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Ibrahima Diarra, Michel Guillard
and Hubert Kempf

CEPS
Centre for Economics at Paris-Saclay

école normale supérieure paris—saclay
université evry val-d'essonne
université PARIS-SACLAY
université PARIS-SACLAY
Maison des Sciences de l'Homme PARIS-SACLAY

Sovereign Defaults and Debt Sustainability: A Joint Analysis

Ibrahima Diarra[*] Michel Guillard[*†]

and Hubert Kempf[‡]

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Abstract

We build a stochastic model of excusable sovereign default which incorporates a simple debt recovery rule. It depends on a single parameter that allows for partial debt recovery. We show that the maximum debt-to-GDP ratio that a country can sustain without defaulting is increasing, nonlinear, and sensitive to the debt-recovery parameter. Using the concept of risky steady state, we study the dynamics of public debt when the default premium is taken into account and offer new definitions of public debt unsustainability. A higher debt recovery parameter increases the fiscal space but worsens the financial position of a borrowing country after a default episode. We show that the estimated debt-recovery parameter is lower for emerging countries than for developed countries.

JEL classification: E44, F34, H6, H62, H63,

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[*] *Université Paris-Saclay, Univ Evry, EPEE, 91025, Evry-Courcouronnes, France.*
[†] *Corresponding author: michel.guillard@univ-evry.fr*
[‡] *Université Paris-Saclay, ENS Paris-Saclay, CEPS, 91190, Gif-sur-Yvette, France.*

1 Introduction.

A stylized fact in historical sovereign defaults data is that default is almost always partial, that is, creditors are able to recover a fraction of the defaulted debt after default.[1] This suggests the existence of a "debt recovery channel" which we define as the link between sovereign defaults, public debt sustainability and the fraction of due debt recovered by lenders after a sovereign default.[2] In this paper we investigate such a channel and show how it affects the dynamics of public debt, its sustainability, and the occurrence of sovereign defaults. Specifically, we show how lenders' expectations of a debt recovery after a potential default contribute to the "snowball effect" related to the default premium included in the interest rate on public debt. Relying on the concept of "excusable default" (see the seminal paper of Grossman and Van Huyck, 1988), we set up a tractable stochastic model of sovereign default with a "debt recovery rule" that allows for partial debt haircuts.[3] We use a simple specification of such a rule which hinges on a unique parameter. Formally, as will be explained later, this parameter is defined as the expected maximum debt recovery rate.

Solving explicitly this model, we show that there is an increasing, nonlinear relationship between the debt recovery parameter and a country's default ratio, namely the maximum debt-to-GDP ratio that can be sustained without default. We show that the default ratio is different from the solvency ratio. The latter corresponds to the maximum debt-to-GDP ratio consistent, in the case of an upper limit to the primary surplus, with the standard transversality condition. The dynamics of public debt is shown to depend on the debt recovery parameter but also on the realizations of the growth shock. Without uncertainty, the issue of hitting the default ratio is irrelevant: public debt is always sustainable as it must meet the no-Ponzi solution.

Given the stochastic nature of the model we use, the non-linear dynamics of public debt is quite complex to address. In order to shed light on this dynamics, we resort to the concept of *risky steady state* (RSS) recently used by Coeurdacier et al. (2011).

[1] See, for instance, Sturzenegger and Zettelmeyer (2008), Cruces and Trebesch (2013), and Arellano et al. (2019).

[2] This refers to what is commonly known as an "haircut". The haircut rate is equal to one minus the fraction of debt-to-GDP recovered by creditors following a sovereign default.

[3] See Edwards, 2015 for empirical evidence supporting the "excusable default" model.

Specifically, we consider the configuration in which agents form their expectations of relevant variables and make decisions knowing the probability distribution of future shocks whereas the realizations of these shocks are equal to their mean values. We show that a RSS debt level does not always exist in this model. In particular, it does not exist under the assumption of a zero debt recovery. It exists only for sufficiently high values of the debt recovery parameter and is always unstable. A related result is that the "snowball effect" of the risk premium on debt dynamics is observable only when the debt recovery parameter is sufficiently high so that there is a RSS. This is against the intuition suggesting that the snowball effect is larger when the risk supported by lenders is higher, that is, when the post-default recovered debt is lower. Still assuming that realizations of shocks are at their mean values, for low values of the debt recovery parameter, default occurs immediately when the debt-to-GDP ratio is above the default ratio.

Building on these results, we introduce a new definition of debt unsustainability: public debt is unsustainable when its trajectory leads to the default ratio at some finite date, assuming that there is no realization of the growth shock higher than the mean. An important result of the paper is that, in practice, the assessment of the sustainability of public debt crucially depends on the value of the debt recovery parameter.

Next, we analyze the post-default dynamics of public debt. We show that there exists a critical value of the debt recovery parameter such that the post-default debt ratio is "sustainable" if this parameter is below this critical value. If not, the ex-post public debt is unsustainable and the defaulting country is led to a new default. This is consistent with what is known in the literature as "serial defaults", that is repetitive defaults.[4]

The definition of debt unsustainability given above allows us to revisit the concept of "fiscal space" introduced by Ghosh et al. (2013). A fiscal space is an indicator of the capacity of a country to uphold bad shocks by means of additional borrowing without defaulting. Given our definition of debt unsustainability, we define the fiscal space as the difference between the actual and RSS debt ratios when this latter exists, or the actual and default ratios when there is no RSS. As both default and RSS ratios depend

[4]On serial defaults and the link with debt recovery, see Asonuma (2016).

positively on the debt recovery parameter, it plays a critical role in the assessment of country fiscal spaces.

Turning to empirical evidence, we assess the magnitude of the debt recovery channel. We show that values of the debt recovery rule parameter can be recovered from historical data on both advanced and emerging countries, but conditionally on the selected value for the ratio of the maximum primary surplus to GDP. Using the different scenarios used by IMF in its *Fiscal Policy and Debt Sustainability Analysis Framework*,[5] we find that these estimated parameters are markedly lower for emerging countries than for advanced countries: on the whole the fiscal spaces for emerging countries are narrower than for advanced countries. Based on these estimates, we compute debt limits and associated fiscal spaces and show that they are much less sensitive than the estimated parameters of the default rule to the scenarios considered for the maximum primary surplus.

Finally, we reproduce the same exercise with a recent and shorter data set in order to reassess the issue of sustainability when the risk-free interest rate is low, possibly lower than the growth rate. In particular, even for high values of the debt recovery parameter, a sovereign default cannot be ruled out as the debt limit and the fiscal space are finite, although the solvency ratio –which corresponds to a more classical definition of sustainability– is infinite in this case: the debt recovery channel is still at work.

Related literature

Willems and Zettelmeyer (2021) provide a recent and up-to-date survey on sovereign debt sustainability which is a useful introduction to this topic. Sturzenegger and Zettelmeyer (2006), Reinhart and Rogoff (2008) and Das et al. (2012) provide a comprehensive survey of historical sovereign defaults and restructurings. In a pioneering work, Sturzenegger and Zettelmeyer (2008) introduce a methodology to compute haircuts on defaulted debt. The haircut is defined as the percentage difference between the present value of old and new debt instruments issued during debt restructuring. Using

[5]See IMF (2011).

data for 14 debt restructurings in 1998-2005, they document average haircuts ranging from 13% to 73%. Cruces and Trebesch (2013) and, more recently, Meyer et al. (2019) use a similar approach to compute haircuts using data on sovereign default events in a larger number of countries and a time period going back to 1815. They find that debt repudiation and debt cancellations (haircuts of, or close to 100%) are the exception rather than the rule.

Following Eaton and Gersovitz (1981), the bulk of theoretical studies on sovereign default address the issue in a strategic framework. Aguiar and Amador (2014) and Mitchener and Trebesch (2021) provide useful surveys on this topic. This literature focus on solving the puzzle of the existence of sovereign debt contracts between fully rational agents when there is no or limited enforcement capacity. The issue is the designing of efficient contracts taking into account the incentive of the sovereign to default. Important references on the subject are Calvo (1988), Cole and Kehoe (2000), Aguiar and Gopinath (2006) and Arellano (2008). The standard assumption in these papers is a full discharge of public debt after default and a sanction by lenders in the form of complete exclusion from financial markets. These assumptions are in contrast with the empirical studies mentioned above and with our work.[6] In particular, we allow for a partial haircut on the defaulted debt and the possibility for the government to reenter the markets after default.

A few recent papers depart from the complete default assumption of early papers in the strategic default paradigm. Yue (2010) develops a model of debt renegotiation with Nash bargaining and complete information. In her setting, the government and creditors bargain to a debt haircut that maximizes the total renegotiation surplus. She shows that the renegotiation outcome affects the expected duration of financial exclusion, and therefore the country's incentive to default. In the same spirit, Benjamin and Wright (2009) and Ghosal et al. (2018) consider a model of debt renegotiation with a dynamic alternating offers framework to analyze the delay observed in some historical

[6] On the assumption of exclusion from financial markets, Gelos et al. (2011) document that, while the average length of exclusion was 4 years in the 1980s, it drops to 2 years during the 1990s. Meyer et al. (2019) note that, in recent period, defaulting countries managed to place bonds quickly post-default. A notable example is Argentina in 2016. The country re-accessed international markets only months after its 7th default.

debt restructurings.[7]

Arellano et al. (2019) emphasize the role of missed payments on debt service preceding sovereign default events. In their setting, each period the sovereign strategically decides whether to fully honor its debt payment or to miss a fraction. The amount of payments missed accumulate as arrears and add to future debt. In their model, the government uses missed payments to inter-temporally transfer resources and to smooth consumption.

Following the seminal paper of Grossman and Van Huyck (1988), a growing strand of the literature takes a different approach and models sovereign defaults as "excusable". Our paper clearly adopts this approach. An "excusable default" excludes any strategic decision by the sovereign to default and is solely associated to identifiable "bad states of the world".[8] Such defaults occur when the government is unable to obtain the necessary funds to refinance its outstanding debt, either by issuing new debt, by decreasing public spending or by raising taxes.[9] In a model of excusable default, Bi (2012) shows that the existence of fiscal limits drastically modify the conditions on the sustainability of debt and contributes to defaults. Ghosh et al. (2013) relate fiscal fatigue to public default and endogenously derive the "debt limit". Assuming that default may occur in one period only, Lorenzoni and Werning (2019) investigate the gradual worsening of public debt position which is due to the presence of long-term debt.

Assuming zero debt recovery (haircut of 100%) by investors in case of a sovereign default, Collard et al. (2015) propose a measure of maximum borrowing for advanced economies. This assumption is at odds with the observations on historical sovereign defaults mentioned before. As we shall see below, it substantially underestimates a country's maximum borrowing, which we find to be a highly non-linear function of (expected) haircut.

Finally, the issue of public debt sustainability has recently been re-examined, taking into account the low risk-free interest rate relative to the growth rate. Blanchard (2019),

[7] See also Sunder-Plassmann (2018), Asonuma and Joo (2020), Dvorkin et al. (2021) and Amador and Phelan (2021).

[8] See Grossman and Van Huyck (1988), p.1088.

[9] Note that sovereign "excusable defaults" are different from "rollover crises" à la Cole and Kehoe (2000), which are driven by sunspot shocks.

Sergeyev and Mehrotra (2020) and Mauro and Zhou (2020) suggest that negative $r - g$ differentials[10] are quite common over the past 200 years and characterize recent years. The authors of these two last papers and Blanchard et al. (2021) nevertheless point to the possibility of abrupt bond yield reversals and subsequent reappearances of public debt sustainability issues.

The remaining of the paper is organized as follows. Section 2 presents the model of economic growth with public debt and financial markets, including a simple yet reasonable debt recovery rule. Section 3 addresses the valuation of public debt and its link with the debt recovery rule. Section 4 analyzes the dynamics of public debt in the presence of stochastic shocks and addresses the issues of unsustainability and fiscal space, showing how the debt recovery rule impinges on these magnitudes. In section 5, using a dataset that covers two groups of countries (advanced and emerging) over the period 1980-2018, we provide estimations of the debt recovery parameter and compute the debt limits and fiscal spaces associated to these estimations. Section 6 concludes.

2 The model.

We consider a small open economy with international financial markets and perfect diversification of risks. Time is discrete $t = 0, 1, 2 \ldots$. In each period t, a quantity Y_t of goods is available and represents the country's GDP. Let $a_t \equiv Y_t / Y_{t-1}$ be the gross rate of growth of output between $t - 1$ and t.[11] We assume that a_t evolves randomly across time and follows a probability law with the following characteristics:

Assumption 1.

1. a_t is an *i.i.d.* random variable with a density function $g(a)$, denoting by $G(a)$ its cumulative distribution function, both defined on the interval $[0, +\infty)$, and $E(a) \equiv \bar{a} < \beta^{-1}$ where $\beta^{-1} = 1 + r$ is the risk-free real gross interest rate;

2. the hazard function $z(a) = \frac{g(a)}{1-G(a)}$ is monotone and non-decreasing.

[10] g refers to the real growth rate of GDP, and r is the real risk-free interest rate.

[11] We will often refer to a_t simply as the growth rate and be more precise when necessary.

Assumption 1.1 makes clear that the productivity follows a random walk and the condition $E(a) < \beta^{-1}$ will guarantee that the long run growth rate is inferior to the risk-free interest rate for this economy.[12] We will relax this assumption in Section 5. Assumptions 1.2 is a regularity assumption which allows us to exclude the possibility of multiple equilibria as it will be made explicit in Section 3.

2.1 Private sector.

We assume that international financial markets allow perfect coverage against risk and therefore investors behave as risk-neutral agents. Consider a one-period maturity security offering – in the absence of default – a promise of one unit of goods in $t + 1$. The price at date t, denoted q_t, of such a security satisfies rational expectations if

$$q_t = \beta E_t h_{t+1}, \tag{1}$$

where h_{t+1} is the fraction of the end-of-period value that will be repaid in a given state of nature in period $t + 1$, with $h_{t+1} = 1$ if there is no default and $h_{t+1} < 1$ in case of default.

2.2 Government.

2.2.1 Fiscal rule and fiscal constraint.

The government generates a sequence of primary fiscal surpluses as fractions of output $\{s_t\}$, representing total taxes collected minus total outlays on government purchases and transfers. A negative value of s_t corresponds to a primary deficit. The government balances its budget by issuing one-period maturity Treasury bonds of facial value 1 at price q_t. The level of debt (which is also the number of bonds emitted in t) is denoted by B_t. In case of default at t, it reimburses a fraction $h_t < 1$ of its debt contracted at

[12] The random walk assumption, already made by Collard et al. (2015), allows an analytical solving of the model. It is often used in studies of other macroeconomic issues. See for example Barro (2006, 2009). On the evidence on the importance of shocks on sovereign defaults, see Sturzenegger and Zettelmeyer (2006) and Cevik and Jalles (2022).

$t-1$, B_{t-1}. The instantaneous government budget constraint writes:

$$q_t B_t = h_t B_{t-1} - s_t Y_t, \tag{2}$$

with $h_t \in [0,1]$. This parameter takes the value of 1 if there is no default in t and a lower value, given by a debt recovery rule, when the government is unable to meet its financial obligations in t and thus defaults.

Following Davig et al. (2011), Bi (2012) and Daniel and Shiamptanis (2013), we assume that the primary surplus s_t increases with the *actually redeemed* debt-to-GDP ratio, up to a limit denoted by $\hat{s}$:

$$s_t = \min\left(\bar{s} + \theta \cdot \left(\frac{h_t B_{t-1}}{Y_t} - \bar{\omega}\right); \hat{s}\right), \tag{3}$$

where $\bar{\omega} \geq 0$ is the long run target for the outstanding debt-to-GDP ratio in period t : B_{t-1}/Y_t. Such a limit to the primary surplus can be justified by the coexistence of tax distortions (leading to a Laffer curve) and inelastic public expenditures.

We make the following assumption:

Assumption 2. *The parameters θ, $\bar{s}$ and $\hat{s}$ satisfy:*

$$\theta > 1 - \beta\bar{a}, \text{ and } \hat{s} > \bar{s} \equiv (1 - \beta\bar{a})\,\bar{\omega}.$$

The presence of the upper bound $\hat{s}$ captures the maximum fiscal effort the government is able to make in order to repay its debt. When the primary surplus has reached its maximum value $\hat{s}$, we refer to this situation as *fiscally constrained* and we will say that the economy is in a *constrained fiscal regime.*[13]

2.2.2 Default and the debt recovery rule.

Default occurs only when the government does not obtain the necessary funds to refinance its outstanding debt. Let us denote by Ω_t^{def} the maximum (face value of) debt which can be redeemed by the Treasury in t: default occurs when $B_{t-1} > \Omega_t^{\text{def}}$. We

[13] Ghosh et al. (2013) assume a smooth transition to a constrained fiscal regime, what they call "fiscal fatigue".

refer to Ω_t^{def} as the "default threshold" for period t. As we will see later, this threshold obtains in equilibrium on the financial markets.

We abstract from specifically studying the bargaining process between the defaulting public borrower and its lenders and consider that it is captured by a simple debt recovery rule, contingent on the level of contractual debt B_{t-1} and on the default threshold Ω_t^{def}. We use the following specification:

$$h_t = \begin{cases} \mathbf{h} \cdot \Omega_t^{\text{def}} / B_{t-1} & \text{if } B_{t-1} > \Omega_t^{\text{def}} \\ 1 & \text{else} \end{cases} \tag{4}$$

with $0 \leq \mathbf{h} \leq 1$.[14]

According to this rule, any realization of the (stochastic) default threshold Ω_t^{def} below the contractual level of debt triggers default and a rescheduling of public debt. This rescheduling is such that the after-default (redeemed) debt level is a fraction of Ω_t^{def}, i.e. $h_t B_{t-1} = \mathbf{h}\Omega_t^{\text{def}}$. Considering the limit case where the overrun is negligible (when Ω_t^{def} is arbitrarily close but inferior to B_{t-1}), $\mathbf{h}$ can be interpreted as the maximum debt recovery rate in a default episode. By extension, $1 - \mathbf{h}$ is the minimal rate of default, or equivalently and loosely speaking, the lowest possible "haircut". This rule displays two important features:

1. This debt recovery rule has the property of ensuring that the government is immediately able to re-enter the bond market as its post-default initial debt is below Ω_t^{def} and the economy functions again according to the set of equations characterizing its dynamics.

2. The possibility of future defaults is not ruled out. Nevertheless the rule allows the defaulting government to withstand adverse shocks in the future. The lower is $\mathbf{h}$, the more room there is to accommodate future adverse shocks.

1. is meant to simplify the analysis of the dynamics and could be relaxed at the cost of cumbersome analytical complexities. 2. is important as it captures the fact that a debt rescheduling is a temporary arrangement. It does not necessarily provide a definitive

[14]Note that, although we use bold notation, $\mathbf{h}$ is a scalar parameter not a vector.

solution to a country's fiscal situation which may worsen due to adverse shocks. Cross-country evidence shows that the ratio of recovered debt to due debt h_t is not unique and markedly differs across countries and circumstances.[15] This evidence is consistent with (4) when considering country-specific values of $\mathbf{h}$. Moreover the realized values of h_t are affected by macroeconomic shocks.

2.2.3 The no-Ponzi condition and the solvency ratio.

The government's budget constraint is subject to a no-Ponzi condition:

$$\lim_{T\to\infty} E_t \beta^T h_{t+T} B_{t+T-1} \leq 0. \tag{5}$$

Using (1) in (2), one gets:

$$\beta E_t h_{t+1} B_t = h_t B_{t-1} - s_t Y_t.$$

Defining $\omega_t \equiv h_t B_{t-1}/Y_t$, and remembering that $a_{t+1} = Y_{t+1}/Y_t$, we obtain:

$$\beta E_t a_{t+1} \omega_{t+1} = \omega_t - s_t, \tag{6}$$

and the no-Ponzi condition (5) is equivalent to:

$$\lim_{T\to\infty} E_t \beta^T \left(\prod_{n=1}^{T} a_{t+n} \right) \omega_{t+T} \leq 0. \tag{7}$$

The no-Ponzi solution is consistent with individual rationality and therefore standard in macro models. In models where the possibility of defaults is a priori excluded, this condition corresponds to a debt sustainability condition. As we shall see below, when taking into account the possibility of defaults and therefore of debt rescheduling, this equivalence does not hold anymore.

Note that ω_t is a stochastic variable which may "jump" in each period according to the growth rate innovation and the possibility of a sovereign default. Using the

[15] See the empirical studies mentioned in Section 1.

definition of ω_t, the fiscal rule (3) rewrites:

$$s_t = \min\left(\bar{s} + \theta \cdot (\omega_t - \bar{\omega})\,;\hat{s}\right). \tag{8}$$

Using (8) and the definition of $\bar{s}$ given in Assumption 2, we obtain from (6) the following dynamic equation for the expected redeemed debt-to-output ratio:

$$E_t \beta a_{t+1} \omega_{t+1} = \begin{cases} (1-\theta)\left(\omega_t - \bar{\omega}\right) + \beta \bar{a} \bar{\omega} & \text{for } \omega_t < \hat{\omega} \\ \omega_t - \hat{s} & \text{for } \omega_t \geq \hat{\omega} \end{cases} \tag{9}$$

with

$$\hat{\omega} \equiv \bar{\omega} + \frac{\hat{s} - \bar{s}}{\theta} > \bar{\omega}, \tag{10}$$

where the last inequality comes from Assumption 2.

Equation (9) makes clear the consequence of a maximum fiscal surplus $\hat{s}$. It creates a kink in the dynamics of expected debt-to-output ratio. If the actually redeemed debt-to-output ratio ω_t is sufficiently low (below $\hat{\omega}$), an increase in the public debt ratio can be partially offset by an increase in the primary surplus ratio s_t. Let us consider a deterministic version of this equation by assuming $a_{t+1} = \bar{a}$. The expected debt ratio is obtained from a linear equation. From Assumption 2, its slope, equal to $(1-\theta)/\beta\bar{a}$, is less than one. When ω_t is above the debt-to-output ratio $\hat{\omega}$ at which the primary surplus ratio reaches its maximum $\hat{s}$, the expected actually redeemed debt ratio is obtained from a linear equation the slope of which, $(\beta\bar{a})^{-1}$, is more than one. Hence the kink at $\hat{\omega}$ creates two (deterministic) steady states, the first of which is $\bar{\omega}_1 = \bar{\omega}$, and the second: $\bar{\omega}_2 = \omega^{\text{sup}}$, with

$$\omega^{\text{sup}} \equiv \frac{\hat{s}}{1 - \beta\bar{a}}. \tag{11}$$

Note that ω^{sup} is equal to the sum of the present and expected discounted primary surpluses (relative to the actual GDP) when they are set at their maximum value. Hence it defines the conventional solvency limit of public debt-to-output ratio in a deterministic environment. It does not depend on the debt recovery parameter. As we will see below this is an important difference with the (equilibrium) default ratio which we find to be very sensitive to the (expected) debt recovery parameter.

When $\omega_t \geq \hat{\omega}$ we obtain from (9):

$$\begin{aligned}\omega_t &= \hat{s} + E_t \beta a_{t+1} \omega_{t+1} \\ &= \omega^{\text{sup}} + \lim_{T\to\infty} E_t \beta^T \left(\prod_{n=1}^{T} a_{t+n} \right) \omega_{t+T},\end{aligned}$$

where the second equality is obtained by iterating the first one, using (11). The no-Ponzi condition (7) implies:

$$\omega_t \leq \omega^{\text{sup}}. \tag{12}$$

This inequality is the solvency condition on government debt in this stochastic environment. In the sequel, we will refer to ω^{sup} as the *solvency ratio* of sovereign debt.

2.3 Market equilibrium.

Let us denote by $b_t \equiv B_t/Y_t$ the level of contractual government debt emitted today relative to GDP at t, and

$$\omega_t^{\text{def}} \equiv \Omega_t^{\text{def}}/Y_t, \tag{13}$$

the "default threshold" for period t as a percentage of GDP. Using these notations and according to (4) default occurs when $b_{t-1} > a_t \omega_t^{\text{def}}$. The market equilibrium is given by the following equations:

$$q_t b_t = \frac{h_t b_{t-1}}{a_t} - \min\left(\bar{s} + \theta \cdot \left(h_t \frac{b_{t-1}}{a_t} - \bar{\omega} \right); \hat{s} \right) \tag{14}$$

$$h_t = \begin{cases} \mathbf{h} \frac{a_t \omega_t^{\text{def}}}{b_{t-1}} & \text{if } b_{t-1} > a_t \omega_t^{\text{def}} \\ 1 & \text{else} \end{cases} \tag{15}$$

$$q_t = \beta E_t h_{t+1}, \tag{16}$$

together with the no-Ponzi condition (7).

Equation (14) is the government budget constraint, obtained by using equations (2) and (3); (15) is the debt recovery rule, and (16) is the pricing equation. Taking the sequence $\left\{\omega_t^{\text{def}}\right\}$ as given, these equations are sufficient to analyze the valuation of public debt and the dynamics of emitted debt-to-output ratio b_t. Of course, the sequence of default ratios $\left\{\omega_t^{\text{def}}\right\}$ is endogenous and ultimately needs to be obtained. We will see below that this sequence is actually deterministic in this setting.

3 Sovereign default and debt recovery.

In this section, we focus on the study of the functioning of this economy in the fiscal constraint regime.[16] Specifically, we suppose that the economy was in a constrained tax regime in $t-1$, remains in this regime in t and will be there in $t+1$. The budget constraint is then written in the following simpler form:

$$q_t b_t = \frac{h_t b_{t-1}}{a_t} - \hat{s}. \tag{17}$$

3.1 Debt valuation.

Assuming that $\omega_{t+1}^{\text{def}}$ is known in t and using (15) the price of public debt (16) rewrites as:

$$q_t = \beta \left[1 - G\left(\frac{b_t}{\omega_{t+1}^{\text{def}}}\right) + \mathbf{h}\frac{\omega_{t+1}^{\text{def}}}{b_t} \int^{b_t/\omega_{t+1}^{\text{def}}} a dG(a) \right]. \tag{18}$$

Notice that the price of bond is a decreasing function of b_t. Lenders include in the price a risk premium linked to the probabilities of expected future defaults, based on the ratio $b_t/\omega_{t+1}^{\text{def}}$, on the probability law of a_t and the debt recovery parameter in case of default. The market value of public debt in t is denoted by $v_t \equiv q_t b_t$. From (18), it

[16]Formally, this leads in particular to neglecting the probability of a shock favorable enough to exit from this regime. Treating this hypothesis more rigorously would require restricting the distribution support of shocks, which would considerably and unnecessarily complicate the analysis (see Guillard and Kempf 2017).

is a function of b_t, parameterized by $\omega^{\text{def}}_{t+1}$ and $\mathbf{h}$:

$$v_t = \beta \left\{ \left[1 - G\left(\frac{b_t}{\omega^{\text{def}}_{t+1}}\right)\right] b_t + \mathbf{h}\omega^{\text{def}}_{t+1} \int^{b_t/\omega^{\text{def}}_{t+1}} a dG(a) \right\} \equiv v\left(b_t; \omega^{\text{def}}_{t+1}, \mathbf{h}\right). \tag{19}$$

The function $v(\cdot)$ is potentially non-monotone. The following proposition formalizes the existence of a unique maximum to this function:

Proposition 1. *Given ω^{def}_{t+1}, under Assumption 1, the market value of debt v_t reaches a unique maximum $v^{\max}_t$ for a quantity of debt $b_t = b^{\max}_t$. Both $v^{\max}_t$ and $b^{\max}_t$ are linearly increasing in ω^{def}_{t+1}: $v^{\max}_t = \beta x_{\mathbf{h}} \omega^{def}_{t+1}$ and $b^{\max}_t = \delta_{\mathbf{h}} \omega^{def}_{t+1}$ where $\delta_{\mathbf{h}}$ is such that*

$$[1 - G(\delta_{\mathbf{h}})] [1 - (1 - \mathbf{h}) \delta_{\mathbf{h}} z(\delta_{\mathbf{h}})] = 0, \tag{20}$$

$z(\delta) = \frac{g(\delta)}{1-G(\delta)}$ being the hazard function and $x_{\mathbf{h}}$ given by

$$x_{\mathbf{h}} = [1 - G(\delta_{\mathbf{h}})] \delta_{\mathbf{h}} + \mathbf{h} \int^{\delta_{\mathbf{h}}} a dG(a). \tag{21}$$

$\delta_{\mathbf{h}}$ and $x_{\mathbf{h}}$ are increasing functions of $\mathbf{h}$, with $0 < x_{\mathbf{h}} \leq \bar{a}$ and $0 < \delta_{\mathbf{h}} \leq +\infty$ for $0 \leq \mathbf{h} \leq 1$.

According to this proposition, the maximum value of public debt $v^{\max}_t$ and the corresponding amount of emitted debt $b^{\max}_t$ are increasing functions of the default ratio $\omega^{\text{def}}_{t+1}$ and the debt recovery parameter $\mathbf{h}$.

The higher the debt recovery parameter $\mathbf{h}$, the higher the maximal market value: Lenders are ready to lend more as they receive more in case of default. Even in the extreme case of no debt recovery ($\mathbf{h} = 0$), lenders are potentially willing to lend to the government, despite complete loss in case of default, because they are compensated by a positive risk premium. In the extreme case of the highest debt recovery parameter ($\mathbf{h} = 1$), the maximum public debt value is equal to the discounted default ratio, that is: $v^{\max}_t = \beta\bar{a}\omega^{\text{def}}_{t+1}$.[17]

[17] Note that, since both $v^{\max}_t$ and $\omega^{\text{def}}_{t+1}$ are expressed in terms of output, the discount rate used is the risk-free real interest rate net of the expected growth rate of output.

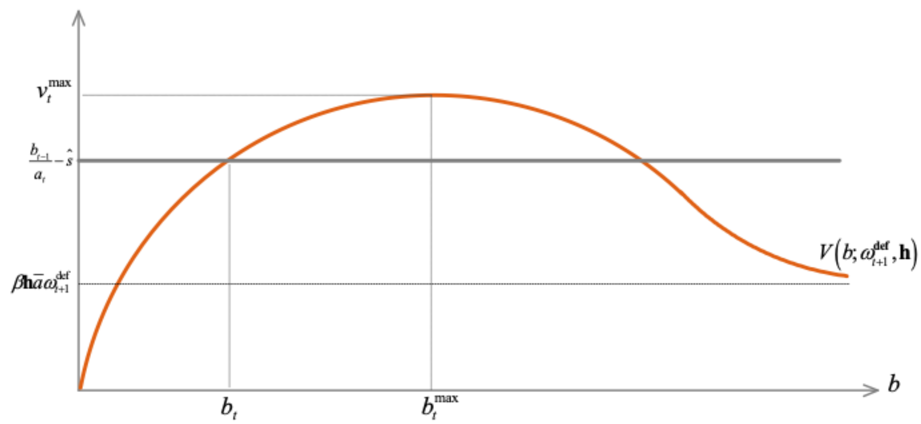

Figure 1: Equilibrium debt valuation.

Figure 1 illustrates this relation for a given value of $\mathbf{h}$ verifying $0 < \mathbf{h} < 1$. For values of b_t below $b_t^{\max}$, the market value of public debt $v_t = q_t b_t$ is increasing in b_t. Above $b_t^{\max}$, the decreasing effect of bond price overcomes the direct effect of increasing debt and makes the public debt value starting to decrease. Because of its "bell"-shaped form, the function $v(\cdot)$ is referred to as the "debt Laffer curve" in the literature (see D'Erasmo et al. 2016, and Lorenzoni and Werning 2019).

An equilibrium debt ratio b_t without default in t is such that (17) holds with $h_t = 1$. The equilibrium displayed in Figure 1 corresponds to the no-default case. For financing needs $b_{t-1}/a_t - \hat{s}$ between $\beta \mathbf{h} \bar{a} \omega_{t+1}^{\text{def}}$ and $v_t^{\max}$, there are two values of b_t which meet this request (as shown in Figure 1). Notice that the equilibrium situated on the decreasing side of the valuation function is "unstable" in the Walrasian sense. In the neighborhood of the high debt equilibrium, in the case of an excess demand a higher bond price increases the gap between demand and supply; the reverse is true in the case of an excess supply.[18] This leads us to select the low debt equilibrium, satisfying $b_t \leq b_t^{\max}$. Excluding the case of default (i.e. assuming $b_{t-1}/a_t \leq \omega_{t+1}^{\text{def}}$), the equilibrium debt-to-output ratio is given by:

$$b_t = \min \left(b \,\middle|\, v\left(b; \omega_{t+1}^{\text{def}}, \mathbf{h}\right) = -\hat{s} + b_{t-1}/a_t \right). \tag{22}$$

[18] Lorenzoni and Werning (2019) develop the same argument and give other reasons justifying the discarding of the "unstable" equilibrium.

3.2 Equilibrium default ratio.

Figure 1 helps us to graphically understand default as a market event. There is default in t when a sufficiently negative shock heightens the horizontal line above the $v\left(b_t;\omega_{t+1}^{\text{def}},\mathbf{h}\right)$ curve, that is, above $v_t^{\max}$. Formally the condition corresponding to default can be written as:

$$\frac{b_{t-1}}{a_t} - \hat{s} > v_t^{\max}. \tag{23}$$

The default condition used in (15) has been defined as: $b_{t-1} > a_t\omega_t^{\text{def}}$. Thus the default ratio ω_t^{def} is necessarily equal to:

$$\omega_t^{\text{def}} = v_t^{\max} + \hat{s}. \tag{24}$$

It is defined as the sum of the maximum value that the government can obtain from the market and the primary surplus of the period.

Since from Proposition 1 we have: $v_t^{\max} = \beta x_{\mathbf{h}}\omega_{t+1}^{\text{def}}$, using (24), we get a dynamic expression for ω_t^{def}:

$$\omega_t^{\text{def}} = \beta x_{\mathbf{h}}\omega_{t+1}^{\text{def}} + \hat{s}. \tag{25}$$

It is a forward-looking equation: how much can at most be redeemed today depends on how much can at most be redeemed tomorrow, because this last one directly determines the opportunities for public funding.

Denoting by $\omega_{\mathbf{h}}$ the stationary solution of (25),the following proposition obtains:

Proposition 2. *The equilibrium default ratio is locally unique and equal to:*

$$\omega_t^{def} = \frac{\hat{s}}{1-\beta x_{\mathbf{h}}} \equiv \omega_{\mathbf{h}},\ \forall t. \tag{26}$$

$\omega_{\mathbf{h}}$ *is a strictly increasing function of* $\hat{s}$ *and* $\mathbf{h}$, with $\omega_{\mathbf{h}} \leq \omega^{\text{sup}}$ for $\mathbf{h} \leq 1$.

Strikingly, even though we reason in a stochastic environment, the default ratio is a constant, $\omega_t^{\text{def}} = \omega_{\mathbf{h}}\,\forall t$, independent from the dynamics of public debt and thus from the history of shocks. We can deduce from Proposition 1 that

$$b_t^{\max} = \delta_{\mathbf{h}}\omega_{\mathbf{h}} \equiv b_{\mathbf{h}}^{\max}, \forall t, \tag{27}$$

and

$$v_t^{\max} = \beta x_{\mathbf{h}} \omega_{\mathbf{h}} \equiv v_{\mathbf{h}}^{\max}, \forall t, \tag{28}$$

which denote respectively the maximum quantity of public bonds in percentage of output that can be emitted and the associated maximum public debt value[19] – again in terms of output – where $\delta_{\mathbf{h}}$ and $x_{\mathbf{h}}$ are given by (20) and (21).

When comparing the value of $\omega_{\mathbf{h}}$ given by equation (26) with $\omega^{\sup}$ given by (11) $\left(\omega^{\sup} \equiv \frac{\hat{s}}{1-\beta\bar{a}}\right)$, we observe that $\beta x_{\mathbf{h}}$ plays in the computation of the default ratio ($\omega_{\mathbf{h}}$) a role similar to $\beta\bar{a}$ in computing the solvency ratio ($\omega^{\sup}$). It thus can be defined as the inverse of the interest factor – when taking into account the partial debt recovery in the case of default – adjusted by the gross rate of growth. It allows to compute the present discounted value of future primary surpluses equal to the default ratio.

From equation (26), we note that, unless $x_{\mathbf{h}}$ is equal to its upper limit $\bar{a}$ corresponding to the case $\mathbf{h} = 1$, the default ratio is lower than the solvency ratio $\omega^{\sup}$. Collard et al. (2015) highlight the same kind of result in the particular case $\mathbf{h} = 0$. Taking into account a positive recovery parameter allows us to generalize their findings while showing the sensitivity of the default ratio to the recovery parameter $\mathbf{h}$. Figure 2 shows the default ratio $\omega_{\mathbf{h}}$ as a function of the (expected) debt recovery parameter $\mathbf{h}$, using a baseline calibration proposed in Section 5.[20]

The default ratio (blue curve) is an increasing, highly nonlinear function of the debt recovery parameter. Recall that when $\mathbf{h} = 1$, the default ratio is equal to the solvency ratio $\omega^{\sup}$, which is evaluated to 238% of GDP (horizontal dash line) with our baseline calibration. As $\mathbf{h}$ moves from 1 to 0.98 the default ratio falls to 197% of GDP and amounts only to 135% at $\mathbf{h} = 0.5$, and 129% when $\mathbf{h} = 0$. The increasing sensitivity of $\omega_{\mathbf{h}}$ to the debt recovery parameter is due to the effect of sovereign risk on debt price: The default premium is decreasing in the debt recovery parameter, the higher $\mathbf{h}$ the lower the prospect of post-default losses and the higher the price of emitted debt. This

[19] What CHR calls, respectively, the maximum sustainable debt (MSD) and the maximum sustainable borrowing (MSB). We prefer to keep the term "sustainable" for another use, proposed in the next section.

[20] To construct Figure 2, we set $\hat{s}$, the maximum primary surplus, to 5%, $\beta = (1+r)^{-1}$ with a risk free rate r equal to 2.93%, and a log-normal distribution for the gross rate of growth, that is: $\ln a \sim N\left(\mu, \sigma^2\right)$ with $\mu = 0.0281$, and $\sigma = 0.0263$. Section 5 provides more details on the choice of parameter values.

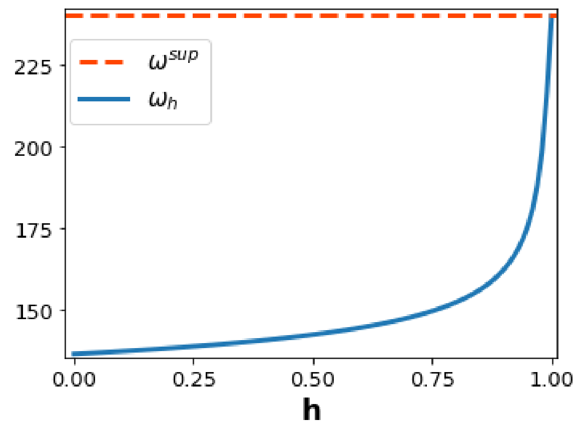

Figure 2: Debt recovery parameter and default ratio

increases the maximum debt value $v_{\mathbf{h}}^{\max}$ and the implied default ratio: $\omega_{\mathbf{h}} = v_{\mathbf{h}}^{\max} + \hat{s}$. The effect of the debt recovery parameter on the default ratio illuminates the debt recovery channel and shows the limitation of assuming no debt recovery, as it is the case in most sovereign default models. It is clear from Figure 2 that such an assumption would substantially underestimate a country's default ratio.

Since this ratio is constant we simplify the notation of the valuation function $\upsilon(b_t; \omega_{\mathbf{h}}, \mathbf{h}) \equiv \upsilon(b_t; \mathbf{h})$. Equation (19) becomes:

$$\upsilon(b_t; \mathbf{h}) = \beta \left\{ \left[1 - G\left(\frac{b_t}{\omega_{\mathbf{h}}} \right) \right] b_t + \mathbf{h}\omega_{\mathbf{h}} \int^{b_t/\omega_{\mathbf{h}}} a dG(a) \right\}. \tag{29}$$

The property of this function is given in the following proposition:

Proposition 3. *The market value of public debt is a strictly increasing function of the debt recovery parameter* $\mathbf{h}$.

This proposition confirms the intuition that lenders expect to be better covered in case of default when the debt recovery parameter increases and thus value more a given amount of public debt.

4 Public debt dynamics and unsustainability.

In this section we study the public debt dynamics when it is subject to market pricing and dependent on the debt recovery rule as explained in the previous section. In general, this dynamics is complex because it depends on many factors: the capacity to implement fiscal adjustments, the recurring shocks hitting the economy and, last but not least, the prospects of haircuts to be applied in case of default. This is true even in the constrained fiscal regime. To overcome this difficulty, we exploit the notion of "Risky Steady State" and offer a new notion of public debt unsustainability in the presence of default. This allows us to reformulate the definition of fiscal space, originally introduced by Ghosh et al. (2013). This notion is central in the management of public debt as it points to the fact that the prospect of default is more or less acute, depending on the capacity of a government to modify its fiscal policy or buffer negative shocks given the probability law governing the relevant random variables. Intuitively, the larger the fiscal space in a given period, the lower the probability of default in the next period. We highlight the impact of the debt recovery rule on the dynamics of public debt and its impact on the fiscal space.

4.1 The dynamics of the public debt.

The debt dynamic process can be formally obtained in our model. Consider a period t where the random variable realization a_t and the debt ratio to be redeemed b_{t-1} are such that no default occurs, that is: $b_{t-1}/a_t < \omega_{\mathbf{h}}$, implying $h_t = 1$. The dynamics of public debt defined by the government budget constraint (17) expressed in the constrained fiscal regime can be written as:

$$b_t = \min\left(b \,|v\left(b;\mathbf{h}\right) = -\hat{s} + b_{t-1}/a_t\right), \tag{30}$$

where the function $v\left(b;\mathbf{h}\right)$ is given by (29).

This formula makes clear that the debt dynamics is stochastic and shifts with the realizations of the productivity shock.

Figure (3) illustrates the dynamics of the public debt for two possible values of

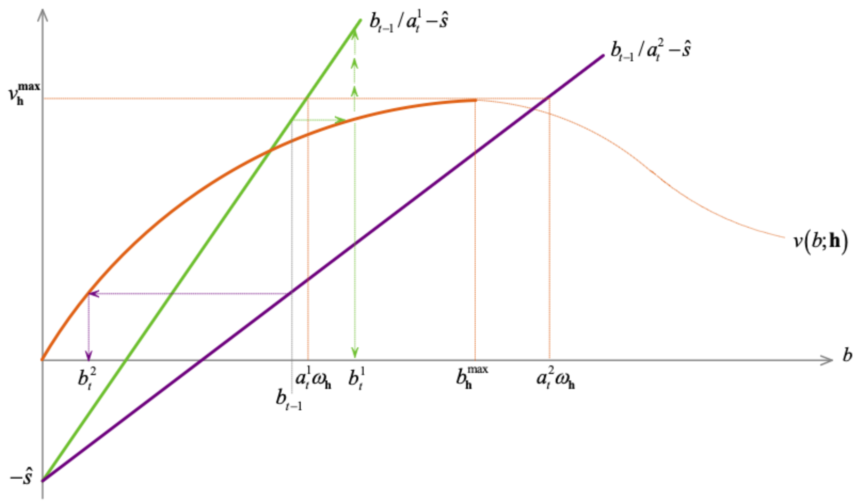

Figure 3: The no-default case dynamics

the realized rate of growth a_t^1 and a_t^2, for which there is no default in t, satisfying: $b_{t-1} < a_t^1\omega_{\mathbf{h}} < a_t^2\omega_{\mathbf{h}}$.

For an initial public debt-to-output ratio b_{t-1}, the straight lines $(b_{t-1}/a_t^1 - \hat{s})$ and $(b_{t-1}/a_t^2 - \hat{s})$ give the government's refinancing requirements in each scenario corresponding to the two states of nature considered. By projecting these values onto the curve $\upsilon\,(b_t;\mathbf{h})$, we get two possible debt-to-output ratios of period t: b_t^1 and b_t^2. For the higher growth rate, a_t^2, the service of the maturing debt b_{t-1}/a_t^2 is low, leading to a reduction of the new emitted debt: $b_t^2 < b_{t-1}$. However this is not so for the lower growth rate a_t^1 and the debt ratio increases: $b_t^1 > b_{t-1}$. Interestingly, even if the growth rate a_t^1 is not low enough to lead to an immediate default, it nevertheless leads to a serious deterioration in the government's financial situation which contributes to a higher default risk premium included in the price of debt. A "snowball effect" comes into play. The increase in a given period t of the amount of emitted debt increases the probability of default and thus the default risk premium. This in turn lowers the price of public bond which increases the quantity of debt to be emitted in the next period for the refinancing of the outstanding debt. This results in a gradual worsening of the financial position of the government. If the same macroeconomic situation is repeated in period $t+1$, *i.e.* $a_{t+1} = a_t^1$, it leads to a sovereign default since the financial needs in $t+1$ now exceed the maximum availability of funds $\upsilon_{\mathbf{h}}^{\max}$.

4.2 The Risky Steady State and the debt recovery rule.

In order to shed more light on the debt dynamics in this stochastic environment, we resort to the concept of "Risky Steady State" (RSS), introduced by Juillard (2011) and Coeurdacier et al. (2011).[21] This concept makes it possible to study the dynamics of public debt by disregarding the realization of shocks but without eliminating the effect of risk on the debt valuation. Let us consider the following

Definition 1. *A Risky Steady State (RSS) is a stationary equilibrium of the dynamic system when the realizations of these shocks are equal to their mean value and agents form their expectations of relevant variables and make decisions on the basis of the probability distribution of future shocks.*

Applying this definition to our problem, the *Risky Steady State* level of debt is the stationary level of the debt-to-output ratio $b_t = b_{t-1}$ in equation (30) with $a_t = \bar{a}$. More precisely, denoting by $b_{\mathbf{h}}^{\text{rss}}$ the RSS-debt-to-output ratio, it is such that:

$$\upsilon\left(b_{\mathbf{h}}^{\text{rss}};\mathbf{h}\right) = \frac{b_{\mathbf{h}}^{\text{rss}}}{\bar{a}} - \hat{s}. \tag{31}$$

The left hand side of (31) represents the market value of debt at the RSS, that is what lenders are willing to lend. The right hand side is the financial needs of the sovereign borrower at the RSS. We formalize the existence of the RSS-debt-to-output ratio in the following

Proposition 4. *In the constrained fiscal regime,*

1. *there exists a unique risky-steady-state-debt ratio, $b_{\mathbf{h}}^{rss}$, satisfying (31) and $b_{\mathbf{h}}^{rss} \leq \bar{a}\omega_{\mathbf{h}} \leq b_{\mathbf{h}}^{max}$, if and only if $\mathbf{h} \geq \underline{\mathbf{h}} = 1 - \frac{1}{\bar{a}z(\bar{a})}$, with strict equalities for $\mathbf{h} = \underline{\mathbf{h}}$.*

2. *When $\mathbf{h} > \underline{\mathbf{h}}$, $b_{\mathbf{h}}^{rss}$ and the difference $b_{\mathbf{h}}^{max} - b_{\mathbf{h}}^{rss}$ are both increasing in $\mathbf{h}$.*

[21] An early reference on this notion is Juillard and Kamenik (2005).

Figure (4) represents the potential existence and determination of the RSS for different values for the recovery parameter: 0, $\underline{\mathbf{h}}$, 1 and a value $\widetilde{\mathbf{h}}$ such that $\underline{\mathbf{h}} < \widetilde{\mathbf{h}} < 1$.

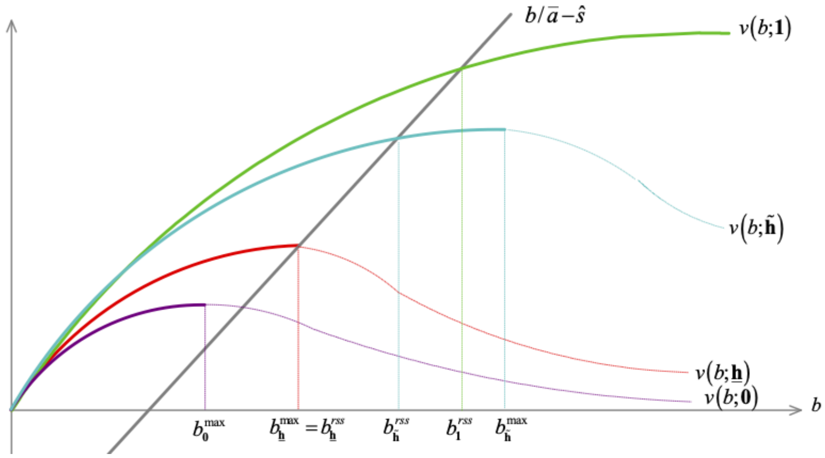

Figure 4: (Non-) Existence of a RSS according to $\mathbf{h}$

A notable result from Proposition 4 is that a RSS does not always exist in this model. Its existence depends on the debt recovery parameter and this parameter must be sufficiently large. In particular a RSS does not exist when $\mathbf{h} = 0$, the case considered for instance by Collard et al. (2015). In this case, and more generally when $\mathbf{h} < \underline{\mathbf{h}}$, defining $b_{\mathbf{h}}^{\max}$ as the debt limit[22] seems to be a good choice for the assessment of public debt sustainability. However, when $\underline{\mathbf{h}} \leq \mathbf{h} \leq 1$ a RSS always exists and it is generally below $b_{\mathbf{h}}^{\max}$. We will propose in Section 4.3 to consider $b_{\mathbf{h}}^{rss}$ as a relevant alternative candidate to define the debt limit ratio in this case.

Given that the value of an emitted public bond is increasing in the debt recovery parameter, the amount of debt which can be rolled over consistent with the RSS is also increasing in $\mathbf{h}$. This explains point 2. of Proposition 4.

Figure 5 illustrates the implied dynamics of the public debt ratio, given by equation (30) when $a_t = \bar{a}$, for the two polar cases $\mathbf{h} = 0$, and $\mathbf{h} = 1$.[23]

[22] That is, using the definition of Ghosh et al. (2013): "the maximum debt level at which the government can rollover its maturing debt and finance the primary deficit at a finite interest rate".

[23] We use the same calibration described in the footnote 20. We limit the scale of the axes for ease of display.

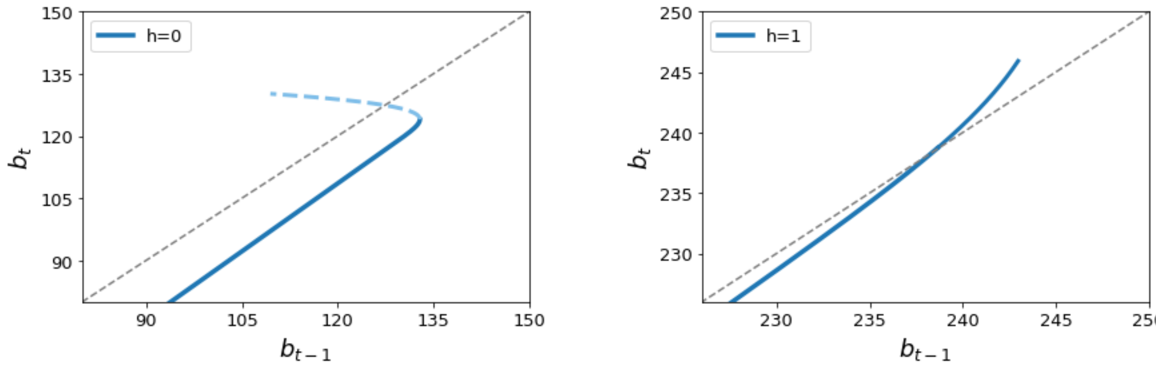

Figure 5: Debt dynamics when $a_t = \bar{a}$.

When $\mathbf{h} = 0$, a turning point of the curve corresponding to the maximum quantity of public debt exists and is below the 45° degree line. Thus the intersection with the 45° degree line does not define a RSS as the part of the curve above the turning point corresponds to the wrong side of the debt Laffer curve and is discarded.
In the other limit case, $\mathbf{h} = 1$, considered for instance by Uribe (2006) and Juessen et al. (2016), there is a RSS but no turning point. The curve is asymptotically vertical and the default ratio is the solvency ratio. In such a configuration, a default makes the post-default indebtedness equal to the solvency ratio. If the post-default value of a_t is at most equal to its mean, this necessarily leads to a renewed default. This captures an extreme case of the feature of serial default.[24]

There is a value of the debt recovery parameter, denoted by $\underline{\mathbf{h}}$, such that the turning point of the curve is exactly on the 45° line. It is the lowest value of $\mathbf{h}$ for which there exists a RSS. For values of $\mathbf{h}$ higher than $\underline{\mathbf{h}}$ but lower than 1, there exists a RSS which is below the solvency ratio. The level of public debt consistent with the RSS is below the maximum debt level $b_{\mathbf{h}}^{\max}$. Lastly, notice that when it exists, a RSS is unstable as the dynamics of public debt is diverging as long as $b_t > b_{\mathbf{h}}^{\max}$ and $a_{t+\tau} \leq \bar{a}$ (for $\tau \geq 0$).

This makes apparent a striking paradox with respect to the snowball effect (as defined above). The intuition is that the snowball effect, understood as the build-up of public debt possibly leading to default, is large when the risk supported by the lenders is high, that is when the post-default recovered debt is low (due to a low recovery

[24]See Reinhart and Rogoff (2004), for instance.

parameter or, loosely speaking, a high haircut). Actually, it happens only when $\mathbf{h}$ is above $\underline{\mathbf{h}}$ and the level of debt is above the RSS: the subsequent debt level is increased and closer to the default ratio (again as long as $a_{t+\tau} \leq \bar{a}$). On the other hand, when $\mathbf{h}$ is below $\underline{\mathbf{h}}$, there is no snowball effect at all: if the due debt level is higher than the level corresponding to the turning point, default is immediate.

4.3 Reassessing unsustainability

Ghosh et al. (2013) define the fiscal space at time t as the difference between the "debt limit", which corresponds to the maximum level of debt $b_{\mathbf{h}}^{\max}$ in the context of our model, and the current debt ratio b_t. Therefore it depends on the minimum debt recovery parameter $\mathbf{h}$. This notion is critical for the management of public debt as it points to the fact that the prospect of default is more or less acute, depending on the capacity of a government to modify its fiscal policy[25] or the capacity to buffer negative shocks given the probability law governing the relevant random variables. The larger the fiscal space, the lower the probability of future default.

However, in line with our discussion in the previous subsection, defining the fiscal space as the difference between $b_{\mathbf{h}}^{\max}$ and the current debt ratio, especially for using it as a criterion of debt sustainability, is of little value when the debt recovery parameter is high and thus a RSS exists. In this case, it is relevant to define the fiscal space as the difference between the RSS debt-to-output ratio $b_{\mathbf{h}}^{\text{rss}}$ and the contemporary debt-to-output ratio b_t. This allows to distinguish two very different situations, depending on whether b_t is below or above $b_{\mathbf{h}}^{\text{rss}}$. In the former case, the fiscal situation can be perilous, especially if the debt level is close to $b_{\mathbf{h}}^{\text{rss}}$, but it is "not critical" in the following sense: if the growth rate is not strictly below its average, the share of debt in GDP should decrease over time. In the latter case, the public debt situation is "critical" given the instability of the RSS: the debt sustainability cannot be taken for granted and default looms in even if the growth rate is equal to its mean.

In order to shed some light on this intuition, we first give an original definition of the unsustainability of public debt:

[25] This is no longer possible in our economy, under the assumption of a constrained fiscal regime.

Definition 2. *A public debt is said to be "unsustainable" at date t when its trajectory reaches the* default ratio *at some finite date, assuming that there is no realization of the (gross) rate of output growth a_{t+s} higher than $\bar{a}$.*

The case of unsustainability refers to the following "non-optimistic" scenario: no future realizations of the shock will be higher than $\bar{a}$. The period t public debt is *"unsustainable"* since, under this scenario, a market-triggered default will unavoidably occur in the future.[26]

This calls for the redefinition of the notion of "debt limit". When there exists a RSS ($\mathbf{h}$ above $\underline{\mathbf{h}}$), trespassing this level implies that public debt is unsustainable and leads to future default (assuming that $a_t = \bar{a}$). Thus the RSS should be considered as the debt limit. When it does not exist ($\mathbf{h}$ below $\underline{\mathbf{h}}$), the debt limit is logically the maximum level of debt. Thus we propose the following

Definition 3. *The debt limit and the fiscal space denoted by FS_t in period t are respectively defined as:* $b_{\mathbf{h}}^{\text{lim}} = \min\left(b_{\mathbf{h}}^{\text{max}}, b_{\mathbf{h}}^{\text{rss}}\right)$ and $FS_t = b_{\mathbf{h}}^{\text{lim}} - b_t$.

As we have just shown that the maximum debt-to-gdp ratio $b_{\mathbf{h}}^{\text{max}}$ and the risky steady state $b_{\mathbf{h}}^{\text{rss}}$ are both increasing functions of the recovery parameter $\mathbf{h}$, so is the fiscal space FS_t. This comes directly from Proposition 3 and the fact that the value of public debt is increasing in $\mathbf{h}$.

Figure 6 represents $b_{\mathbf{h}}^{\text{max}}$ and $b_{\mathbf{h}}^{\text{rss}}$, and implicitly the debt limit $b_{\mathbf{h}}^{\text{lim}} = \min\left(b_{\mathbf{h}}^{\text{max}}, b_{\mathbf{h}}^{\text{rss}}\right)$, with the basic calibration already used for Figures 2 and 5.

We shall see in the next section how this dual definition of the debt limit can be used in empirical analyses to shed light on the public finance positions of different countries, both advanced and emerging.

In line with Definition 2, a worrisome case is when, in the event of a default, the post-default debt ratio is unsustainable. The following proposition establishes that this outcome is possible when the recovery parameter is sufficiently high:

[26] Symmetrically we could said that a public debt is "sustainable" at date t when its trajectory does not reach the default ratio at any future date, assuming that there is no realization of the (gross) rate of output growth a_{t+s} lower than $\bar{a}$. It is a very weak definition of sustainability given the very optimistic nature of the considered scenario.

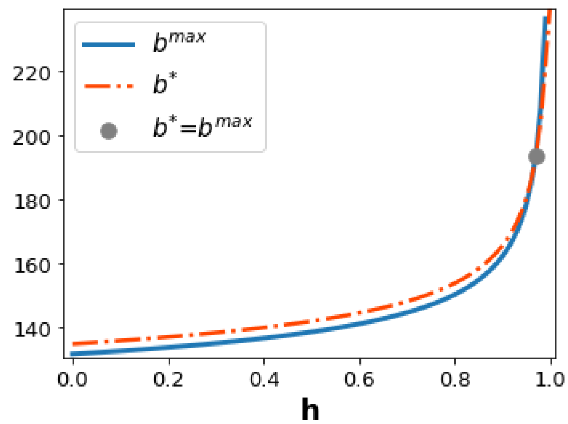

Figure 6: Debt limit: $\min(b_{\mathbf{h}}^{\max}, b_{\mathbf{h}}^{\text{rss}})$

Proposition 5. *When a public default has occurred, the post-default debt-to-GDP ratio* $\mathbf{h}\omega_{\mathbf{h}}$ *is unsustainable if the debt recovery parameter* $\mathbf{h}$ is above a critical value $\mathbf{H}$: $\mathbf{h} > \mathbf{H} > \underline{\mathbf{h}}$, *where* $\mathbf{H}$ *is implicitly defined by:*

$$\mathbf{H}\omega_{\mathbf{H}} = \frac{b_{\mathbf{H}}^{rss}}{\bar{a}}.$$

When $\mathbf{h} > \mathbf{H}$, the post-default debt ratio $\mathbf{h}\omega_{\mathbf{h}}$ is superior to the level $b_{\mathbf{h}}^{\text{rss}}/\bar{a}$ that makes it possible to maintain the debt ratio at its RSS level at the next period when the realization of the shock a_{t+1} is equal to its mean value $\bar{a}$. In other words, according to Definition 2, public debt is unsustainable. In such a situation, except in the case where a very favorable macroeconomic shock allows the economy to leave the zone of unsustainability, the economy could suffer a series of repeated defaults, i.e. serial defaults. Post default, a higher value of the recovery parameter $\mathbf{h}$ increases the debt burden. Above the threshold value $\mathbf{H}$, this burden is so high that public debt becomes unsustainable. This is in stark contrast with the ex ante perspective adopted in the previous sub-sections where a high value of $\mathbf{h}$ was viewed as favorable.

5 Numerical / Empirical analysis.

The previous analysis provided a better understanding of the dynamics of public debt in a stochastic environment where default is not a priori excluded. It highlighted the role played by the debt recovery rule on the dynamics of public debt, both before and after default has occurred. This allows us to offer new instruments so as to assess the soundness of the financial position of a country at a given date, by redefining the debt limit and the fiscal space.

In this section, we show how these notions can be put in practical use to empirically investigate the link between public default and the debt recovery parameter.

5.1 Data.

We use a dataset that covers two groups of countries over the period 1980-2018. The first one ("advanced") contains 31 advanced economies. The second one ("emerging") contains 13 emerging economies. We restrict the sample of countries to those with sufficient historical observations for our variables of interest.[27] Appendix A.2 presents the definition of the variables, data sources and gives relevant descriptive statistics.[28] For each country, we compute the annualized gross growth rate, its standard deviation, and the average public debt maturity over the considered period.

5.2 Baseline calibration.

An important issue that arises when calibrating the model is the choice of the duration of one period in the theoretical model, that is, from the period t at which debt is issued to period $t+1$ at which it is repaid. To align the model with annual data, we adapt the methodology used by Collard et al. (2015) to deal with country specific maturity and match the theoretical period to the average maturity drawn from the data for a

[27]We limit our analysis to countries with at least ten consecutive years of observations. We use the IMF's World Economic Outlook definition to classify countries between emerging and advanced groups.

[28]Table A.2.1 presents the definition of the variables, and data sources. Tables A.2.2 to A.2.6 presents descriptive statistics of the data.

given borrowing country. Let us denote by m_i the average maturity for country i: one period t in the model corresponds to m years in the data for a country with an average debt maturity m.

Neglecting the index i for simplicity of notation we denote by $a_{t,m}$ the (gross) rate of output growth at date t for any country characterized by maturity m. We assume that $a_{t,m}$ follows a log-normal distribution:

$$\ln a_{t,m} \sim N\left(\mu_m,\, \sigma_m^2\right),$$

where μ_m is the mean and σ_m^2 the variance of $\ln a_{t,m}$. Defining μ and σ^2 as the mean and the variance of the annual growth rate, we have $\sigma_m^2 = m\sigma^2$ and $\mu_m = m\mu$. Note that the ratio of the mean to the standard deviation of $\ln a_{t,m}$, thus corresponding of a period of m years is equal to:

$$\frac{\mu_m}{\sigma_m} = \frac{m\mu}{\sqrt{m\sigma^2}} = \sqrt{m}\frac{\mu}{\sigma}, \tag{32}$$

and is an increasing function of m for a given ratio μ/σ.

Let us denote by $S_a(\cdot)$ the function defining the annualized theoretical spread:

$$S_a\left(b_\tau; \mu_m, \sigma_m, m, \hat{s}, \mathbf{h}\right) \equiv \left(\frac{1}{q_\tau}\right)^{\frac{1}{m}} - (1+r), \tag{33}$$

where r is the annual risk free interest rate, and q_τ is the annualized theoretical price of a bond of maturity m. It is derived from (18) by making explicit the role of maturity; b_τ is the annual observation of the debt-to-GDP ratio and b_τ/m is its theoretical equivalent in the model:

$$q_\tau = \beta_m \left[1 - G_m\left(\frac{b_\tau}{m \cdot \omega_{\mathbf{h},m}}\right) + \mathbf{h}\frac{\omega_{\mathbf{h},m}}{b_{m,\tau}} \int^{b_\tau/m \cdot \omega_{\mathbf{h},m}} a g_m(a)\, da\right], \tag{34}$$

with:

$$\omega_{\mathbf{h},m} \equiv \frac{\hat{s}}{1 - \beta_m x_{\mathbf{h},m}},$$

where $\beta_m = 1/(1+r)^m$, and :

$$x_{\mathbf{h},m} = [1 - G_m(\delta_{\mathbf{h},m})]\,\delta_{\mathbf{h},m} + \mathbf{h}\int^{\delta_{\mathbf{h},m}} a g_m(a)\,da,$$

$$[1 - G_m(\delta_{\mathbf{h},m})]\,[1 - (1-\mathbf{h})\,\delta_{\mathbf{h},m} z_m(\delta_{\mathbf{h},m})] = 0.$$

Table 1 presents the baseline parameter values used in the calibration exercises to follow. We report these values on an annual basis computed from our annual data set. We also report our simulation results on an annual basis.

Table 1: Baseline calibration (annual basis).

	Advanced	Emerging
Risk-free rate, r	0.0293^a	0.0293^a
Maximum primary surplus, $\hat{s}$	0.04 or 0.05^b	0.02 or 0.03^b
Mean of the growth rate, μ	0.0281^c	0.0364^c
Volatility of the growth rate, σ	0.0263^d	0.0333^d

Notes: a : Average annual rate on 5-year-maturity German bonds (1980-2018); b : IMF(2011; 2018); c : Historical average growth rate (1980-2018); d : Historical Standard deviation of the growth rates (1980-2018).

The growth mean and volatility are computed over the whole country-time sample. The risk-free rate r is set to the average real yield on German Treasury bond.[29] The maximum primary surplus $\hat{s}$ is calibrated following IMF (2011, 2018), with two possible values capturing different degrees of fiscal limit.

5.3 Conditional estimates of h.

To provide conditional estimates of the debt recovery parameter **h** for both groups of countries, we examine the relation between a country i's actual sovereign yield spread in year τ, denoted $s_{i,\tau}$, and its theoretical spread in that same year. Because this theoretical spread is conditional to the assumption concerning the maximum primary

[29] Calibration results that we shall report below are similar when we use the US Government rate as the risk-free rate. We prefer the German rate as it appears to be a fairly better benchmark over the past few decades than the US rate (see also Mitchener and Trebesch 2021).

surplus, our estimates are conditional to this assumption. We will see in the next subsection that the computed fiscal spaces are much less sensitive to this assumption than the estimates of $\mathbf{h}$.

We estimate the recovery parameter $\mathbf{h}$ by nonlinear least squares, minimizing the sum of squared deviations of theoretical yield spreads from actual spreads.[30] That is, our estimated parameter, denoted $\hat{\mathbf{h}}$, solves:

$$\min_{\mathbf{h}} \quad \sum_{\tau} \sum_{i} \left[S_a \left(b_{i,\tau}; \mu_i, \sigma_i, m_i, \hat{s}, \mathbf{h} \right) - s_{i,\tau} \right]^2 . \tag{35}$$

where $s_{i,\tau} = r_{i,\tau} - r_{G,\tau}$ is the difference between the annual long-term real interest rate of country i and the German long-term real rate. μ_i and σ_i are the mean and volatility of the growth rate, respectively, and are calibrated to their sample values at country level. $\hat{s}$ is the maximum primary surplus, which is calibrated according to IMF (2011, 2018). Debt maturities for advanced countries (1980-2010) are taken from OECD Statistical Database on Central Government Debt and for emerging countries (1980-2012) from Perez (2017).

We estimate equation (35) for both groups of countries separately. The dataset for each country group is an unbalanced panel because sovereign yields and debt-to-GDP ratios are not available for all countries over the time period considered, 1980-2018. Table 2 reports the obtained values for $\hat{\mathbf{h}}$ for each group of countries, considering the two different values for the primary surplus given in Table 1. The last column of Table 2 shows the mean absolute deviation of theoretical spreads from actual spreads in percentage point.

We obtain a higher $\hat{\mathbf{h}}$ for advanced countries than for emerging countries, assuming either a high primary surplus or a low one. The estimated values for both country groups are positive and well above zero, suggesting that *ex-ante* lenders do expect a partial debt recovery in the case of sovereign default. This finding is in line with historical estimates of post-default debt haircuts documented in the empirical studies mentioned in Section 1. In the case of emerging countries, the estimated value of $\mathbf{h}$ is

[30] Arellano and Ramanarayanan (2012) use a similar method to estimate the parameters of the yield curve of long-term Government debt for four emerging countries.

Table 2: Debt recovery parameter estimation.

	$\hat{s}$	$\hat{\mathbf{h}}$	Mean absolute yield spread error (%)[a]
advanced economies	0.05	0.63	0.46
	0.04	0.81	0.47
emerging economies	0.03	0.22	0.15
	0.02	0.75	0.29

Notes: $\hat{s}$ is calibrated following IMF(2011; 2018). $\hat{\mathbf{h}}$ is the value of $\mathbf{h}$ that solves (35). a: Average (absolute) difference between theoretical spreads and actual spreads when $\mathbf{h} = \hat{\mathbf{h}}$. Data for country debt maturities are from OECD Statistical Database on Central Government Debt and Perez (2017) advanced and emerging countries, respectively.

very sensitive to the calibration of the maximum primary surplus $\hat{s}$: $\hat{\mathbf{h}}$ is respectively equal to 0.75 and 0.22 for $\hat{s}$ equal to 3% and 2% of GDP.[31]

5.4 Sustainability and the debt recovery rule.

Section 4.3 introduced a more precise measure of the debt limit than the one proposed by Ghosh et al. (2013). We showed that this measure depends crucially on the debt recovery rule. In this section, we illustrate the role of the debt recovery parameter $\mathbf{h}$ by computing debt limits for the advanced and emerging groups of countries in our dataset. More precisely, for each country i, we calibrate the mean μ_i and volatility σ_i of the log growth rate of GDP to their historical values while setting the risk-free rate r and the primary surplus $\hat{s}$ to their baseline values defined in Table 1. We solve the model numerically and compute the debt limit for four different values of $\mathbf{h}$: the case of no debt recovery $\mathbf{h} = 0$ (haircut of 100%), the case of maximum debt recovery $\mathbf{h} = 1$ (haircut of 0%), an intermediate case: $\mathbf{h} = 0.5$ and the conditional estimated values of $\mathbf{h} = \hat{\mathbf{h}}$ (for each group of countries).

[31] An alternative strategy would be to fix the value of $\mathbf{h}$ and estimate the maximum primary surplus but the same type of sensitivity of the obtained estimates would probably be found.

Tables 3 and 4 present the results of this exercise for advanced and emerging countries, respectively. For comparison, we also report the debt-to-GDP ratio of each country in 2018, the last year in our dataset.

First, consider the group of advanced countries. Assume, as in Collard et al. (2015), zero debt recovery by creditors in case of a sovereign default (that is $\mathbf{h} = 0$) and a primary surplus of 5%. This case corresponds to Column 2 of Table 3. Under this assumption, Greece has the lowest debt limit at 110% of GDP, followed by Finland (129%) and Czech Republic with 138%. On the other hand, Singapore has no debt limit, while Korea and Israel present a debt limit of 916% and 760%, respectively.[32]

Moving from $\mathbf{h} = 0$ to $\mathbf{h} = 0.5$, again setting $\hat{s} = 5\%$, the debt limit is still infinite in Singapore and increases from 916% to 951% in Israel and from 760% to 4260% in Korea, respectively. At the same time, the debt limits for Greece, Finland and the Czech Republic increase only to 123%, 145%, and 158%, respectively. Assuming a maximum debt recovery parameter ($\mathbf{h} = 1$), default is no more a concern for 1/3 of advanced countries which have no finite limit in this case.[33] The lowest debt limits obtain for Greece (260%) and Italy (320%). A similar pattern occurs when we set $\hat{s} = 4\%$. When $\mathbf{h}$ is equal to its estimated value 0.63, debt limits in advanced countries are reduced from 926% to 311% on average with respect to the case $\mathbf{h} = 1$. Two countries only, Singapore and Korea, benefit from an infinite debt limit.

Comparing the debt limit of each country to its debt-to-GDP ratio in 2018 (Table 3 Column 1) so as to have a measure of its fiscal space at this date, Greece and Japan are associated with a negative fiscal space when $\mathbf{h} = 0.63$ whereas they benefit from a positive fiscal space when $\mathbf{h} = 1$. This again illustrates the sensitivity of the assessment of public debt sustainability to the debt recovery parameter and the need to improve the estimation of this rate. In the case of Japan, which have not defaulted and doesn't appear on the verge of default, this may be due to a value of $\mathbf{h}$ included in the market

[32]The large variation of the debt limit across countries reflects differences in theirs economic fundamentals, in particular the mean growth rate which is positively related to the debt limit. For instance, over the period 1980-2018, Greece presents an annual growth rate of 0.7% on average while that of Singapore is 8 times larger (6.21% on average). See Table A.2.2 in Appendix A.2 for the mean growth of the list of advanced countries.

[33]For these countries, the growth rate is higher than the risk-free interest rate and the solvency ratio is infinite.

Table 3: Debt limit, $b^{\lim}_{\mathbf{h}} = \min\left(b^{\max}_{\mathbf{h}},\, b^{rss}_{\mathbf{h}}\right)$: advanced countries.

	b_{2018}	$b^{\lim}_{\mathbf{h}}$ ($\hat{s} = 5\%$, $\hat{\mathbf{h}} = 0.63$)				$b^{\lim}_{\mathbf{h}}$ ($\hat{s} = 4\%$, $\hat{\mathbf{h}} = 0.81$)			
		$\mathbf{h} = 0$	$\mathbf{h} = 0.5$	$\mathbf{h} = \hat{\mathbf{h}}_{5\%}$	$\mathbf{h} = 1$	$\mathbf{h} = 0$	$\mathbf{h} = 0.5$	$\mathbf{h} = \hat{\mathbf{h}}_{4\%}$	$\mathbf{h} = 1$
Country	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Australia*	41.37	602.80	693.84	746.37	∞	482.24	555.07	729.01	∞
Austria	73.75	263.01	282.42	292.51	633.45	210.41	225.94	255.96	506.76
Belgium	102.03	237.07	254.06	262.85	573.73	189.66	203.25	229.31	458.98
Canada	89.94	301.76	340.01	361.42	4159.84	241.41	272.01	340.50	3327.87
Czech Republic	32.56	138.11	158.70	170.62	629.85	110.49	126.96	166.29	503.88
Denmark	34.26	191.92	208.08	216.63	496.21	153.53	166.47	192.28	396.97
Finland	59.26	129.18	145.91	155.34	752.51	103.35	116.73	147.03	602.01
France	98.39	230.49	245.60	253.35	509.17	184.39	196.48	219.27	407.34
Germany	61.69	169.87	184.19	191.75	460.51	135.90	147.35	170.19	368.41
Greece	184.85	110.27	123.10	130.22	260.81	88.21	98.48	121.01	208.65
Hong Kong*	0.05	397.80	566.69	706.33	∞	318.24	453.35	1358.43	∞
Iceland*	37.62	175.59	210.92	232.80	∞	140.47	168.73	247.50	∞
Ireland*	63.65	287.98	415.45	522.71	∞	230.39	332.36	1058.16	∞
Israel*	60.78	760.43	951.61	1077.97	∞	608.34	761.29	1263.00	∞
Italy	132.16	146.42	157.04	162.56	321.73	117.13	125.63	142.06	257.39
Japan	237.13	184.66	203.06	213.00	569.54	147.73	162.45	193.14	455.63
Korea*	37.92	916.24	4260.37	∞	∞	732.99	3408.29	∞	∞
Latvia*	35.93	164.44	213.55	247.85	∞	131.55	170.84	317.04	∞
Lithuania*	34.17	202.22	265.29	310.20	∞	161.78	212.23	411.74	∞
Luxembourg*	21.43	300.64	375.49	424.87	∞	240.51	300.39	495.07	∞
Netherlands	52.39	225.22	245.41	256.15	687.55	180.17	196.33	228.99	550.04
New Zealand	29.84	216.78	240.86	254.03	1867.68	173.42	192.69	233.83	1494.15
Norway	39.97	213.41	234.80	246.31	1188.31	170.73	187.84	223.26	950.65
Portugal	120.13	153.92	171.27	180.82	588.64	123.14	137.01	167.01	470.91
Singapore*	113.63	∞	∞	∞	∞	∞	∞	∞	∞
Slovak Republic*	48.94	177.61	213.40	235.58	∞	142.09	170.72	250.72	∞
Spain	97.09	194.88	215.92	227.41	872.25	155.90	172.74	208.55	697.80
Sweden	38.46	165.98	182.91	192.06	720.28	132.79	146.33	174.60	576.22
Switzerland	40.53	195.88	210.63	218.30	503.26	156.70	168.50	191.36	402.61
United Kingdom	86.82	199.30	218.52	228.85	727.05	159.44	174.82	206.48	581.64
United States	104.26	257.08	286.07	301.96	1996.75	205.67	228.85	278.67	1597.40
Sample average	71.32	195.37	215.33	226.27	925.96	156.30	172.27	206.69	740.77

Notes. ∞: Cases where $b^{\max}_{\mathbf{h}} = \infty$ and no positive value exists for $b^{rss}_{\mathbf{h}}$. $\hat{\mathbf{h}}$ is the estimated value for $\mathbf{h}$. For each country, the mean μ and volatility σ of the growth rate are calibrated to their historical values. The other parameters (r and $\hat{s}$) are set to their baseline values in Table 1. *Excluded from the computation of the sample average in columns (2) to (9).

Table 4: Debt limit, $b_{\mathbf{h}}^{\lim} = \min\left(b_{\mathbf{h}}^{\max},\, b_{\mathbf{h}}^{rss}\right)$: emerging countries.

	b_{2018}	$b_{\mathbf{h}}^{\lim}$ ($\hat{s} = 3\%$, $\hat{\mathbf{h}} = 0.22$)				$b_{\mathbf{h}}^{\lim}$ ($\hat{s} = 2\%$, $\hat{\mathbf{h}} = 0.75$)			
		$\mathbf{h} = 0$	$\mathbf{h} = 0.5$	$\mathbf{h} = \hat{\mathbf{h}}_{3\%}$	$\mathbf{h} = 1$	$\mathbf{h} = 0$	$\mathbf{h} = 0.5$	$\mathbf{h} = \hat{\mathbf{h}}_{2\%}$	$\mathbf{h} = 1$
Country	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Brazil	87.89	90.47	103.77	94.68	675.10	60.32	69.18	82.70	450.07
Chile*	25.56	35.57	41.11	37.32	∞	23.71	27.41	33.14	∞
China*	50.64	∞	∞	∞	∞	∞	∞	∞	∞
Colombia*	52.16	135.66	156.76	142.33	∞	90.44	104.51	126.24	∞
Hungary	70.85	107.46	120.72	111.72	458.93	71.64	80.48	93.31	305.95
Malaysia*	55.57	110.58	144.84	120.48	∞	73.72	96.56	146.99	∞
Mexico	53.62	51.60	58.39	53.77	827.49	34.40	38.92	45.58	551.66
Nigeria*	27.26	36.77	43.52	38.86	∞	24.52	29.01	36.42	∞
Pakistan*	71.69	1249.72	8934.74	1783.60	∞	833.15	5956.49	∞	∞
Philippines*	38.92	94.77	113.31	100.48	∞	63.18	75.54	96.48	∞
Poland*	48.89	157.57	189.88	167.47	∞	105.05	126.59	163.83	∞
Russia	14.61	39.12	46.72	41.47	140.17	26.08	31.15	39.56	93.44
South Africa	56.71	197.28	218.35	204.11	589.75	131.52	145.57	165.28	393.17
Sample average	50.34	97.19	109.59	101.15	538.29	64.79	73.06	85.28	358.86

Notes. ∞: Cases where $b_{\mathbf{h}}^{\max} = \infty$ and no positive value exists for $b_{\mathbf{h}}^{rss}$. $\hat{\mathbf{h}}$ is the estimated value for $\mathbf{h}$. For each country, the mean μ and volatility σ of the growth rate are calibrated to their historical values. The other parameters (r and $\hat{s}$) are set to their baseline values in Table 1. *Excluded from the computation of the sample average in columns (2) to (9).

risk premium higher than 0.63. In the case of Greece, its negative fiscal space may suggest that its recent default has not been fully resolved.

Turning to emerging countries[34] (Table 4), we observe a pattern similar to advanced countries. Setting $\hat{s} = 3\%$, the debt limit increases on average from 96% when $\mathbf{h} = 0$ to 538% when $\mathbf{h} = 1$. Under the latter case, default would not be an issue for any emerging country, when considering their 2018 debt-to-GDP ratios (Column 1).

Finally, we note that despite the difference between the two conditional estimations of $\mathbf{h}$, especially for the group of emerging countries, the two computed values for the debt limit are sufficiently close to provide a fairly good approximation or, at least, a reasonable range for this financial sustainability indicator. Figure 7 allows us to

[34]See Table A.2.3 in Appendix A.2 for the mean growth of the list of emerging countries.

compare the two evaluations of the debt limit for advanced countries[35] according to the case $\mathbf{h} = .63$ and $\hat{s} = 5\%$, or $\mathbf{h} = .81$ and $\hat{s} = 4\%$.

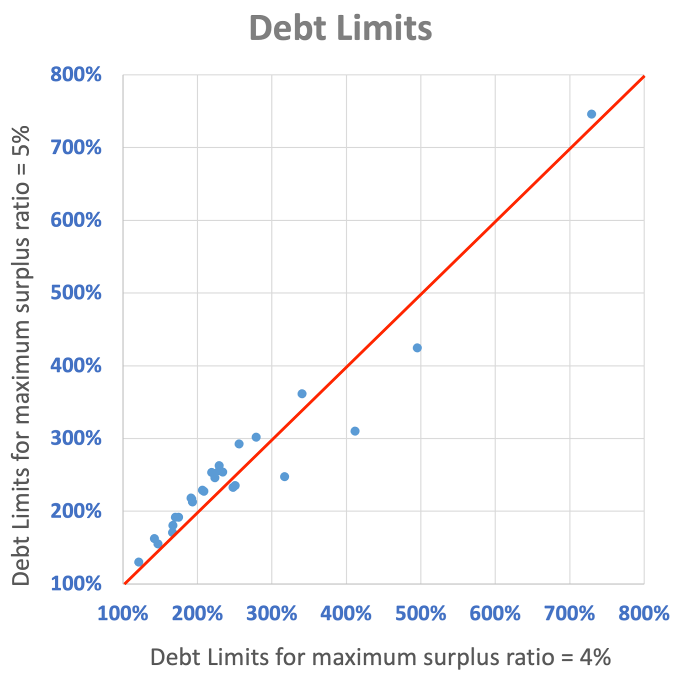

Figure 7: Computation of the debt limits of advanced countries for $\hat{s} = 4\%$ and $\hat{s} = 5\%$.

To sum up, our methodology, even though it does not provide a precise assessment of the parameter $\mathbf{h}$ since its numerical values depend on the calibration chosen for the maximum (unobservable) level of primary surplus, enables us to obtain estimates of debt limits that are stable and consistent with the available data on interest rate spreads.

[35]Only countries with computed debt limits below 350% are included here. The difference between the two evaluation is greater for countries with a computed debt limit above 400% but the risks associated with these cases are negligible.

5.5 Role of debt maturity

Unlike Lorenzoni and Werning (2019), our theoretical model does not explicitly take debt maturity into account.[36] However, ignoring the (average) maturity of public debt when estimating the debt limit is tantamount to neglecting an important determinant of sovereign risk.[37]

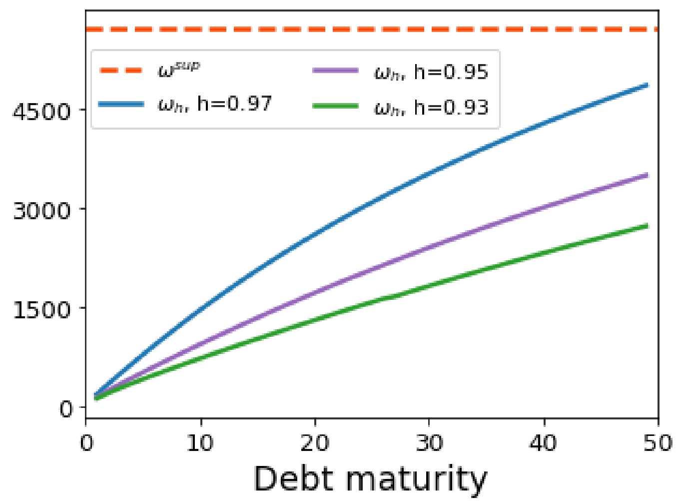

Figure 8: Debt maturity and default ratio.

Figure 8 shows the simulated default ratio (a variable that is highly correlated with the debt limit) as a function of "maturity", i.e. the length of a theoretical period expressed in number of years, for $\mathbf{h} = 1$ and three other recovery ratios close to 1. When $\mathbf{h} = 1$, the default ratio merges with the solvency ratio and does not depend on maturity. For values of $\mathbf{h}$ less than 1, the default ratio is an increasing function of maturity. For low values of $\mathbf{h}$, the gap with the solvency ratio is significant. This can be explain as follows. Given the assumed lognormal distribution of growth rates, as shown in equation (32), increasing the length of the reference period, m, reduces the relative volatility of the shock $\left(\frac{\sigma_m}{\mu_m} = \frac{\sigma}{\mu\sqrt{m}}\right)$. In other words, over a longer period, cumulative growth increases faster than the risk arising from the accumulation of shocks. In our

[36]The authors are obliged to assume that the default risk is concentrated over a single period in order to simulate their theoretical model taking into account maturity.

[37]The interplay between debt maturity and sovereign default is now well documented. See Arellano and Ramanarayanan (2012), Sánchez et al. (2018).

model, this relative smoothing effect of macroeconomic risk translates into a fall in the risk premium and, consequently, a rise in the default ratio. Except for $\mathbf{h} = 1$, the higher $\mathbf{h}$, the greater the effect.

Table 5 provides a more empirical illustration of the effect of maturity on debt limits for developed countries.[38] Assuming that debt maturity is identical for all these countries, it shows the calculated values of debt limits for two maturities, 4 years and 5 years.

On average, the debt limits increase by 20.9%, 23.3% and 25.5% for $\mathbf{h}$ values of 0, 0.5 and $\mathbf{h} = 0.63$ respectively. These are significant increases given that the maturity is only increased by one year. It confirms the importance of considering debt maturity on the fiscal space and the risk of sovereign default, even in the context of our parsimonious treatment of this variable.

5.6 Sovereign default and debt sustainability when $r - g < 0$.

In his presidential lecture to the American Economic Association, Blanchard (2019) argues that "public debt may have no fiscal cost" if interest rates remain below the rate of growth. With close to zero interest rates, governments can potentially borrow and roll over their debts despite the existence of an upper bound on future primary surplus. In a recent contribution Blanchard et al. (2021) are more cautious and recognize the potentially important role of default risk in assessing the sustainability of public debt. In the same vein, Sergeyev and Mehrotra (2020) and Mauro and Zhou (2020) suggest that negative $r - g$ differentials[39] are quite common over the past 200 years. But both papers also point to the large uncertainty over future interest rates and shocks, including the possibility of abrupt bond yield reversals and subsequent defaults.

In this sub-section, we relax the condition $\bar{a} < \beta^{-1}$ in Assumption 1,[40] which was equivalent to $r - g > 0$, in order to reassess the question of the sustainability of public

[38] Remember that Table 3 provides values for debt limits based on country specific maturities. This explains the differences with the results of table 5.

[39] Here g refers to the (net) real growth rate of GDP, and r is the real risk-free rate as before.

[40] Remember that Assumption 1 is a sufficient but not a necessary condition for the validity of Proposition 1. In the case of lognormal distribution function, the condition $\bar{a} < \beta^{-1}$ is not necessary for computing the value of $x_{\mathbf{h}}$.

Table 5: Impact of maturity on debt limits. Advanced countries ($\hat{s} = 5\%$, $\hat{\mathbf{h}} = 0.63$).

	b_{2018}	$m_i = m = 4$ years			$m_i = m = 5$ years		
		$\mathbf{h} = 0$	$\mathbf{h} = 0.5$	$\mathbf{h} = \hat{\mathbf{h}}_{5\%}$	$\mathbf{h} = 0$	$\mathbf{h} = 0.5$	$\mathbf{h} = \hat{\mathbf{h}}_{5\%}$
Country	(1)	(2)	(3)	(4)	(6)	(7)	(8)
Australia	41.37	347.90	393.60	419.23	426.38	485.40	518.83
Austria	73.75	194.69	209.80	217.68	218.79	235.52	244.23
Belgium	102.03	192.66	206.98	214.40	215.64	231.39	239.55
Canada	89.94	220.28	247.07	261.94	260.73	293.21	311.33
Czech Republic	32.56	106.09	121.34	130.10	122.35	140.32	150.69
Denmark	34.26	155.29	168.65	175.72	174.24	189.09	196.94
Finland	59.26	129.18	145.91	155.34	149.26	168.92	180.03
France	98.39	189.75	202.77	209.46	210.93	225.07	232.34
Germany	61.69	151.71	164.62	171.44	169.87	184.19	191.75
Greece	184.85	84.77	94.40	99.72	94.03	104.83	110.81
Hong Kong	0.05	274.23	364.67	430.93	397.80	566.69	706.33
Iceland	37.62	175.59	210.92	232.80	220.13	268.01	298.30
Ireland	63.65	201.93	273.28	326.98	287.98	415.45	522.71
Israel	60.78	366.06	431.79	471.21	474.30	569.48	628.23
Italy	132.16	133.07	142.88	147.98	146.42	157.04	162.56
Japan	237.13	146.48	161.13	169.05	166.08	182.70	191.68
Korea*	37.92	916.24	4260.37	∞	411080.56	∞	∞
Latvia	35.93	128.18	162.83	186.21	164.44	213.55	247.85
Lithuania	34.17	154.16	196.70	225.67	202.22	265.29	310.20
Luxembourg	21.43	230.65	281.72	314.22	300.64	375.49	424.87
Netherlands	52.39	178.27	194.47	203.10	202.34	220.64	230.38
New Zealand	29.84	216.78	240.86	254.03	253.63	282.21	297.90
Norway	39.97	213.41	234.80	246.31	246.69	271.54	284.93
Portugal	120.13	134.90	150.00	158.30	153.92	171.27	180.82
Singapore*	113.63	2039.93	∞	∞	∞	∞	∞
Slovak Republic	48.94	241.91	297.79	333.79	319.20	403.25	459.74
Spain	97.09	169.37	187.54	197.44	194.88	215.92	227.41
Sweden	38.46	165.98	182.91	192.06	189.80	209.19	219.70
Switzerland	40.53	175.30	188.71	195.70	195.88	210.63	218.30
United Kingdom	86.82	174.76	191.64	200.69	199.30	218.52	228.85
United States	104.26	219.59	243.98	257.32	257.08	286.07	301.96
Sample average	71.32	188.72	217.03	234.44	228.10	267.62	293.77

Notes. ∞: Cases where $b_{\mathbf{h}}^{\max} = \infty$ and no positive value exists for $b_{\mathbf{h}}^{rss}$. $\hat{\mathbf{h}}$ is the estimated value for $\mathbf{h}$. For each country, the mean μ and volatility σ of the growth rate are calibrated to their historical values. The other parameters (r and $\hat{s}$) are set to their baseline values in Table 1.

debt when the risk-free interest rate is lower than the growth rate. Using our notations we identify $r - g$ to $-\ln\beta\bar{a}$ and we define:

$$(r-g)_{\mathbf{h}} \equiv -\ln\beta x_{\mathbf{h}},$$

as the interest rate net of growth, adjusted for the possibility of partial debt recovery in case of default. In the sequel, we refer to this term as the "adjusted net-of-growth-interest rate". From Proposition 1 we know that $x_{\mathbf{h}}$ is an increasing function of $\mathbf{h}$, and thus the adjusted net-of-growth-interest rate is a decreasing function of $\mathbf{h}$. Assuming $\bar{a} > \beta^{-1}$ (that is $r - g < 0$), there may exist a critical value $\widetilde{\mathbf{h}}$, such that $x_{\widetilde{\mathbf{h}}} = \beta^{-1}$ (that is $(r-g)_{\widetilde{\mathbf{h}}} = 0$). When $\mathbf{h} \geq \widetilde{\mathbf{h}}$, the corresponding debt limit is infinite. Otherwise, when $\mathbf{h} < \widetilde{\mathbf{h}}$, we may have: $\bar{a} < \beta^{-1} < x_{\mathbf{h}}$, that is

$$r - g < 0 < (r-g)_{\mathbf{h}}$$

and the debt limit is thus finite.

To illustrate this possibility, we compute the country-specific $\tilde{\mathbf{h}}_{\mathbf{i}}$ for Eurozone countries in the last years of our time period (2009-2018), assuming that the specific debt maturities remain unchanged.[41] Specifically, we consider the 4-Year German (risk-free) bond rate for this period and we compute for each country the term $\beta\bar{a}_i \simeq exp\left(g_i - r\right)$.[42] The computed values are reported in Table A.2.4 in Appendix A.2. Except for Greece and Italy, for which $\beta\bar{a}_i$ is equal to 0.97 and to 0.99, respectively, this term is higher than 1 and the solvency ratio is infinite for all other countries over the considered period. Nevertheless, the default ratio, given by $\omega_{\mathbf{h}} = \frac{\hat{s}}{1-\beta x_{\mathbf{h}}}$, takes a positive and finite value as long as $x_{\mathbf{h}}$ satisfies $x_{\mathbf{h}} < \beta^{-1}$. In the same Table A.2.4, we compute for each country the critical value of $\mathbf{h}$, denoted by $\tilde{\mathbf{h}}_{\mathbf{i}}$, which satisfies $x_{\tilde{\mathbf{h}}_{\mathbf{i}}} = \beta^{-1}$. These values $\tilde{\mathbf{h}}_{\mathbf{i}}$ are reported on Figure 9. The vertical red line corresponds to the value of the maximum recovery parameter estimated over 1980-2018 for the group of advanced countries, $\hat{\mathbf{h}} = 0.63$. We

[41] See table A.2.2. We only consider the 15 countries of the euro zone present in our database of advanced countries which excludes Cyprus, Estonia, Malta and Slovenia for data limitations for these countries.

[42] Here, the value for β is given by $(1 + r_G)^{-1}$ where $r_G = 0.31\%$ is the annualized German interest rate for this period.

use this value as the best proxy for the country-specific value of the recovery rate.[43]

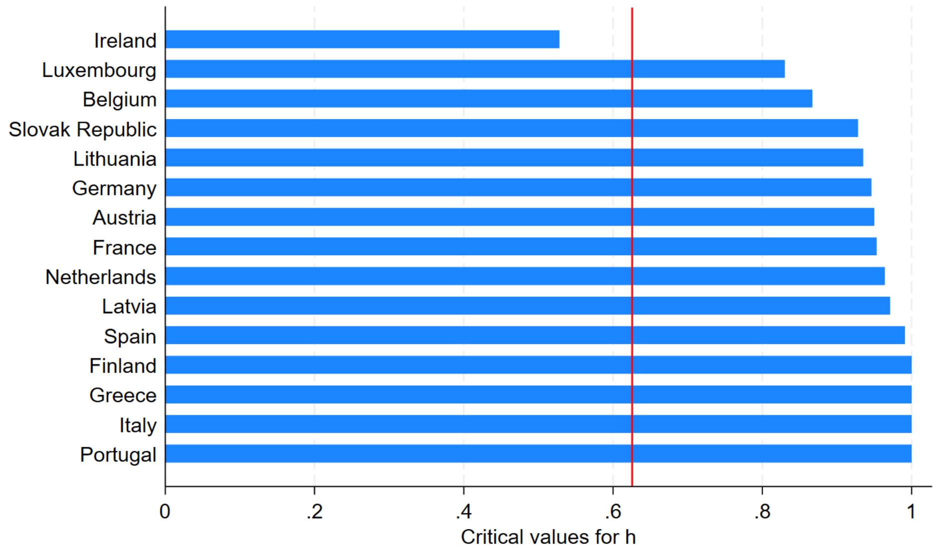

Figure 9: Sustainability with low interest interest rate ($r = 0.31\%$) in the Euro zone

It is immediate to observe that all countries except Ireland would be characterized by a positive and finite default ratio since their values $\tilde{\mathbf{h}}_{\mathbf{i}}$ are higher than $\hat{\mathbf{h}}$. On the opposite Ireland would be considered immune to the risk of default, likely so because of its high growth rate over the same period.

Remember that this exercise is based on the country-specific debt-maturities observed on the period 1980-2012, not on the period 2009-2018. Notice that, as shown on Figure 8, the default ratio is increasing with debt maturity. Actually, there has been a large increase in public debt maturities in recent years: the average maturity for the European countries considered has risen from 4.6 years for the period 1980-2012 to 7 years in 2018. Using the debt maturities for 2018, there is no positive solution $\tilde{\mathbf{h}}_{\mathbf{i}}$ to the equation $x_{\tilde{\mathbf{h}}_{\mathbf{i}}} = \beta^{-1}$, applied to each country. In other words the adjusted net of growth interest rate becomes negative, even in the case of zero recovery rate ($\mathbf{h} = \mathbf{0}$), and, based on this computation, there is neither risk of default nor sustainability issue

[43] The panel data set based on a small number of countries as well as the short period (2009-2018) does not allow us to obtain a new reliable estimate of $\mathbf{h}$.

for any of these countries.

In brief, the point made by Blanchard is sustained insofar as European countries took advantage of the low long term interest rate environment and increased the maturity of their public debt.

6 Conclusion.

We have developed a tractable stochastic model of excusable sovereign default allowing us to highlight the relevance of debt recovery, that is, the impact of the expected debt recovery rule to be applied in the case of default on the whole dynamics of public debt, and in particular on its sustainability. We use a simple specification of such a rule which depends on a single parameter **h**, the (maximum) debt recovery rate. We show that the *default ratio*, namely the maximum debt-to-GDP ratio that a country can reach without defaulting, depends on this debt recovery parameter. It differs from the *solvency ratio* which corresponds to the transversality condition obtained when the possibility of default is neglected. The two quantities are equal only under the extreme, non realistic, assumption of a debt recovery parameter equal to one.

We provide a new definition of debt unsustainability and a new measure of fiscal space. We show that the assessment of the unsustainability of public debt depends crucially on the debt recovery rule that is applied following a sovereign default. This finding provides some insights on the current debate on the sustainability of public debt in the context of low real interest rates.

We illustrate these findings by means of several empirical analyses based on a dataset covering advanced and emerging countries. First we provide some evaluations of the debt recovery parameter. It appears that its magnitude is higher for advanced countries than for emerging ones. Second we assess the extent of fiscal spaces for the various countries of the dataset. Fiscal spaces for advanced economies are fairly large. Greece and Italy (to a lesser extent and in the event of a future increase in the risk-free interest rate) are notable exceptions. The estimated values of the fiscal spaces for emerging countries are much lower. The sensitivity of the estimated fiscal spaces to the debt recovery parameter shows clearly that it plays a major role in the assessment of the financial position of a country. These analyses illustrate the necessity to take into account the partial recovery of public debt when studying its dynamics.

The excusable default model we use, excluding any strategic behavior from the sovereign government, relies on two exogenous rules: a fiscal rule generating a constrained fiscal regime characterized by the maximum primary surplus ratio $\widehat{s}$, and a debt recovery rule depending on a single parameter $\mathbf{h}$. Empirically, we had to choose a given value of $\widehat{s}$ to estimate the parameter $\mathbf{h}$. However, our results show that the computed fiscal spaces based on these estimations are rather insensitive to this choice. A more complex model highlighting the interplay between these two parameters would be a theoretical advance, with the possible advantage of simultaneously determining the coefficients of these rules. This would lead to a better understanding of the interplay between sovereign default, debt recovery and the dynamics of public debt. This is left to further research.

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A Appendix

A.1 Proof of Propositions.

A.1.1 Proof of Proposition 1

Proposition 1: *Given ω_{t+1}^{def}, under Assumption 1, the market value of debt v_t reaches a unique maximum $v_t^{\max}$ for a quantity of debt $b_t = b_t^{\max}$. Both $v_t^{\max}$ and $b_t^{\max}$ are linearly increasing in ω_{t+1}^{def}: $v_t^{\max} = \beta x_{\mathbf{h}} \omega_{t+1}^{def}$ and $b_t^{\max} = \delta_{\mathbf{h}} \omega_{t+1}^{def}$ where $\delta_{\mathbf{h}}$ is such that*

$$[1 - G(\delta_{\mathbf{h}})][1 - (1 - \mathbf{h})\delta_{\mathbf{h}} z(\delta_{\mathbf{h}})] = 0,$$

where $z(\delta) = \frac{g(\delta)}{1-G(\delta)}$ is the hazard function, and $x_{\mathbf{h}}$ is given by

$$x_{\mathbf{h}} = [1 - G(\delta_{\mathbf{h}})]\delta_{\mathbf{h}} + \mathbf{h}\int^{\delta_{\mathbf{h}}} adG(a).$$

These two coefficients are increasing functions of $\mathbf{h}$, with $0 < x_{\mathbf{h}} \leq \bar{a}$ and $0 < \delta_{\mathbf{h}} \leq +\infty$ for $0 \leq \mathbf{h} \leq 1$.

Proof. By denoting $\delta_t = b_t/\omega_{t+1}^{\text{def}}$, from (19) we can rewrite v_t as:

$$v_t = \beta\chi(\delta_t, \mathbf{h})\,\omega_{t+1}^{\text{def}}, \tag{A.1}$$

where $\chi(\delta, \mathbf{h})$ is a non-monotonic function defined by:

$$\chi(\delta, \mathbf{h}) \equiv [1 - G(\delta)]\delta + \mathbf{h}\int^{\delta} a \cdot dG(a). \tag{A.2}$$

Let us define $\Phi(\delta, \mathbf{h}) \equiv \partial\chi(\delta, \mathbf{h})/\partial\delta$, the derivative of $\chi(\delta, \mathbf{h})$ with respect to δ, we get:

$$\Phi(\delta, \mathbf{h}) = [1 - G(\delta)][1 - (1 - \mathbf{h})\delta z(\delta)], \tag{A.3}$$

where the function $z(\delta)$ is the hazard function:

$$z(\delta) \equiv \frac{g(\delta)}{1 - G(\delta)}.$$

Assuming that there exists a positive value $\delta_{\mathbf{h}}$ such that:

$$\Phi\left(\delta_{\mathbf{h}}, \mathbf{h}\right) = 0, \tag{A.4}$$

we can then define

$$x_{\mathbf{h}} \equiv \chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right). \tag{A.5}$$

By denoting $\Phi_z\left(\delta, \mathbf{h}\right) \equiv \partial\Phi\left(\delta, \mathbf{h}\right)/\partial z$, the partial derivatives of $\Phi\left(\delta, \mathbf{h}\right)$ for $z = \delta, \mathbf{h}$, we get, for any $\mathbf{h} \in [0, 1)$:

$$\Phi_{\mathbf{h}}\left(\delta_{\mathbf{h}}, \mathbf{h}\right) = \delta_{\mathbf{h}} g\left(\delta_{\mathbf{h}}\right) > 0, \tag{A.6}$$

$$\Phi_{\delta}\left(\delta_{\mathbf{h}}, \mathbf{h}\right) = -\left[1 - G\left(\delta_{\mathbf{h}}\right)\right]\left(1 - \mathbf{h}\right)\left[z\left(\delta_{\mathbf{h}}\right) + \delta_{\mathbf{h}} z'\left(\delta_{\mathbf{h}}\right)\right] < 0, \tag{A.7}$$

where the last inequality is implied by Assumption *1.* Hence, from (A.1), (A.3), (A.4) and (A.7), $v_t^{\max} = \beta\chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right)\omega_{t+1}^{\text{def}} = \beta x_{\mathbf{h}} \omega_{t+1}^{\text{def}}$ is a maximum reached for $b_t^{\max} = \delta_{\mathbf{h}} \omega_{t+1}^{\text{def}}$. From the definition of $\delta_{\mathbf{h}}$, implicitly given by (A.3) and (A.4), and using (A.2), (A.6) and (A.7), we find that:

$$\frac{\partial\delta_{\mathbf{h}}}{\partial\mathbf{h}} = -\frac{\Phi_{\mathbf{h}}\left(\delta_{\mathbf{h}}, \mathbf{h}\right)}{\Phi_{\delta}\left(\delta_{\mathbf{h}}, \mathbf{h}\right)} > 0. \tag{A.8}$$

$$\frac{\partial\chi_{\mathbf{h}}}{\partial\mathbf{h}} = \left.\frac{\partial\chi\left(\delta, \mathbf{h}\right)}{\partial\mathbf{h}}\right|_{\delta=\delta_{\mathbf{h}}} = \int^{\delta_{\mathbf{h}}} a \cdot dG\left(a\right) > 0. \tag{A.9}$$

Furthermore, from (A.2) we compute:

$$x_{\mathbf{0}} = \chi\left(\delta_{\mathbf{0}}, \mathbf{0}\right) = \left[1 - G\left(\delta_{\mathbf{0}}\right)\right]\delta_{\mathbf{0}}$$

where, from (A.3) and (A.4), $\delta_{\mathbf{0}}$ is such that:

$$\delta_{\mathbf{0}} z\left(\delta_{\mathbf{0}}\right) = 1.$$

From the same equations (A.2), (A.3), and (A.4), where $\Phi\left(\delta, \mathbf{h}\right)$ is given by we get

$\delta_{\mathbf{1}} = +\infty$ and

$$x_{\mathbf{1}} = \chi\left(\delta_{\mathbf{1}}, \mathbf{1}\right) = \int a \cdot dG\left(a\right) = \bar{a},$$

which ends the proof of Proposition 1. □

A.1.2 Proof of Proposition 2

Proposition* 2*: *The equilibrium* default ratio ω_t^{def} *is locally unique and equal to:*

$$\omega_t^{def} = \frac{\hat{s}}{1 - \beta x_{\mathbf{h}}} \equiv \omega_{\mathbf{h}},\ \forall t. \tag{A.10}$$

$\omega_{\mathbf{h}}$ *is a strictly increasing function of* $\hat{s}$ *and* $\mathbf{h}$, *with* $\omega_{\mathbf{h}} \leq \omega^{sup}$ *for* $\mathbf{h} \leq 1$.

Proof. Using (25)

$$\omega_t^{\text{def}} = \beta x_{\mathbf{h}} \omega_{t+1}^{\text{def}} + \hat{s}, \tag{A.11}$$

we obtain the stationary value for ω_t^{def} that we denote $\omega_{\mathbf{h}}$. It is given by:

$$\omega_{\mathbf{h}} = \frac{\hat{s}}{1 - \beta x_{\mathbf{h}}}. \tag{A.12}$$

From Proposition 1, we know that $x_{\mathbf{h}}$ is an increasing function of $\mathbf{h}$ with a maximum $x_{\mathbf{h}} = \bar{a}$ for $\mathbf{h} = 1$. It immediately follows that $\omega_{\mathbf{h}}$ is a growing function of $\mathbf{h}$ with a maximum

$$\omega_{\mathbf{1}} = \frac{\hat{s}}{1 - \beta \bar{a}} \equiv \omega^{\text{sup}}.$$

From Assumption 1, we have $\bar{a} < 1 + r$ with $1 + r = \beta^{-1}$ and hence, from Proposition 1, $\beta x_{\mathbf{h}} \leq \beta x_{\mathbf{1}} = \beta \bar{a} < 1$. This implies that, by rewriting the dynamics of equation (A.11) in a more conventional backward-looking form, it is unstable around the unique stationary equilibrium, $\omega_{\mathbf{h}}$. Since ω_t^{def} is not predetermined, $\omega_{\mathbf{h}}$ is a determinate, *i.e.* locally unique, equilibrium. □

A.1.3 Proof of Proposition 3

Proposition* 3*: *The market value of public debt is a strictly increasing function of the debt* recovery parameter $\mathbf{h}$.

Proof. Using equation (A.1) the market value of public debt (29) can be rewritten

$$\upsilon\left(b_t;\mathbf{h}\right)=\beta\chi\left(\frac{b_t}{\omega_{\mathbf{h}}},\mathbf{h}\right)\omega_{\mathbf{h}}, \tag{A.13}$$

where $\chi\left(\delta_t,\mathbf{h}\right)$ is given by (A.2). We compute

$$\frac{\partial\upsilon\left(b_t;\mathbf{h}\right)}{\partial\mathbf{h}}=\beta\omega_{\mathbf{h}}\int^{\delta_{\mathbf{h}}}a\cdot dG\left(a\right)+\beta\left[\chi\left(\frac{b_t}{\omega_{\mathbf{h}}},\mathbf{h}\right)-\frac{b_t}{\omega_{\mathbf{h}}}\Phi\left(\frac{b_t}{\omega_{\mathbf{h}}},\mathbf{h}\right)\right]\frac{\partial\omega_{\mathbf{h}}}{\partial\mathbf{h}},$$

where is $\Phi\left(\frac{b_t}{\omega_{\mathbf{h}}},\mathbf{h}\right)$ given by (A.3) is the derivative of $\chi\left(\delta,\mathbf{h}\right)$ with respect to δ. Since $\chi\left(\frac{b_t}{\omega_{\mathbf{h}}},\mathbf{h}\right)$ is strictly concave, with $\chi\left(0,\mathbf{h}\right)=0$, the term in square brackets is strictly positive, as is $\frac{\partial\omega_{\mathbf{h}}}{\partial\mathbf{h}}$ from Proposition 2, which makes it possible to conclude that $\frac{\partial\upsilon(b_t;\mathbf{h})}{\partial\mathbf{h}}>0\ \forall b_t$. □

A.1.4 Proof of Proposition 4

Proposition 4: *In the constrained fiscal regime,*

1. *there exists a unique risky-steady-state-debt ratio,* $b_{\mathbf{h}}^{rss}=b_{\mathbf{h}}^{*}$*, satisfying (31) and* $b_{\mathbf{h}}^{rss}\leq\bar{a}\omega_{\mathbf{h}}\leq b_{\mathbf{h}}^{max}$*, if and only if* $\mathbf{h}\geq\underline{\mathbf{h}}=1-\frac{1}{\bar{a}z(\bar{a})}$*, with strict equality for* $\mathbf{h}=\underline{\mathbf{h}}$.

2. *when* $\mathbf{h}>\underline{\mathbf{h}}$*, we have:*

   (a) $b_{\mathbf{h}}^{rss}$ *is increasing in* $\mathbf{h}$*,*

   (b) $b_{\mathbf{h}}^{max}-b_{\mathbf{h}}^{rss}$ *is increasing in* $\mathbf{h}$.

Proof. **1.** Recalling equation (31) for convenience:

$$\upsilon\left(b_{\mathbf{h}}^{*};\mathbf{h}\right)=\frac{b_{\mathbf{h}}^{*}}{\bar{a}}-\hat{s}, \tag{A.14}$$

a Risky Steady State (RSS) exists and is defined by: $b_{\mathbf{h}}^{\mathrm{rss}}=b_{\mathbf{h}}^{*}$, if and only if $b_{\mathbf{h}}^{*}<b_{\mathbf{h}}^{\max}$, since $b_{\mathbf{h}}^{\max}$ is the maximum level of debt that can be issued on the market. Figure A.1 represents two curves $\upsilon\left(b;\mathbf{h_1}\right)$ and $\upsilon\left(b;\mathbf{h_2}\right)$ corresponding to two different recovery parameters $\mathbf{h_1}$ and $\mathbf{h_2}$, and the line $b/\bar{a}-\hat{s}$. The figure is sufficient to prove the existence of a RSS for $\mathbf{h}=\mathbf{h_2}$, and its non-existence for $\mathbf{h}=\mathbf{h_1}$. In the first case, we observe that

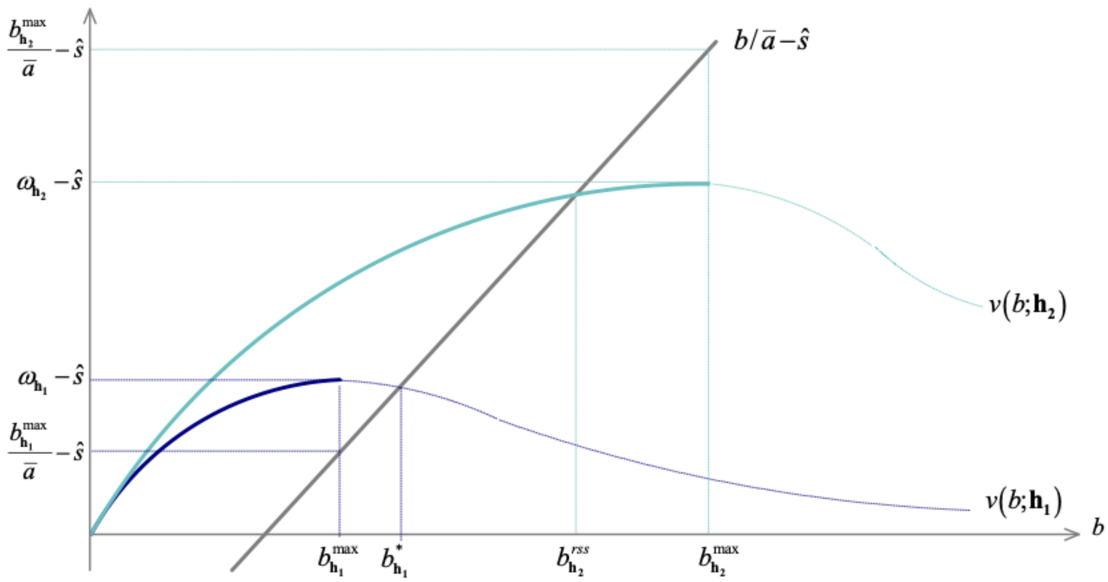

Figure A.1: Existence of a RSS

$b_{\mathbf{h_1}}^{\max} < b_{\mathbf{h_1}}^*$ and $b_{\mathbf{h_1}}^{\max} < \bar{a}\omega_{\mathbf{h_1}}$, and we also simply check that: $b_{\mathbf{h}}^*/\bar{a} - \hat{s} < \omega_{\mathbf{h_1}} - \hat{s}$. This can be summarized as follows:

$$b_{\mathbf{h_1}}^{\max} < b_{\mathbf{h_1}}^* < \bar{a}\omega_{\mathbf{h_1}}.$$

In the other case, we obtain:

$$b_{\mathbf{h_2}}^{\text{rss}} = b_{\mathbf{h_2}}^* < \bar{a}\omega_{\mathbf{h_2}} < b_{\mathbf{h_2}}^{\max}.$$

It remains to be shown that $\mathbf{h_1} < \mathbf{h_2}$ and that there exists $\underline{\mathbf{h}}$, satisfying $\mathbf{h_1} < \underline{\mathbf{h}} < \mathbf{h_2}$, and such that $b_{\underline{\mathbf{h}}}^* = \bar{a}\omega_{\underline{\mathbf{h}}} = b_{\underline{\mathbf{h}}}^{\max}$. Note that, from Proposition 1, we can express the difference $b_{\mathbf{h}}^{\max} - \bar{a}\omega_{\mathbf{h}}$ as:

$$b_{\mathbf{h}}^{\max} - \bar{a}\omega_{\mathbf{h}} = (\delta_{\mathbf{h}} - \bar{a})\,\omega_{\mathbf{h}}. \tag{A.15}$$

This difference is positive for $\mathbf{h} = \mathbf{h_2}$, and negative for $\mathbf{h} = \mathbf{h_1}$. From Proposition 1, we know that $\delta_{\mathbf{h}}$ is an increasing function of $\mathbf{h}$, which is sufficient to conclude that $\mathbf{h_1} < \mathbf{h_2}$. Furthermore, when $\delta_{\mathbf{h}} = \bar{a}$, we necessarily have: $b_{\mathbf{h}}^{\max} = b_{\mathbf{h}}^* = \bar{a}\omega_{\mathbf{h}}$, or equivalently $\delta_{\mathbf{h}} = \delta_{\mathbf{h}}^* = \bar{a}$, with $\delta_{\mathbf{h}}^* \equiv b_{\mathbf{h}}^*/\omega_{\mathbf{h}}$. Thus, there is a value $\underline{\mathbf{h}}$ such that $\delta_{\underline{\mathbf{h}}} = \delta_{\underline{\mathbf{h}}}^* = \bar{a}$. From (A.3) and (A.4), $\delta_{\mathbf{h}}$ is implicitly given by: $(1-\mathbf{h})\,\delta_{\mathbf{h}} z\,(\delta_{\mathbf{h}}) = 1$, which implies, when

$\delta_{\underline{\mathbf{h}}} = \delta^*_{\underline{\mathbf{h}}} = \bar{a}$:

$$\underline{\mathbf{h}} = 1 - \frac{1}{\bar{a}z(\bar{a})}.$$

A necessary and sufficient condition to have $0 < \underline{\mathbf{h}} < 1$ is therefore $\bar{a}z(\bar{a}) > 1$.

2. a. We now seek to show that, for $\mathbf{h} > \underline{\mathbf{h}}$, the RSS debt ratio, $b^{\text{rss}}_{\mathbf{h}} = b^*_{\mathbf{h}}$, is an increasing function of $\mathbf{h}$. By looking for the derivative $\frac{\partial b^*_{\mathbf{h}}}{\partial \mathbf{h}}$ from equation (A.14), one find:

$$\frac{\partial b^*_{\mathbf{h}}}{\partial \mathbf{h}} = \frac{\bar{a}\frac{\partial v\left(b^*_{\mathbf{h}};\mathbf{h}\right)}{\partial \mathbf{h}}}{1 - \bar{a}\frac{\partial v\left(b^*_{\mathbf{h}};\mathbf{h}\right)}{\partial b}}.$$

Remembering that $v(b_t;\mathbf{h}) = q_t b_t$ with $\frac{\partial q}{\partial b} < 0$, and $q_t < \beta$, we necessarily have $\frac{\partial v\left(b^*_{\mathbf{h}};\mathbf{h}\right)}{\partial b} = q^*_{\mathbf{h}} + \frac{\partial q}{\partial b} b^*_{\mathbf{h}} < \beta$ which implies $1 - \bar{a}\frac{\partial v\left(b^*_{\mathbf{h}};\mathbf{h}\right)}{\partial b} > 1 - \beta\bar{a} > 0$, where the last inequality comes from assumption 1. Using this result with $\frac{\partial v\left(b^*_{\mathbf{h}};\mathbf{h}\right)}{\partial \mathbf{h}} > 0$, from Proposition 3, we obtain $\frac{\partial b^*_{\mathbf{h}}}{\partial \mathbf{h}} > 0$.

2. b. Finally, we have to prove that $b^{\max}_{\mathbf{h}} - b^{rss}_{\mathbf{h}}$ is increasing in $\mathbf{h}$ when $\mathbf{h} \geq \underline{\mathbf{h}}$. Note first that:

$$b^{\max}_{\mathbf{h}} - b^{rss}_{\mathbf{h}} = \left(b^{\max}_{\mathbf{h}} - \bar{a}\omega_{\mathbf{h}}\right) + \left(\bar{a}\omega_{\mathbf{h}} - b^{rss}_{\mathbf{h}}\right).$$

From (A.15), the first term of the right-hand side of this equality can be written:

$$b^{\max}_{\mathbf{h}} - \bar{a}\omega_{\mathbf{h}} = \left(\delta_{\mathbf{h}} - \bar{a}\right)\omega_{\mathbf{h}},$$

where, $\delta_{\mathbf{h}}$ and $\omega_{\mathbf{h}}$ are both increasing in $\mathbf{h}$, from Propositions 1 and 2 and $\delta_{\mathbf{h}} - \bar{a} > 0$ when $\mathbf{h} \geq \underline{\mathbf{h}}$,

Next, we have to prove that $\bar{a}\omega_{\mathbf{h}} - b^{rss}_{\mathbf{h}}$ is increasing in $\mathbf{h}$. As $b^*_{\mathbf{h}} = \delta^*_{\mathbf{h}}\omega_{\mathbf{h}}$, and knowing, from Proposition 2, that $\omega_{\mathbf{h}}$ is a strictly increasing function of $\mathbf{h}$, we only have to show that $\delta^*_{\mathbf{h}}$ is decreasing in $\mathbf{h}$ for $\mathbf{h} > \underline{\mathbf{h}}$. Using again $\delta = b/\omega_{\mathbf{h}}$, (A.1) and (A.12), we can express the difference between the value function $v(b;\mathbf{h})$ and the refinancing needs, $\frac{b}{\bar{a}} - \hat{s}$, as follows:

$$v(b;\mathbf{h}) - \frac{b}{\bar{a}} - \hat{s} = \left(\frac{\hat{s}}{\bar{a}}\right)\frac{\varphi(\delta;;\mathbf{h})}{1 - \beta x_{\mathbf{h}}}, \tag{A.16}$$

where the function $\varphi(\delta;\mathbf{h})$ is defined by

$$\varphi(\delta;\mathbf{h}) \equiv \bar{a} - \delta - \beta\bar{a}\left[x_{\mathbf{h}} - \chi(\delta,\mathbf{h})\right], \tag{A.17}$$

and verifies

$$\varphi(\delta^*_{\mathbf{h}};\mathbf{h}) = 0. \tag{A.18}$$

Let us denote $\varphi_z(\delta,\mathbf{h}) \equiv \partial\varphi(\delta;\mathbf{h})/\partial z$, the partial derivatives of $\varphi(\delta;\mathbf{h})$ for $z = \delta, \mathbf{h}$. Using again the notation $\Phi(\delta,\mathbf{h}) \equiv \partial\chi(\delta,\mathbf{h})/\partial\delta$, introduced in Appendix A.1.1, we obtain:

$$\varphi_\delta(\delta,\mathbf{h}) = -\left[1 - \beta\bar{a}\Phi(\delta,\mathbf{h})\right] < 0, \tag{A.19}$$

$$\varphi_{\mathbf{h}}(\delta,\mathbf{h}) = \beta\bar{a}\left[\int^{\delta} a\cdot dG(a) - \int^{\delta_{\mathbf{h}}} a\cdot dG(a)\right] \gtrless 0 \quad \text{iff} \quad \delta \geq \delta_{\mathbf{h}}. \tag{A.20}$$

The first derivative is negative since $\Phi(\delta,\mathbf{h})$, given by (A.3), is such that $\Phi(\delta,\mathbf{h}) \leq 1$ for $\delta \geq 0$, and $\beta\bar{a} < 1$ by assumption 1. The second one is negative (respect. positive) if $\delta > \delta_{\mathbf{h}}$ (respect. $\delta < \delta_{\mathbf{h}}$) From (A.18) , (A.19), and (A.20) we then obtain:

$$\frac{\partial\delta^*_{\mathbf{h}}}{\partial\mathbf{h}} = -\frac{\varphi_{\mathbf{h}}(\delta^*_{\mathbf{h}},\mathbf{h})}{\varphi_\delta(\delta^*_{\mathbf{h}},\mathbf{h})} \leq 0 \quad \text{iff} \quad \delta^*_{\mathbf{h}} \leq \delta_{\mathbf{h}}, \quad i.e. \text{ iff} \quad \mathbf{h} \geq \underline{\mathbf{h}}, \tag{A.21}$$

which ends the proof. □

A.1.5 Proof of Proposition 5

***Proposition* 5**: *In case of default, the post-default debt-to-GDP ratio* $\mathbf{h}\omega_{\mathbf{h}}$ *is unsustainable when* $\mathbf{h} > \mathbf{H} > \underline{\mathbf{h}}$, *where* $\mathbf{H}$ *is implicitly defined by:*

$$\mathbf{H}\omega_{\mathbf{H}} = \frac{b^{rss}_{\mathbf{H}}}{\bar{a}}.$$

Proof. Let us introduce the function $\Delta(\mathbf{h})$ implicitly defined by the condition (A.18): $\varphi(\delta^*_{\mathbf{h}};\mathbf{h}) = 0$, such that $\delta^*_{\mathbf{h}} = \Delta(\mathbf{h})$. From (A.21), we know that $\Delta'(\mathbf{h}) \leq 0$ for $\mathbf{h} \geq \underline{\mathbf{h}}$. Note that the condition $\mathbf{H}\omega_{\mathbf{H}} = \frac{b^{rss}_{\mathbf{H}}}{\bar{a}}$, or equivalently $\frac{\delta^*_{\mathbf{H}}}{\bar{a}} = \mathbf{H}$, is reached when we have $\Delta(\mathbf{h}) = \bar{a}\mathbf{h}$. We represent on Figure A.2 the functions $\Delta(\mathbf{h})$ and $\bar{a}\mathbf{h}$.

The two functions intersect for $\mathbf{h} = \mathbf{H}$ which unambiguously satisfies: $\underline{\mathbf{h}} < \mathbf{H} < 1$. When $\mathbf{h} > \mathbf{H}$, $\delta^*_{\mathbf{h}} = \Delta(\mathbf{h}) < \bar{a}\mathbf{h}$, or equivalently $\frac{b^{rss}_{\mathbf{h}}}{\bar{a}} < \mathbf{h}\omega_{\mathbf{h}}$. □

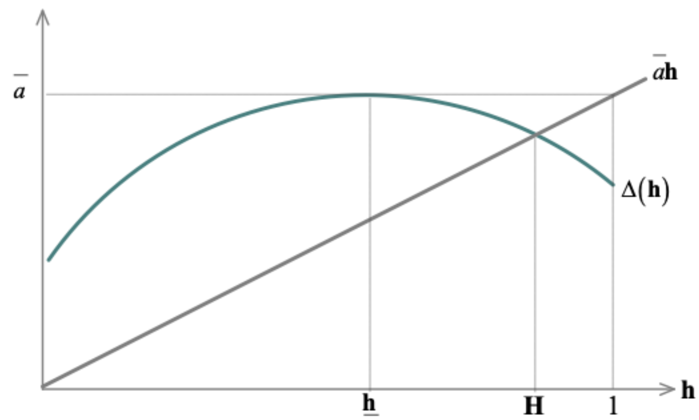

Figure A.2: $\underline{\mathbf{h}} < \mathbf{H} < 1$

A.2 Data and supplementary results.

Table A.2.1: Definition of variables and data sources.

Variable	Definition	Source
Debt-to-GDP ratio	General government gross debt to GDP ratio.	World Economic Outlook (IMF, October 2019)
Growth rate	Gross growth rate of real GDP	World Development Indicators (World Bank)
Inflation rate	Consumer price inflation (percentage, average)	World Development Indicators (World Bank)
Debt maturity	Average maturity of outstanding public debt	OECD database and Perez (2017)
Yield spread	Difference between long term country real interest rate and German rate. Real rate is equal to nominal rate minus three-year average of future consumer price inflation rate.	OECD database and Reuters (for nominal interest rates)

Table A.2.2: Data : advanced countries (1980-2018).

Country	$\bar{a}$	σ_a	Debt-maturity*	$\underline{\mathbf{h}}$
Australia	1.0314	0.0148	6.86	0.96
Austria	1.0201	0.0148	6.77	0.96
Belgium	1.0193	0.0144	6.25	0.96
Canada	1.0279	0.0199	5.79	0.95
Czech Republic	1.0207	0.0378	5.56	0.90
Denmark	1.0180	0.0190	6.47	0.95
Finland	1.0223	0.0306	4.45	0.93
France	1.0181	0.0136	6.38	0.96
Germany	1.0174	0.0191	5.47	0.95
Greece	1.0085	0.0349	7.27	0.90
Hong Kong	1.0470	0.0373	5.34	0.92
Iceland	1.0359	0.0358	4.27	0.92
Ireland	1.0499	0.0502	5.33	0.89
Israel	1.0356	0.0184	6.59	0.95
Italy	1.0124	0.0184	5.33	0.95
Japan	1.0195	0.0225	5.56	0.94
Korea	1.0621	0.0400	4.05	0.92
Latvia	1.0413	0.0570	5.34	0.88
Lithuania	1.0432	0.0509	5.34	0.89
Luxembourg	1.0390	0.0315	4.87	0.93
Netherlands	1.0210	0.0181	6.43	0.95
New Zealand	1.0264	0.0187	4.25	0.96
Norway	1.0247	0.0173	3.58	0.96
Portugal	1.0201	0.0262	4.73	0.94
Singapore	1.0648	0.0392	5.34	0.92
Slovak Republic	1.0399	0.0314	3.48	0.94
Spain	1.0230	0.0217	5.03	0.95
Sweden	1.0218	0.0210	3.68	0.95
Switzerland	1.0183	0.0158	5.34	0.96
United Kingdom	1.0217	0.0193	5.34	0.95
United States	1.0265	0.0185	5.15	0.96
Sample	1.0284	0.0310	5.34	0.94

Notes: μ and σ are the mean and standard deviation of the log gross growth rate of GDP per capita expressed in %. $\underline{\mathbf{h}}$ is the minimum value of $\mathbf{h}$ above which a risky steady state exists (see Proposition 4). Growth data are from the World Bank database and cover the period 1980-2018 . *Average maturity of outstanding public debt (1980-2012).

Table A.2.3: Data : emerging countries (1980-2018).

Country	$\bar{a}$	σ_a	Debt-maturity*	$\underline{\mathbf{h}}$
Brazil	1.0246	0.0334	5.22	0.92
Chile	1.0438	0.0397	1.27	0.95
China	1.0953	0.0271	4.84	0.95
Colombia	1.0348	0.0210	2.79	0.96
Hungary	1.0220	0.0276	6.10	0.93
Malaysia	1.0585	0.0354	2.18	0.95
Mexico	1.0257	0.0326	2.23	0.95
Nigeria	1.0320	0.0547	1.70	0.91
Pakistan	1.0493	0.0202	4.10	0.96
Philippines	1.0387	0.0335	3.35	0.94
Poland	1.0377	0.0267	3.53	0.94
Russia	1.0082	0.0643	4.59	0.84
South Africa	1.0228	0.0226	11.37	0.93
Sample	1.0390	0.0408	4.10	0.93

Notes: μ and σ are the mean and standard deviation of the log gross growth rate of GDP per capita. $\underline{\mathbf{h}}$ is the minimum value of $\mathbf{h}$ above which a risky steady state exists (see Proposition 4). Growth data are from the World Bank database and cover the period 1980-2018. *Average maturity of outstanding public debt (1980-2012).

Table A.2.4: Sustainability and low interest interest rate: Euro zone ($r = 0.31\%$).

Country	$\beta\bar{a}_i$	$\tilde{\mathbf{h}}_\mathbf{i}$
Austria	1.007	0.95
Belgium	1.009	0.87
Finland	1.000	1.00
France	1.006	0.95
Germany	1.010	0.95
Greece*	0.970	1.00
Ireland	1.049	0.53
Italy*	0.994	1.00
Latvia	1.006	0.97
Lithuania	1.013	0.94
Luxembourg	1.021	0.83
Netherlands	1.006	0.96
Portugal	1.000	1.00
Slovak Republic	1.019	0.93
Spain	1.002	0.99

Notes. *: Countries where $\beta\bar{a}_i < 1$, that is $g < r$. The time period for the growth rate $\bar{a}_i$ and the risk-free rate r is 2009-2018.

Table A.2.5: Debt-to-GDP ratios and real yield spreads (%): advanced countries (1980-2018).

	Debt-to-GDP ratio (b_t)				Yield spread ($\tilde{s}_t$)			
Country	Mean	Std	Min	Max	Mean	Std	Min	Max
Australia	23.71	9.68	9.69	41.37	1.36	1.90	-1.03	7.00
Austria	69.1	9.32	55.93	84.4	0.08	0.29	-0.33	1.02
Belgium	110.73	15.37	76.36	138.14	0.38	0.90	-0.79	2.58
Canada	78.24	14.12	44.91	100.25	0.67	1.00	-0.59	3.19
Czech Republic	28.63	10.47	11.65	44.91	0.18	0.96	-1.67	1.46
Denmark	48.9	14.13	27.35	78.63	0.61	1.09	-0.85	4.16
Finland	38.33	17.38	10.89	63.45	1.02	1.87	-1.34	5.50
France	58.83	24.52	20.83	98.42	0.54	0.78	-0.75	2.48
Germany	63.26	11.27	38.99	82.31	–	–	–	–
Greece	101.97	48.44	22.53	184.85	4.10	6.67	-1.78	22.90
Hong Kong	0.97	1.02	0.05	3.52	0.20	3.24	-4.34	7.13
Iceland	47.61	19.64	24.48	92.03	1.46	2.51	-3.77	4.91
Ireland	61.15	30.82	23.62	120.04	1.25	2.35	-2.80	7.60
Israel	74.39	10.74	60.41	92.89	1.86	2.13	-3.70	5.68
Italy	112.52	12.31	92.91	132.16	1.25	1.68	-1.13	4.34
Japan	136.42	66.77	48.81	237.13	-0.72	1.05	-3.46	0.94
Korea	22.64	10.19	7.98	37.92	1.06	1.13	-0.95	3.04
Latvia	26.12	14.09	8.12	46.91	-0.04	4.50	-8.04	8.60
Lithuania	28.59	9.74	14.57	42.58	0.71	3.39	-4.90	9.54
Luxembourg	13.46	6.83	6.49	23.69	-0.58	0.87	-2.39	0.54
Netherlands	60.76	10.56	41.97	76.78	-0.01	0.95	-2.28	1.89
New Zealand	38	14.66	16.3	68.58	1.88	1.60	-0.48	6.74
Norway	36.85	8.09	22.94	52.56	0.60	1.34	-1.27	3.98
Portugal	78.89	30.42	50.34	130.61	1.58	2.90	-2.16	9.56
Singapore	90.48	13.31	69.82	113.63	-0.09	2.23	-3.45	3.55
Slovak Republic	41.59	9.52	21.67	54.74	0.31	1.44	-2.27	3.72
Spain	55.84	22.88	16.58	100.37	1.03	2.02	-1.88	5.18
Sweden	49.68	11.49	37.24	69.15	0.93	1.29	-1.07	4.16
Switzerland	47.69	7	34.35	59.16	-0.48	1.08	-3.21	1.23
United Kingdom	50.4	19.93	28.57	87.91	0.55	0.93	-1.10	2.81
United States	84.29	20.85	53.15	106.82	-0.05	0.93	-1.66	1.96
Sample	60.14	38.13	0.05	237.13	0.68	2.19	-8.04	22.90

Notes: Min, Max and Std are the minimum, maximum and standard deviation. The time period of the sample is 1980-2018. Sources: Debt-to-GDP ratios correspond to general government gross debt from the IMF World Economic Outlook database (October 2019). Yield spreads are the difference between the country's long-term government real interest rates and the German rates, hence the dash (–) in the table for the German spread. Sources: see Table A.2.1.

Table A.2.6: Debt-to-GDP ratios and real yield spreads (%) : emerging countries (1980-2018).

	Debt-to-GDP ratio (b_t)				Yield spread ($\tilde{s}_t$)			
Country	Mean	Std	Min	Max	Mean	Std	Min	Max
Brazil	69.32	8.07	60.2	87.89	4.39	3.83	-2.31	13.31
Chile	15.31	8.17	3.88	37.37	1.60	1.54	-1.33	3.65
China	30.59	8.87	20.45	50.64	-0.10	1.92	-3.07	3.16
Colombia	38.63	7.99	23.36	52.16	4.34	2.23	0.19	8.15
Hungary	68.43	9.75	51.58	84.06	1.53	2.31	-1.33	6.84
Malaysia	46.82	11.03	29.62	74.13	0.66	1.87	-1.37	3.96
Mexico	44.21	5.46	37.21	56.76	2.58	1.46	-0.02	6.19
Nigeria	33.64	21.97	7.28	74.96	0.40	3.75	-5.65	5.21
Pakistan	65.62	7.3	52.44	81.23	3.98	2.43	-0.44	7.12
Philippines	55.39	11.14	38.92	76.08	3.04	2.85	-1.74	9.90
Poland	46.85	5.63	36.38	55.69	2.10	1.58	-0.80	5.00
Russia	29.15	31.45	7.44	135.06	0.60	5.97	-6.55	15.84
South Africa	39.67	8.87	26.51	56.71	2.41	2.41	-2.57	7.62
Sample	44.32	20.47	3.88	135.06	2.22	2.84	-5.65	13.31

Notes: Min, Max and Std are the minimum, maximum and standard deviation. The time period of the sample is 1980-2018. Sources: Debt-to-GDP ratios correspond to general government gross debt from the IMF World Economic Outlook database (October 2019). Yield spreads are the difference between the country's long-term real government interest rates and the German rates. See Table A.2.1 for data sources.