.

We now show that even if there exists a minority worker $w\in m_E\setminus\mu_F(f)$, with $w\neq w_m$, who could be hired by firm f at a subsidy strictly less than $t_{w_m\to f}$, this worker remain assigned to $\mu_F(w)$ under $\mu_F^{\mathbf{T}}$. By the stability of $(\mu_F,\mathbf{s}_F)$, there exists no salary vector $\mathbf{s}$ and set of workers $W'\subseteq W$ such that

$$V_f(W';\mathbf{s},R_f,\mathbf{T}_0)>V_f(\mu_F;\mathbf{s}_F,R_f,\mathbf{T}_0),\ \text{and}\ u_w(f,s_{w,f})\geq u_w(\mu_F,\mathbf{s}_F), \quad (9)$$

for $w\in W'$.

Using the additivity and separability of revenue functions, construct a transfer $\mathbf{T}^w$ such that for all $w'\in W\setminus\{w\}$ and all $f'\in F$, we have $t^w_{w',f'}=0$, and $t^w_{w,f'}=t_{w,f'}$. Now, define a salary vector $\mathbf{s}'$ such that, for each $w'\in W$,

$$s'_{w',\mu_F(w')}=s_{Fw',\mu_F(w')}+t^w_{w',\mu_F(w')}.$$

That is, only the salary of w is adjusted, and only by the amount of the transfer.

Since $t_{w,\mu_F(w)}\geq t_{w,f}$ it follows, from (9), that $w\in W'$ for some $W'\in D_{\mu_F(w)}(\mathbf{s}';R_{\mu_F(w)},\mathbf{T}^w)$ and $w\notin W''$ for any $W''\in D_f(\mathbf{s}';R_f,\mathbf{T}^w)$. That is, under the transfer $\mathbf{T}^w$, worker w remains assigned to $\mu_F(w)$. By the additive separability of revenue functions, the same assignment holds under the transfer $\mathbf{T}$. We conclude that, under $\mu_F^{\mathbf{T}}$, only w_m is hired by firm f.

(ii) From Lemma 1, we decompose $\mathbf{T}$ into three transfers:

- $\mathbf{T}_1$, defined such that for each $w\in m_E\setminus\{w_m\}$ and each $f\in F$, $t_{1_{w,f}}=t_{w,f}$, and all other entries are zero;
- $\mathbf{T}_2$, defined such that $t_{2_{w_m,f}}=t_{w_m\to f}$, and all other entries are zero;
- $\mathbf{T}_3$, defined such that $t_{3_{w_m,f}}=t_{w_m,f}-t_{w_m\to f}$, and all other entries are zero.

It is clear that $\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_3=\mathbf{T}$.

Consider $(\mathbf{T}_0) \overset{\mathbf{T}_1}{\to} (\mathbf{T}_1)$. From Theorem 2, the introduction of $\mathbf{T}_1$ increases the utility of every $w \in \mu_W(f) \cap m$ by at least $t_{1_{w,f}}$, while preserving their assignment; that is $\mu_W^{\mathbf{T}_1}(w) = f$.

Next, $(\mathbf{T}_1) \overset{\mathbf{T}_2}{\to} (\mathbf{T}_1 + \mathbf{T}_2)$, Lemma 2 guarantees that the utility of any worker $w \in \mu_W(f) \cap m$ can decrease by at most $t_{w_m \to f}$. However, since the worker-optimal matching ensures that workers in $\mu_W(f) \cap m$ already attain their highest utility at firm f, and because the constructed transfers satisfy $t_{w,f} \geq \max_{f' \in F \setminus \{f\}}(t_{w,f'}) + t_{w_m \to f}$, these workers remain assigned to firm f in $(\mathbf{T}_1 + \mathbf{T}_2)$. Therefore, for each $w \in \mu_W(f) \cap m$, $\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w) = f$. Moreover, from (i), we know that w_m is now assigned to firm f.

When we consider $(\mathbf{T}_1 + \mathbf{T}_2) \overset{\mathbf{T}_3}{\to} (\mathbf{T})$, Theorem 2 ensures that for each $w \in \mu_W^{\mathbf{T}_1+\mathbf{T}_2}(f) \cap m$, we have $\mu_W^{\mathbf{T}}(w) = f$.
From Lemma 3 we know that there exist $\mu_F^{\mathbf{T}}$ and $\mu_W^{\mathbf{T}}$ such that $\mu_F^{\mathbf{T}} = \mu_W^{\mathbf{T}}$, which completes the proof. ■

We now show that the transfer $\mathbf{T}$ does not reduce the welfare of minority workers under φ_W.

Claim 4. $\mathbf{T}$ does not reduce the welfare of minority workers under φ_W.

Proof. We rely on the decomposition introduced in the proof of Claim 3 (ii).

First, consider $(\mathbf{T}_0) \overset{\mathbf{T}_1}{\to} (\mathbf{T}_1)$. Since for each $w \in m_E \setminus \{w_m\}$, $t_{1_{w,\mu_W(w)}} = \max_{f \in F}(t_{1_{w,f}}) \geq t_{w_m \to f}$, it follows from Theorem 2 that

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) \geq u_w(\mu_W, \mathbf{s}_W) + t_{1_{w,\mu_W(w)}} \quad \text{for each } w \in m_E \setminus \{w_m\}.$$

Next, consider $(\mathbf{T}_1) \overset{\mathbf{T}_2}{\to} (\mathbf{T}_1 + \mathbf{T}_2)$. From Lemma 2 (ii), for each $w \in m_E \setminus \{w_m\}$, we have:

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) - u_w(\mu_W^{\mathbf{T}_1+\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_1+\mathbf{T}_2}) \leq t_{w_m \to f}.$$

Hence,

$$u_w(\mu_W^{\mathbf{T}_1+\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_1+\mathbf{T}_2}) \geq u_w(\mu_W, \mathbf{s}_W).$$

In addition, Claim 3 implies that $\mu_F^{\mathbf{T}_1+\mathbf{T}_2}(w_m) = f$, and by Lemma 3, it follows that $\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w_m) = f$ with $u_{w_m}(\mu_W^{\mathbf{T}_1+\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_1+\mathbf{T}_2}) \geq u_{w_m}(\mu_W, \mathbf{s}_W)$.

Finally, consider $(\mathbf{T}_1 + \mathbf{T}_2) \overset{\mathbf{T}_3}{\to} (\mathbf{T})$, since $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$. We know that $\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w_m) = f$, and in $\mathbf{T}_3$, for each $w \in m_E^{\mathbf{T}_1+\mathbf{T}_2}$, we have $t_{3_{w,\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w)}} = \max_{f \in F}(t_{3_{w,f}})$, by Theorem 2, it follows that

$$u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W^{\mathbf{T}_1+\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_1+\mathbf{T}_2}) + t_{3_{w,\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w)}} \geq u_w(\mu_W, \mathbf{s}_W).$$

■

■

D.7 Proof of Proposition 3

Proof. Consider a market $\overline{G} \in \overline{\mathbb{G}}$, and let $\mathbf{T} \in \mathbb{T}$ be a transfer such that $t_{w,f} > 0$ only if $w \in m$, and $t_{w,f} \geq 0$ for all (w, f). Furthermore:

- For each $w \in \{w \in m_E : \mu_W(w) = f\}$, we have $t_{w,f} \geq \max(0, R_f(w_M) - s_{Fw_M,f} - (R_f(w) - s_{Fw,f}))$;
- For each $w' \in W \setminus (m_E \cap (\mu_F(f) \cup \{w_m\}))$, we have for each $f' \in F$, $t_{w',f'} = 0$;
- For each $f \in F \setminus \{f\}$, we have $t_{w_m,f} = 0$, and $t_{w_m,f} \geq t_{w_m \to f}$.

We examine the firm-optimal stable allocations $(\mu_F, \mathbf{s}_F)$ and $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$. From Claim 3, we know that $\mu_F^{\mathbf{T}}(w_m) = f$. It remains to show that for each $w \in m_E \cap \mu_F(f)$, we have $\mu_F^{\mathbf{T}}(w) = f$.

Recall that $w_M \equiv \arg\min_{w \in \mu_F(f) \cap M} (R_f(w) - s_{Fw,f})$. By the construction of $\mathbf{T}$ and the optimality of $(\mu_F, \mathbf{s}_F)$ and $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$, the assignment $\mu_F^{\mathbf{T}}(w) = f$ for each $w \in m_E \cap \mu_F(f)$ holds if the following inequality is satisfied:

$$R_f(w) + t_{w,f} - s^{\mathbf{T}}_{Fw,f} \geq R_f(w_M) - s_{Fw_M,f}. \tag{10}$$

We now verify this condition by considering two possible cases for each such worker $w \in m_E \cap \mu_F(f)$.

- **Case 1:** Suppose $R_f(w_M) - s_{Fw_M,f} > R_f(w) - s_{Fw,f}$. Then, by construction,

  $$t_{w,f} \geq R_f(w_M) - s_{Fw_M,f} - (R_f(w) - s_{Fw,f}).$$

  Since $(\mu_F, \mathbf{s}_F)$ is stable, there is no firm $f' \neq f$ and salary vector $\mathbf{s}'$ such that $u_w(f', s'_{w,f'}) > u_w(\mu_F, \mathbf{s}_F)$ and $V_{f'}(W', \mathbf{s}', R_{f'}, \mathbf{T}_0, \mathbf{q}) \geq V_{f'}(\mu_F, \mathbf{s}_F, R_{f'}, \mathbf{T}_0, \mathbf{q})$ with $w \in W'$.
  Since $\mathbf{T}$ only affects firm f, firm f continues to offer the highest utility to w under the firm-optimal allocation, implying $s^{\mathbf{T}}_{Fw,f} = s_{Fw,f}$. Therefore, inequality (10) is satisfied.
- **Case 2:** Suppose $R_f(w) - s_{Fw,f} \geq R_f(w_M) - s_{Fw_M,f}$. Then, by definition, $t_{w,f} \geq 0$. As in Case 1, the stability of $(\mu_F, \mathbf{s}_F)$ and the fact that $\mathbf{T}$ only affects firm f implies that $s^{\mathbf{T}}_{Fw,f} = s_{Fw,f}$. Hence,

  $$R_f(w) - s_{Fw,f} \geq R_f(w_M) - s_{Fw_M,f},$$

  and inequality (10) again holds.

We conclude that for all $w \in m_E \cap \mu_F(f)$, it must be that $\mu_F^{\mathbf{T}}(w) = f$. Finally, by Lemma 3, there exists $\mu_W^{\mathbf{T}}$ such that $\mu_F^{\mathbf{T}} = \mu_W^{\mathbf{T}}$, which establishes the desired property for φ_W. ■

D.8 Proof of Proposition 4

Proof. Consider a market $\overline{G} \in \overline{\mathbb{G}}$, and let $\mathbf{T} \in \mathbb{T}$ be a transfer described in Proposition 4. We first consider the firm-optimal stable allocation $(\mu_F, \mathbf{s}_F)$, and examine the condition under which the unemployed minority worker w_m is hired by firm f^*. Since w_m is unemployed, we have $u_{w_m}(\mu_F, \mathbf{s}_F) = 0$. At the firm-optimal stable allocation, w_m is hired by f^* if

$$R_{f^*}(w_m) + t_{w,f^*} + \sigma_{w,f^*} \geq R_{f^*}(w_M) - s_{Fw_M,f^*}.$$

where

$$w_M = \arg\min_{w \in \mu_F(f^*) \cap M} (R_{f^*}(w) - s_{Fw,f^*} - R_{f^*}(w_m) - \sigma_{w,f^*}),$$

i.e., w_M is the majority worker generating the lowest profit for firm f^* under $(\mu_F, \mathbf{s}_F)$.

We first establish that subsidizing w_m by $t_{w_m \to m_E}$ induces f^* to hire w_m, as this transfer ensures that w_m generates at least as much profit for f^* as w_M does.

Claim 5. $R_{f^*}(w_m) + t_{w_m \to m_E} + \sigma_{w_m,f^*} \geq R_{f^*}(w_M) - s_{Fw_M,f^*}$.

Proof. By definition,

$$t_{w_m \to m_E} \equiv R_{f^*}(w_M) + \sigma_{w_M,f^*} - (R_{f^*}(w_m) + \sigma_{w_m,f^*}),$$

and hence

$$R_{f^*}(w_m) + t_{w_m \to m_E} + \sigma_{w_m,f^*} = R_{f^*}(w_M) + \sigma_{w_M,f^*}.$$

Since $u_{w_M}(\mu_F, \mathbf{s}_F) = \sigma_{w_M,f^*} + s_{Fw_M,f^*}$ and $(\mu_F, \mathbf{s}_F)$ is individually rational, it follows that $\sigma_{w_M,f^*} \geq -s_{Fw_M,f^*}$, and therefore:

$$R_{f^*}(w_M) + \sigma_{w_M,f^*} \geq R_{f^*}(w_M) - s_{Fw_M,f^*}.$$

■

Next, we show that the utility of w_M under $(\mu_W, \mathbf{s}_W)$ is below the subsidy $t_{w_m \to m_E}$.

Claim 6. $t_{w_m \to m_E} \geq u_{w_M}(\mu_W, \mathbf{s}_W)$.

Proof. Because $(\mu_W, \mathbf{s}_W)$ is stable, and $\mu_W(w_M) = f^* \neq \emptyset$ while $\mu_W(w_m) = \emptyset$, stability implies

$$R_{f^*}(w_M) - s_{Ww_m,f^*} \geq R_{f^*}(w_m) + \sigma_{w_m,f^*}.$$

From the definition of $t_{w_m \to m_E}$, we have

$$R_{f^*}(w_m) + \sigma_{w_m,f^*} = R_{f^*}(w_M) + \sigma_{w_M,f^*} - t_{w_m \to m_E}.$$

Substituting, we get:

$$R_{f^*}(w_M) - s_{Ww_m,f^*} \geq R_{f^*}(w_M) + \sigma_{w_M,f^*} - t_{w_m \to m_E}.$$

Rewriting the right-hand side as:[40]

$$R_{f^*}(w_M) - s_{Ww_m,f^*} + \sigma_{w_M,f^*} + s_{Ww_m,f^*} - t_{w_m \to m_E},$$

we obtain:

$$0 \geq \sigma_{w_M,f^*} + s_{Ww_m,f^*} - t_{w_m \to m_E},$$

[40]This decomposition reflects the redistribution of the surplus generated by w_M at f^*.

and therefore,

$$t_{w_m \to m_E} \geq \sigma_{w_M, f^*} + s_{W w_m, f^*} = u_{w_M}(\mu_W, \mathbf{s}_W).$$

■

The underlying intuition of Claim 6 is that the competition resulting from w_M seeking employment at an alternative firm cannot diminish the utility of existing workers by more than $t_{w_m \to m_E}$, given that the maximum utility that w_M can achieve under any stable allocation is lover than $t_{w_m \to m_E}$. To ensure that workers remain employed by a firm, their utility must remain positive after the introduction of the transfer. We decompose the transfer $\mathbf{T}$ using Lemma 1:

- $\mathbf{T}_1$, defined such that for each $w \in m_E$ and each $f \in F$, $t_{1_{w,f}}$, and all other entires are zero;
- $\mathbf{T}_2$, defined such that $t_{2_{w_m,f^*}} = t_{w_m \to m_E}$, and all other entries are zero;
- $\mathbf{T}_3$, defined such that $t_{3_{w_m,f^*}} = t_{w_m,f^*} - t_{w_m \to m_E}$, and all other entries are zero.

First, consider $(\mathbf{T}_0) \overset{\mathbf{T}_1}{\to} (\mathbf{T}_1)$. In this proof, we consider a strict inequality for the transfer concerning employed minority workers. This does not alter the reasoning and simplifies the construction of the proof. Since for each $w \in m_E$, $t_{1_{w,\mu_W(w)}} = \max_{f \in F}(t_{1_{w,f}}) > \max(0, t_{w_m \to m_E} - (s_{Ww,\mu_W(w)} + \sigma_{w,\mu_W(w)}))$, it follows from Theorem 2 that

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) > t_{w_m \to m_E}.$$

Next, consider $(\mathbf{T}_1) \overset{\mathbf{T}_2}{\to} (\mathbf{T}_1 + \mathbf{T}_2)$. From Lemma 2 (ii), for each $w \in m_E$, we have:

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) - u_w(\mu_W^{\mathbf{T}_1+\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_1+\mathbf{T}_2}) \leq t_{w_m \to m_E}.$$

Hence,

$$u_w(\mu_W^{\mathbf{T}_1+\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_1+\mathbf{T}_2}) > 0.$$

In addition, Claim 5 implies that either $\mu_F^{\mathbf{T}_1+\mathbf{T}_2}(w_m) = f^*$, and by Lemma 3, it follows that $\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w_m) = f^*$.

Finally, consider $(\mathbf{T}_1+\mathbf{T}_2) \overset{\mathbf{T}_3}{\to} (\mathbf{T})$, since $\mathbf{T} = \mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_3$. We know that $\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w_m) = f^*$, and in $\mathbf{T}_3$, for each $w \in m_E^{\mathbf{T}_1+\mathbf{T}_2}$, we have $t_{3_{w,\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w)}} = \max_{f \in F}(t_{3_{w,f}})$, by Theorem 2, it follows that

$$u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W^{\mathbf{T}_1+\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_1+\mathbf{T}_2}) + t_{3_{w,\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w)}} > 0.$$

Since for each $w \in m_E$, $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) > 0$, we know that $\mu_W^{\mathbf{T}}(w) \neq \emptyset$. Furthermore, $\mu_W^{\mathbf{T}_1+\mathbf{T}_2} = \mu_W^{\mathbf{T}}$, and thus $\mu_W^{\mathbf{T}}(w_m) = f^*$. From Lemma 3, we conclude that $\mu_W^{\mathbf{T}} = \mu_F^{\mathbf{T}}$, and thus $\mathbf{T}$ promotes the employment of minority workers under both φ_W and φ_F. ■

D.9 Proof of Theorem 4

Proof. Consider a market $\overline{G} \in \overline{\mathbb{G}}$, and let $\mathbf{T} \in \mathbb{T}$ be a transfer such that $t_{w,f} > 0$ only if $w \in m$, and $t_{w,f} \geq 0$ for all (w, f). Furthermore:

- For each $w \in m_E$, we have $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f}) \geq t_{w_m \to m_E}$;
- For each $w' \in W \setminus (m_E \cup \{w_m\})$ and each $f \in F$, we have $t_{w',f} = 0$;
- For each $f \in F \setminus \{f^*\}$, we have $t_{w_m,f} = 0$, and $t_{w_m,f^*} \geq t_{w_m \to m_E}$.

From Lemma 1, we decompose $\mathbf{T}$ into three transfers:

- $\mathbf{T}_1$, defined such that for each $w \in m_E$ and each $f \in F$, $t_{1_{w,f}} = t_{w,f}$, and all other entries are zero;
- $\mathbf{T}_2$, defined such that $t_{2_{w_m,f^*}} = t_{w_m \to m_E}$, and all other entries are zero;
- $\mathbf{T}_3$, defined such that $t_{3_{w_m,f^*}} = t_{w_m,f^*} - t_{w_m \to m_E}$, and all other entries are zero.

First, consider $(\mathbf{T}_0) \xrightarrow{\mathbf{T}_1} (\mathbf{T}_1)$. Since for each $w \in m_E$, $t_{1_{w,\mu_W(w)}} = \max_{f \in F}(t_{1_{w,f}}) \geq t_{w_m \to m_E}$, it follows from Theorem 2 that

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) \geq u_w(\mu_W, \mathbf{s}_W) + t_{1_{w,\mu_W(w)}} \quad \text{for each } w \in m_E.$$

Next, consider $(\mathbf{T}_1) \xrightarrow{\mathbf{T}_2} (\mathbf{T}_1 + \mathbf{T}_2)$. From Lemma 2 (ii), for each $w \in m_E$, we have:

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) - u_w(\mu_W^{\mathbf{T}_1+\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_1+\mathbf{T}_2}) \leq t_{w_m \to m_E}.$$

Hence,

$$u_w(\mu_W^{\mathbf{T}_1+\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_1+\mathbf{T}_2}) \geq u_w(\mu_W, \mathbf{s}_W).$$

In addition, Claim 5 implies that $\mu_F^{\mathbf{T}_1+\mathbf{T}_2}(w_m) = f^*$, and by Lemma 3, it follows that $\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w_m) = f^*$.

Finally, consider $(\mathbf{T}_1+\mathbf{T}_2) \xrightarrow{\mathbf{T}_3} (\mathbf{T})$, since $\mathbf{T} = \mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_3$. We know that $\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w_m) = f^*$, and in $\mathbf{T}_3$, for each $w \in m_E^{\mathbf{T}_1+\mathbf{T}_2}$, we have $t_{3_{w,\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w)}} = \max_{f \in F}(t_{3_{w,f}})$, by Theorem 2, it follows that

$$u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W^{\mathbf{T}_1+\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_1+\mathbf{T}_2}) + t_{3_{w,\mu_W^{\mathbf{T}_1+\mathbf{T}_2}(w)}} \geq u_w(\mu_W, \mathbf{s}_W).$$

Furthermore, $\mu_W^{\mathbf{T}_1+\mathbf{T}_2} = \mu_W^{\mathbf{T}}$, and thus $\mu_W^{\mathbf{T}}(w_m) = f^*$. From Lemma 3, we conclude that $\mu_W^{\mathbf{T}} = \mu_F^{\mathbf{T}}$, and thus $\mathbf{T}$ promotes the employment of minority workers under both φ_W and φ_F. ■

D.10 Proof of Proposition 5

Proof. Consider a market $\overline{G} \in \overline{\mathbb{G}}$, and let $\mathbf{T} \in \mathbb{T}$ be a transfer such that $t_{w,f} > 0$ only if $w \in m$, and $t_{w,f} \geq 0$ for all (w, f). Suppose there exists a firm $f \in F$ such that $|\mu_W^{\mathbf{T}}(f) \cap m| > |\mu_W(f) \cap m|$. Let $w^* \in m$ be a worker such that $\mu_W(w^*) \neq f$ and $\mu_W^{\mathbf{T}}(w^*) \neq f$ under any worker-optimal stable allocation. Let $f^* \equiv \mu_W(w^*)$.

By Lemma 4, we know that for each $f' \in F$ and for any pair of workers $w, w' \in \mu_W(f')$, the following equality holds:

$$R_{f'}(w) - s_{Ww,f'} = R_{f'}(w') - s_{Ww',f'}.$$

Since $(\mu_W, \mathbf{s}_W)$ is a worker-optimal stable allocation, it follows that:

$$u_{w^*}(\mu_W, \mathbf{s}_W) = R_{f^*}(w^*) + \sigma_{w^*,f^*} - (R_{f^*}(w) - s_{Ww,f^*}),$$

for any $w \in \mu_W(f^*)$.

Because $f \neq f^*$, it must be that:

$$R_f(w') - s_{Ww',f} > R_f(w^*) + \sigma_{w^*,f} - u_{w^*}(\mu_W, \mathbf{s}_W). \tag{11}$$

for any $w' \in \mu_W(f)$. That is, at fixed utility, the profit generated by hiring w^* would be insufficient for firm f.

Now consider a worker-optimal stable allocation with $\mathbf{T}$, $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$. By the same logic, we know that:

$$u_{w^*}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = R_f(w^*) + t_{w^*,f} + \sigma_{w^*,f} - (R_f(w') + t_{w',f} - s_{Ww',f}), \tag{12}$$

for any $w' \in \mu_W^{\mathbf{T}}(f)$.

Rewriting (12), we obtain:

$$R_f(w') + t_{w',f} - s_{Ww',f} = R_f(w^*) + t_{w^*,f} + \sigma_{w^*,f} - u_{w^*}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}). \tag{13}$$

Since $t_{w',f} \geq 0$, it follows that:

$$R_f(w') + t_{w',f} - s_{Ww',f} \geq R_f(w') - s_{Ww',f}.$$

Combining this inequality with (11) and (13) yields:

$$R_f(w^*) + t_{w^*,f} + \sigma_{w^*,f} - u_{w^*}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) > R_f(w^*) + \sigma_{w^*,f} - u_{w^*}(\mu_W, \mathbf{s}_W). \tag{14}$$

Simplifying (14), we obtain:

$$t_{w^*,f} > u_{w^*}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) - u_{w^*}(\mu_W, \mathbf{s}_W). \tag{15}$$

We now distinguish two cases:

- If $u_{w^*}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) > u_{w^*}(\mu_W, \mathbf{s}_W)$, then w^* obtains a higher utility when employed by f under the transfer $\mathbf{T}$. However, by (15), the gain in utility is strictly less than the amount of the transfer. Hence, part of the subsidy $t_{w^*,f}$ is absorbed by firm f.
- If $t_{w^*,f} = 0$, then (15) implies that $u_{w^*}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) < u_{w^*}(\mu_W, \mathbf{s}_W)$. That is, the introduction of $\mathbf{T}$ has intensified competition for workers at f^*, such that w^* can no longer attain the

same utility as under $(\mathbf{T}_0)$. This implies that the per-worker profit of firm f^* has increased:

$$R_{f^*}(w) + t_{w,f^*} - s^{\mathbf{T}}_{Ww,f^*} > R_{f^*}(w') - s_{Ww',f^*},$$

for any $w \in \mu^{\mathbf{T}}_W(f^*)$ and $w' \in \mu_W(f^*)$.

In both cases, either firm f or firm f^* captures a portion of the surplus generated by the transfer. ■

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