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Optimal insurance under risk and smooth ambiguity revisited

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Abstract

We revisit the problem of optimal insurance contract design under risk and ambiguity in an optimal control framework where the indemnity function and the premium are to be solved for simultaneously. Our approach generalizes the analyses carried out so far in the context of the smooth ambiguity model. We prove the existence of an optimal insurance policy under the standard assumption of a risk-averse policyholder and a risk-neutral insurer, both of whom can be averse or neutral to ambiguity. We characterize not only the risk-sharing but also the ambiguity-sharing rule between an insurer and a policyholder. Under one-sided risk and ambiguity aversion, we show that a straight deductible policy cannot be optimal when ambiguity leads to the incompleteness of the insurance contract.

Keywords: optimal insurance, smooth ambiguity aversion.

JEL classification: D82, D86.

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1 Introduction

We know from the seminal work of Arrow (1974) that a straight deductible is optimal for a risk-averse policyholder facing a risk-neutral insurer and linear cost of indemnity provision. Since this pioneering work, the efficiency of deductible contracts has become one of the basics of Insurance Economics and has proved to be particularly robust to economic contexts and generalizations.

Raviv (1979) made the first attempt to generalize the work of Borch (1960) and Arrow (1965; 1974) to demonstrate that the existence of co-insurance contracts is due to either the nonlinear cost of indemnity provision, or risk-aversion on the part of the insurer. Huberman et al. (1983) shows that a disappearing deductible is optimal in the presence of concave transaction costs.

Subsequently, some contributions focused on the generalization of these results beyond the Expected Utility (EU) model (Zilcha and Chew (1990), Karni (1992), Machina (1995), Chateauneuf et al. (2000)). For example, Gollier and Schlesinger (1996) showed that the optimality of deductibles is not exclusively reserved for the EU model since it springs from first- and second-degree stochastic dominance. In the presence of fixed cost, an issue initially raised by Gollier (1987), Carlier and Dana (2003) prove the existence of an optimal insurance contract relying on a stochastic dominance hypothesis.

Recently, Bernard et al. (2015) question the relevance of a straight deductible contract for a decision maker whose preferences are described by the Rank Dependent Expected Utility (RDEU) model. In contrast with the mainstream results they showed that the optimal contract insures not only large losses above a deductible but also small ones. Similar results are obtained by Xu (2018), Xu et al. (2019) and Ghossoub (2019b). As the RDEU model results in a better fit to real human behavior than the EU model, these authors challenge Arrow's result.

In this paper, we continue to investigate the robustness of the efficiency of deductible insurance contracts under ambiguity. For this purpose, we characterize the efficient design of an insurance contract under ambiguity and provide a comprehensive treatment of the relationship between the insurer and the policyholder, in a principal-agent framework, both under risk and ambiguity.

The concept of ambiguity motivated by the Ellsberg's paradox (Ellsberg, 1961) has led to the development of several competing ambiguity models (See Gilboa and Marinacci (2016), Machina and Siniscalchi (2014) or Etner et al. (2012)). This concept contributes to the understanding of a growing number of economic topics and puzzles, such as the stock market participation puzzle (Dow and Werlang (1992), Bossaerts et al. (2010), Collard et al. (2018)), portfolio choice and ambiguity aversion (Gollier (2011)), the low take-up of freely available genetic tests (Hoy et al. (2014)), the decision to trust (Corcos et al. (2012), Li et al. (2019)), the value of statistical life (Treich (2010), Bleichrodt et al. (2019), Berger et al. (2013)).

In the case of insurance behavior, ambiguity makes sense since many risks are either objectively poorly defined (e.g. environmental risks) or subjectively poorly perceived by the insured (e.g. health risk). Several competing approaches have been adopted to model this phenomenon. Among them there are contributions that follow the Choquet Expected Utility (CEU) approach proposed by (Schmeidler, 1989), including Carlier et al. (2003), Amarante et al. (2015); Amarante and Ghossoub (2016), Ghossoub (2019a). In the context of CEU, Ghossoub and colleagues consider only fixed-premium contracts. These authors also allow for the possibility of an ambiguity-seeking insurer. The more recent smooth ambiguity approach of Klibanoff et al. (2005) is also adopted to address the question of optimal demand for prevention and insurance when risks are ambiguous. Two recent contributions in this line of research include Alary et al. (2013) and Gollier (2014). While Alary et al. (2013) emphasize the role of ambiguity aversion on the demands for insurance, self-insurance and self-protection, Gollier (2014) characterizes optimal insurance contracting under linear transaction costs.

In this study, we also follow the smooth model of ambiguity of Klibanoff et al. (2005) owing to its ability to separate ambiguity and attitudes towards ambiguity. Moreover, the smooth ambiguity model has received significant support from a number of experimental studies, such as Halevy (2007), Chakravarty and Roy (2009), Conte and Hey (2013), Ahn et al. (2014), Baillon and Bleichrodt (2015), Cubitt et al. (2019), Cubitt et al. (2018).

In the context of smooth ambiguity, we implement a comprehensive approach of the problem of optimal insurance contracting under symmetric information and convex cost. This framework allows us to extend and revisit the analysis of optimal insurance design under risk and ambiguity. We find ambiguity might be a source of contract incompleteness which challenges the efficiency of straight deductibles.

First, we explore the idea that both parties could be ambiguous averse. If risk aversion on insurer's side has already been studied in Raviv (1979), we found relevant to assume ambiguity aversion not only on the policyholder's side but also on insurer's side. In the context of environmental and catastrophic risks, several studies documented the fact that insurers are ambiguity averse (See Kunreuther and Hogarth (1992), Kunreuther et al. (1993), Kunreuther et al. (1995), Cabantous (2007), Cabantous et al. (2011)). Moreover, the substantial growth of insurance-linked securities (Cat bonds), which provide capital market-based insurance against the risk of natural catastrophes, in addition to standard reinsurance mechanisms, also argues for the benefit of the general assumption of ambiguity aversion.

Second, to our knowledge, following the seminal contribution of Arrow (1974), most papers tackle the optimal insurance problem in a two-step approach. First, they solve for the form of an optimal indemnity function for a fixed premium, and then search for the Pareto efficient contracts by determining the premium. In generalizing Arrow's results, Raviv (1979) also follows this approach, and has found, after the first step, two solution candidates: one policy with a deductible and the other with an upper limit coverage. The ultimate form of the indemnity schedule depends on the fixed premium. Nevertheless, this premium is defined in an implicit manner in the second step of Raviv's proof, making it impossible to be analyzed in practice. To resolve this issue, we propose an alternative approach whereby the optimization is done with respect to the pair of the premium and the indemnity function *simultaneously*.

The rest of the paper is organized as follows. Section 2 introduces the optimal insurance problem under risk and ambiguity and key assumptions. Section 3 provides an existence proof and employs the Pontryagin Maximum Principle to study the general features of an optimal contract under ambiguity. Section 4 examines two special important cases in details: the case where both contracting parties are ambiguity-neutral and the case of two ambiguous states. Section 5 concludes and suggests a roadmap for future research. All proofs can be found in the appendices.

2 The optimal insurance problem

In this paper, we are interested in the problem where a potential policyholder considers an optimal insurance policy $(I(\cdot), \pi)$ where π is the premium the policyholder pays to the insurer to obtain an indemnity schedule $I(\cdot)$. Let the subscripts A and P denote, respectively, the policyholder and the insurer. The problem of the policyholder can be formulated as:

$$
\begin{aligned}
\max_{(I(\cdot),\pi)} \quad & \sum_{i=1}^{n} p_i \phi_A \left(\int_{I_x} u(W_A - \pi - x + I(x)) f_i(x) dx \right) && \text{(1a)} \\
s.t. \quad & I(x) \in [0, x], \quad \forall x \in I_x, && \text{(1b)} \\
& \pi \in I_\pi = [\underline{\pi}, \overline{\pi}] \subseteq I_x, && \text{(1c)} \\
& \sum_{i=1}^{n} p_i \phi_P \left(\int_{I_x} v(W_P + \pi - I(x) - \psi(I(x)) f_i(x) dx \right) \geq \bar{V}, && \text{(1d)}
\end{aligned}
$$

where W_A and W_P stand for the initial wealth of the policyholder and the insurer, respectively. The last inequality is often called the participation constraint (of the insurer). In the program above, x stands for the loss faced by the policyholder, which is a continuous random variable. Ambiguity enters through the unknown *second-order state* i taking values in a finite second-order state space $\mathcal{I}$. Notice that the density of the loss is i-conditional. The DMs have perfect knowledge of $\mathcal{I}$ and each conditional distribution $f_i(\cdot)$ of the loss, but faces ambiguity on the distribution of the second-order states. The set $\{p_i\}_{i \in \mathcal{I}}$ is the set of priors the DMs have on the distribution of the second-order states. We assume that the priors are symmetric, in the sense that both DMs have the same information on the distribution of the second-order states, and thus the same conditional loss densities. We can view this assumption as the equilibrium outcome of some information market, an issue that is interesting but is beyond the scope of this paper. Both DMs exhibit attitudes towards risk and towards ambiguity. In particular, the attitude towards risk of the policyholder and the insurer is captured by the convexity of the utility function $u(\cdot)$ and $v(\cdot)$, respectively. Typically, the policyholder is risk-averse, implying that u is strictly concave, and the insurer is risk-neutral, implying that v is linear . Without loss of generality (WLOG), we can let v be the identity function. The policyholder's attitude towards ambiguity, according to Klibanoff et al. (2005), is described by the convexity of the functional ϕ_A. This functional being concave, linear, or convex corresponds to an ambiguity-averse, ambiguity-neutral, or ambiguity-loving policyholder. Typically, the policyholder is ambiguity-averse. Likewise, the insurer's attitude towards ambiguity is captured by the convexity of the functional ϕ_P. Let us now state these assumptions more concretely.

Assumption 1 (Finite second-order state space)**.** *The second-order state space $\mathcal{I}$ is finite. In particular $\mathcal{I} = \{1, 2, \ldots, n\}$, for some positive integer n.*

Assumption 2 (Common priors)**.** *Let p_i denotes the common prior probability of state $i \in \mathcal{I}$ for each DM. Assume $p_i \in (0, 1)$ for all $i \in \mathcal{I}$ and $\sum_{i=1}^{n} p_i = 1$.*

Assumption 3 (Common bounded support)**.** *The loss $\tilde{x}$ is a continuous random variable whose state-conditional densities have a bounded common support $I_x = [0, \bar{x}]$, where $\bar{x} > 0$.*

Assumption 4 (Strictly positive conditional densities). *The state-conditional cumulative density functions (cdfs)* $F_i : I_x \to [0,1]$, *for each* i *in* $\mathcal{I}$. *The cdfs are* C^2 *on the common domain* I_x. *Let* $f_i : I_x \to (0,\infty)$ *stand for the state-i probability density function (pdf) of* $\tilde{x}$ *defined by* $f_i(x) = \frac{\partial F_i(x)}{\partial x}$, *for* $x \in I_x$. *Denote* $f = (f_i)_{i\in\mathcal{I}}$ *the n-dimensional vector of conditional densities.*

Assumption 5 (Ordering of conditional distributions). *Assume that there exists an ordering criterion of the ambiguous states. In particular, we establish that state* i *is better than state* j *whenever* $i < j \in \mathcal{I}$ *in the sense that* F_j *dominates* F_i *in the sense of likelihood ratio dominance (LRD). In particular, let* $\ell_{ij} : I_x \to \mathbb{R}_+^*$ *be the LR defined by* $\ell_{ij}(x) = \frac{f_i(x)}{f_j(x)}$, *for* $x \in I_x$. *Let* $i, j \in \mathcal{I}$. *Then* $i < j$ *implies* $\ell_{ij}^{'}(x) \leq 0$ *for all* $x \in I_x$, *with strict inequality in some subset of positive measure of* I_x. *In other words, the cdf* F_j *dominates the cdf* F_i *in the sense of LRD whenever* $i < j$.[1]

Assumption 6 (Bounded indemnity). *The measurable indemnity function* $I : I_x \to I_x$ *satisfies* $I(x) \in [0, x]$ *for all* $x \in I_x$.

Assumption 7 (Convex cost). *The cost of indemnity provision* $\psi(\cdot)$ *is a* C^2 *function* $\psi : I_x \to \mathbb{R}_+$ *satisfying* $\psi(0) = 0$, $\psi^{'} > 0$, $\psi^{''} \geq 0$, *and* $\psi(I) \leq I$ *for all* $I \geq 0$.

As mentioned earlier, the policyholder is risk-averse, namely that her preference can be modeled with a strictly increasing and concave utility function as follows. By contrast, the insurer is risk-neutral.

Assumption 8 (Risk aversion of the policyholder). *The utility function of the policyholder* $u : \mathbb{R}_+^* \to \mathbb{R}$ *is at least* C^2, *strictly increasing and strictly concave:* $u' > 0$, *and* $u^{''} < 0$.

To ensure that u is always well-defined, let us assume that the initial wealth W_A of the policyholder satisfies:

$$W_A \geq \overline{\pi} + \bar{x}, \tag{2}$$

where $\overline{\pi}$ is the upperbound for the premium. Let $r_u : \mathbb{R}_+^* \to \mathbb{R}_+^*$ denote the familiar Arrow-Pratt measure of absolute risk aversion, defined by:

$$r_u(w) = -\frac{u''(w)}{u'(w)}. \tag{3}$$

Assumption 9 (Risk neutrality of the insurer). *The insurer has identity utility function, namely that* $v : \mathbb{R}_+^* \to \mathbb{R}$ *is the map* $x \mapsto x$ *for all* $x \in \mathbb{R}_+^*$.

In light of Assumption 9, the participation constraint of the insurer can be rewritten as:

$$\sum_{i=1}^{n} p_i \phi_P \left(\int_{I_x} (W_P + \pi - I(x) - \psi(I(x)) f_i(x) dx \right) \geq \bar{V}. \tag{4}$$

The phenomenon known as "ambiguity aversion" revived by Ellsberg (1961) is modeled in the smooth sense of Klibanoff et al. (2005) via a strictly monotone concave second-order utility functional.

[1] Recall that LRD is a special case of first-order stochastic dominance (FSD). Thus F_j dominates F_i in the LRD sense implies $F_j(x) \leq F_i(x)$ for all $x \in I_x$, with strict inequality on some subset of I_x of positive measure. See, for example, Wolfstetter (1999) for a discussion.

Assumption 10 (Ambiguity aversion)**.** *Let the second-order utility functional be* $\phi_J : \mathbb{R} \to \mathbb{R}$*, where* $J \in \{A, P\}$*. Then* ϕ_J *is at least* C^2*, strictly increasing and concave on its domain. Assume that the* ϕ_J*'s have bounded first-order derivatives, so that* $0 < \phi_J' < +\infty$*,* $\phi_J'' \leq 0$*, for each* $J \in \{A, P\}$*. Whenever* ϕ_J *is linear, we assume without loss of generality that* ϕ_J *is the identity function.*

This assumption means that the DMs are either ambiguity-neutral (ϕ_J is the identity function), or is (strictly) ambiguity-averse (ϕ_J is strictly concave).

Finally, let us make the following assumption regarding the initial wealth levels of the DMs.

Assumption 11 (Other parameters)**.** *We assume that* $\bar{V}$ *is equal to the reservation second-order utility of the insurer (i.e., the utility obtained without participating in the contract), namely that:*

$$\bar{V} = \phi_P(W_P). \tag{5}$$

Furthermore, we assume that the bounds for the premium satisfy:

$$\underline{\pi} = 0, \tag{6}$$

$$\overline{\pi} = \int_{I_x} (x + \psi(x))\, \bar{f}(x) dx, \tag{7}$$

$$\tag{8}$$

where $\bar{f} : I_x \to \mathbb{R}_+^*$ *is the ambiguity-neutral density defined by*

$$\bar{f}(x) = \sum_{i=1}^{n} p_i f_i(x), \quad x \in I_x. \tag{9}$$

Equation (7) says that the premium cannot exceed the expected total cost of providing uniformly full insurance with respect to the ambiguity-neutral density.

3 Existence and general characterization of an optimal insurance contract under ambiguity

In this section, we employ techniques from optimal control theory to first prove that under standard conditions, an optimal insurance contract exists. We next use the Pontryagin Maximum Principle (PMP) to derive a set of necessary conditions that must be satisfied by such contracts.

3.1 Existence

We first reformulate the policyholder's problem as an optimal control problem (OCP). To this end let the control be the function $J : I_x \to [0, 1]$ defined by:

$$xJ(x) = I(x), \quad x \in I_x. \tag{10}$$

Observe that J is simply the insurance coverage rate. Since $I(x) \in [0, x]$ for all $x \in I_x$, the admissible control set $\mathcal{U}$ is:

$$\mathcal{U} = \{J : I_x \to [0, 1],\ J \text{ measurable}\}. \tag{11}$$

Note that $\mathcal{U}$ is compact with respect to the weak-$\star$ topology. To describe the "dynamic" of the system, let $X \equiv (y, z, \pi)$ be the state vector defined on the state space $\mathcal{X} = \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times I_\pi$ satisfying:

$$\dot{X}(x) = \begin{pmatrix} \dot{y}(x) \\ \dot{z}(x) \\ \dot{\pi}(x) \end{pmatrix} = \begin{pmatrix} u(W_A - \pi - x + xJ(x))f(x) \\ (W_P + \pi - xJ(x) - \psi(xJ(x)))\, f(x) \\ 0 \end{pmatrix}, \qquad X(0) = \begin{pmatrix} y(0) = 0 \\ z(0) = 0 \\ \pi(0) = \pi \end{pmatrix}, \tag{12}$$

where $f(x) = (f_i(x))_{i \in \mathcal{I}}$ is the n-dimensional vector of conditional densities defined in Assumption 4.

Let the cost functional faced by the policyholder be

$$g(J, \pi) = -\sum_{i=1}^{n} p_i \phi_A\left(y_i(\bar{x})\right), \tag{13}$$

which is just minus her objective function. Define the risk-neutral insurer's net welfare functional:

$$h(J, \pi) = \sum_{i=1}^{n} p_i \phi_P\left(z_i(\bar{x})\right) - \bar{V}. \tag{14}$$

The policyholder's problem can be rewritten more compactly as the following optimal control problem (OCP):

$$\min_{\{J,\pi\}} g(J, \pi) \quad s.t. \quad h(J, \pi) \geq 0. \tag{OCP}$$

Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be measurable subsets of $\mathcal{X}$ defined as

$$\mathcal{M}_0 = \{0\} \times \{0\} \times I_\pi, \tag{15}$$

$$\mathcal{M}_1 = \mathbb{R}^n \times S_{z,\pi}, \tag{16}$$

where $S_{z,\pi} = \{z \in \mathbb{R}^n \times I_\pi \mid h(J, \pi) \geq 0\}$ is the admissible set. The sets $\mathcal{M}_0$ and $\mathcal{M}_1$ are often called the source and target sets of the control system (12). The OCP admits an optimal pair (J, π). In other words, there exits an optimal insurance contract (I, π) such that $I(x) = xJ(x)$ for $x \in I_x$.

Proof. See Subsection 6.1. ■

Lemma 1. *The insurer's participation constraint is saturated at an optimum, that is if (J, π) is optimal, then $h(J, \pi) = 0$.*

Proof. See Subsection 6.2. ■

Before proceeding, let us consider two corner cases of the problem: the case of uniformly zero insurance $J = 0$, and the case of uniformly full insurance $J = 1$.

First, consider the case of uniformly zero insurance $J(x) = 0$ for all $x \in I_x$. In this case,

$$h(0, \pi) = \phi_P(W_P + \pi) - \bar{V} = \phi_P(W_P + \pi) - \phi_P(W_P), \tag{17}$$

where the second equality comes from (5). Furthermore by (6) and (5),

$$h(0, \underline{\pi}) = 0. \tag{18}$$

Observe that h is strictly monotone in π and $\pi \geq \underline{\pi}$ for all $\pi \in I_\pi$. Hence the pair $(0, \pi)$ is admissible for all $\pi \in I_\pi$. Nevertheless by Lemma 1, only the pair $(0, \underline{\pi})$ is a candidate for an optimum.

Next, consider the case of uniformly full insurance, namely the case $J(x) = 1$ for all $x \in I_x$. We have:

$$h(1, \pi) = \sum_{i=1}^{n} p_i \phi_P \left(W_P + \pi - \int_{I_x} (x + \psi(x)) f_i(x) dx \right). \tag{19}$$

Note that since h is strictly increasing in π,

$$h(1, \pi) \leq h(1, \overline{\pi}), \quad \forall \pi \in I_\pi. \tag{20}$$

Let us consider two subscases.

- If ϕ_P is strictly concave (the insurer is ambiguity-averse), then by Jensen inequality and condition (7),
$$h(1, \overline{\pi}) < \phi_P(W_P) - \bar{V} = 0. \tag{21}$$
Then (20) and (21) imply
$$h(1, \pi) < 0, \quad \forall \pi \in I_\pi, \tag{22}$$
implying that there is no admissible pair.
- If ϕ_P is identity (the insurer is ambiguity-neutral), then also by condition (7) we have
$$h(1, \overline{\pi}) = 0, \tag{23}$$
implying that $h(1, \pi) \leq 0$ for all $\pi \in I_\pi$. Hence the only admissible pair is $(J = 1, \pi = \overline{\pi})$.

To sum up, a contract involving uniformly zero insurance is always admissible, but can be an optimum if and only if the associated premium is zero. By contrast a contract involving uniformly full insurance is admissible if and only if the insurer is ambiguity-neutral.

3.2 General shape of an optimal contract

Let us now use the PMP to derive the necessary conditions and investigate some general features of an optimal insurance contract. The statement of the PMP applied to the OCP is provided in the following theorem.[2]

Theorem 1. *Suppose (X, J) is an optimal pair for the OCP. There exists an absolutely continuous vector-valued function $\lambda : I_x \to \mathbb{R}^{2n+1}$ and a real number $\lambda_0 \geq 0$ with $(\lambda, \lambda_0) \neq 0 \in \mathbb{R}^{2n+2}$ such that:*

[2] This version of the PMP is adapted from Trélat (2008).

1. *λ satisfies the canonical equations:*

$$\begin{aligned} \dot{X}(x) &= \nabla_\lambda H(X(x), J(x), \lambda(x), \lambda_0, x), &(24)\\ \dot{\lambda}(x) &= -\nabla_X H(X(x), J(x), \lambda(x), \lambda_0, x), &(25) \end{aligned}$$

for almost every $x \in I_x$, where the real-valued function $H : \mathbb{R}^{2n+1} \times \mathbb{R} \times \mathbb{R}^{2n+1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, called the Hamiltonian, is defined by:

$$\begin{aligned} H(X, \omega, \lambda, \lambda_0, x) = u(W_A - \pi - x + x\omega) \langle \lambda_y, f(x) \rangle \\ + (W_P + \pi - x\omega - \psi(x\omega)) \langle \lambda_z, f(x) \rangle, \end{aligned} \tag{26}$$

where $\lambda \equiv (\lambda_z, \lambda_y, \lambda_\pi)^T \in \mathbb{R}^{2n+1}$ is the adjoint vector whose components $\lambda_z \in \mathbb{R}^n$, $\lambda_y \in \mathbb{R}^n$ and $\lambda_\pi \in \mathbb{R}$ themselves are the adjoint vectors corresponding to the state variables z,y and π, respectively.

2. *The maximum condition:*

$$H(X(x), J(x), \lambda(x), \lambda_0, x) = \max_{\omega \in [0,1]} H(X(x), \omega, \lambda(x), \lambda_0, x) \tag{27}$$

is satisfied for almost every $x \in I_x$.

3. *The transversality conditions (TCs) hold:*

$$\begin{aligned} \lambda(0) &\in N_{\mathcal{M}_0}(X(0)), &(28)\\ -\lambda_0 \nabla_X g(J, \pi) - \lambda(\bar{x}) &\in N_{\mathcal{M}_1}(X(\bar{x})), &(29) \end{aligned}$$

where $N_{\mathcal{M}_i}(X(x))$ denotes the normal cone to $\mathcal{M}_i$ at $X(x)$, for $i \in \{0, 1\}$.

Let $\mu = (\mu_h, \mu_{\underline{\pi}}, \mu_{\overline{\pi}}) \in \mathbb{R}^3_+$ be the vector of Lagrange multipliers, where μ_h is associated to the constraint $h(J, \pi) \geq 0$, and $(\mu_{\underline{\pi}}, \mu_{\overline{\pi}})$ is associated to the constraint $\pi \in I_\pi$.

The adjoint vectors λ_y and λ_z are constant with respect to x. In particular,

$$\begin{aligned} \lambda_y &= \lambda_0 \left(p_i \phi_A'(y_i(\bar{x}))\right)_{i \in \mathcal{I}}, &(30)\\ \lambda_z &= \mu_h \left(p_i \phi_P'(z_i(\bar{x}))\right)_{i \in \mathcal{I}}. &(31) \end{aligned}$$

The adjoint vector λ_π satisfies

$$\begin{aligned} \lambda_\pi(\bar{x}) &= \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle + \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle + \mu_{\underline{\pi}} - \mu_{\overline{\pi}}, &(32)\\ \lambda_\pi(0) &= 2\left(\left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle + \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle\right) + \mu_{\underline{\pi}} - \mu_{\overline{\pi}}. &(33) \end{aligned}$$

Moreover if $\pi \in (\underline{\pi}, \overline{\pi})$, then $(\mu_{\underline{\pi}}, \mu_{\overline{\pi}}) = 0$ and

$$\lambda_\pi(0) = \lambda_\pi(\bar{x}) = 0. \tag{34}$$

Proof. See Subsection 6.3. ■

Lemma 2. *The non-triviality condition $(\lambda_0, \mu_h) \neq 0$ holds.*

Proof. See Subsection 6.4. ■

Let us now consider the maximum condition. Denote $H_\omega \equiv \frac{\partial H}{\partial \omega}$, and $H_{\omega\omega} \equiv \frac{\partial^2 H}{\partial \omega^2}$. For every fixed $x \in I_x$, we have

$$H_\omega = x\left[u'(W_A - \pi - x + x\omega)\langle \lambda_y, f(x)\rangle - (1 + \psi'(x\omega))\langle \lambda_z, f(x)\rangle\right], \quad (35)$$
$$H_{\omega\omega} = x^2\left[u''(W_A - \pi - x + x\omega)\langle \lambda_y, f(x)\rangle - \psi''(x\omega)\langle \lambda_z, f(x)\rangle\right]. \quad (36)$$

Furthermore, denote

$$K(x) = u'(W_A - \pi - x)G(x) - (1 + \psi^{'}(0)), \quad (37)$$

and

$$L(x) = u'(W_A - \pi)G(x) - (1 + \psi'(x)). \quad (38)$$

Notice that the signs of $K(x)$ and $L(x)$ are a.e. identical to the signs of $H_\omega(x)|_{\omega=0}$ and $H_\omega(x)|_{\omega=1}$, respectively.

Lemma 3. *At an optimum,*

- $\lambda_0 = 0$ *if and only if the optimal contract is the trivial pair* $(J = 0, \pi = \underline{\pi})$*;*
- $\mu_h = 0$ *if and only if* ϕ_P *is linear, in which case the optimal contract entails full insurance is the pair* $(J = 1, \pi = \overline{\pi})$*.*

Proof. See Subsection 6.5. ■

In the remaining analyses of the paper we assume, whenever not explicitly stated, that both λ_0 and μ_h are strictly positive. Since all the prior probabilities and the densities are strictly positive (assumptions 2 and 4), we can define:

$$A(x) = \frac{\langle \lambda_y, f(x)\rangle}{\langle p, f(x)\rangle}, \quad x \in I_x, \quad (39)$$
$$P(x) = \frac{\langle \lambda_z, f(x)\rangle}{\langle p, f(x)\rangle}, \quad x \in I_x. \quad (40)$$

Observe that both A and P are strictly positive since $\lambda_0 > 0$ and $\mu_h > 0$. Hence we can define their ratio $G : I_x \to \mathbb{R}_+^*$ by

$$G(x) = \frac{A(x)}{P(x)} = \frac{\langle \lambda_y, f(x)\rangle}{\langle \lambda_z, f(x)\rangle}. \quad (41)$$

Clearly G is strictly positive. To interpret these terms, rewrite $A(x)$ as:

$$A(x) = \lambda_0 \frac{\sum_{i=1}^n p_i \phi_A^{'}(y_i(\bar{x}))f_i(x)}{\sum_{i=1}^n p_i f_i(x)} = \lambda_0 \sum_{i=1}^n p_i(x)\phi_A^{'}(y_i(\bar{x})), \quad (42)$$

where $p_i(x) = \frac{p_i f(x)}{\sum_{i=1}^n p_i f(x)}$ is the Bayesian posterior probability on the occurrence of the second-order state i given that the loss is x. This *inference* is a direct consequence of the uncertainty on the distribution of the loss and the fact that the second-order state is not a contractible parameter. If it were, then it would be possible to specify a contract for each of these states, which would essentially bring us back to the unambiguous setting. Here we have implicitly assumed (as in the rest of the literature) that it is not be feasible to verify the second-order state ex-post. We shall see later that it is essentially this problem that leads to the general sub-optimality of straight deductibles under one-sided risk and ambiguity aversion (i.e. in the case where only the policyholder is averse to risk and ambiguity but the insurer is neutral to both).

Hence $A(x)$ so defined can be interpreted as the expected marginal second-order utility, or expected marginal welfare (EMW) of the policyholder with respect to the posterior distribution (up to a positive constant). Analogously $P(x)$ can be interpreted as the EMW of the insurer with respect to the posterior distribution. Finally, $G(x)$, the expected marginal welfare ratio (EMWR), reflects the relative strength of the policyholder's ambiguity aversion to that of the insurer (see equation (47) below).

For each $\pi \in I_\pi$, let $\Sigma_x : [0, x] \to \mathbb{R}_+^*$ be the function defined by:

$$\Sigma_x(I) = \frac{1 + \psi'(I)}{u'(W_A - \pi - x + I)}. \tag{43}$$

It is easy to see that by the convexity of ψ and the concavity of u, this function strictly increasing in I for each $x \in I_x$. Hence for each fixed π, the function Σ_x is strictly increasing for all $I(x) \in [0, x]$, implying that Σ_x has a well-defined inverse $\Sigma_x^{-1} : \mathbb{R}_+^* \to [0, x]$, which is also strictly increasing. An optimal coverage rate function J is such that

$$xJ(x) \in \left\{0,\ 1\ , \Sigma_x^{-1}(G(x))\right\}, \quad x \in I_x. \tag{44}$$

where $\Sigma_x^{-1} : \mathbb{R}_+^* \to [0, x]$ is the inverse map of Σ_x defined in (43), and G is the ratio of expected marginal welfare given in (41). Equivalently, the corresponding indemnity function $I(x) = xJ(x)$ satisfies

$$I(x) \in \left\{0,\ x,\ \Sigma_x^{-1}(G(x))\right\}, \quad \forall x \in I_x. \tag{45}$$

Moreover, for $x \in (0, \bar{x}]$ such that $J(x)$ takes an interior value, the indemnity function takes values in $(0, x)$, is differentiable at x and satisfies differential equation:

$$I'(x) = \frac{r_u(W_A(x)) + \frac{G'(x)}{G(x)}}{r_u(W_A(x)) + \frac{\psi''(I(x))}{1+\psi'(I(x))}}, \tag{46}$$

where $r_u(\cdot) > 0$ is the policyholder's Arrow-Pratt degree of absolute risk aversion and $W_A(x) = W_A - \pi - x + I(x)$ denotes her final wealth.

Proof. See Subsection 6.6. ∎

Notice that equation (46) characterizes both risk and ambiguity sharing between the policyholder and the insurer. The risk-sharing part is explained by the degree of risk aversion: the higher it is, the higher is the coverage rate. Thus, the behavior of the coverage rate with respect to risk aversion is robust to the introduction of ambiguity. In the same way, the presence of ambiguity does not modify the relationship between the coverage rate and the cost of indemnity provision. The ambiguity sharing component is marked by the term G'/G in the numerator; the larger is this ratio, the more the burden of ambiguity bearing is shifted towards the insurer.

To understand the behavior of G'/G, recall that since $G(x) = \frac{A(x)}{P(x)}$. Differentiating with respect to x and re-arranging yield:

$$\frac{G'(x)}{G(x)} = \frac{A'(x)}{A(x)} - \frac{P'(x)}{P(x)}. \tag{47}$$

Observe that

$$G'(x) = \frac{\sum_{1\le i<j\le n} \left(\lambda_y^i \lambda_z^j - \lambda_y^j \lambda_z^i\right) f_j^2(x) \ell'_{ij}(x)}{\langle \lambda_z, f(x)\rangle^2}. \tag{48}$$

Hence the EMWR varies with respect to x in general. Nevertheless, we can show that in the case of two ambiguous states ($n = 2$), the monotonic behavior of this important term is independent of the value of the loss. We defer the treatment of this case to Subsection 4.2. Furthermore, the behavior of the EMWR is inconclusive with respect to the degree of ambiguity aversion.

4 Two important special cases

4.1 Ambiguity-neutral DMs

Let us study the case of ambiguity-neutral DMs. In the context of the smooth model, this translates to ϕ_A and ϕ_P being linear. Without loss of generality, we can let ϕ_A and ϕ_P be identity functions. The OCP of interest is thus:

$$\max_{(I(\cdot),\pi)} \quad \int_{I_x} u(W_A - \pi - x + I(x))\bar{f}(x)dx \tag{49a}$$
$$s.t. \quad I(x) \in [0, x], \quad \forall x \in I_x, \tag{49b}$$
$$\pi \in I_\pi \equiv [\underline{\pi}, \overline{\pi}], \tag{49c}$$
$$\pi \ge \int_{I_x} (I(x) + \psi(I(x))\bar{f}(x)dx, \tag{49d}$$

where $\bar{f} \equiv \sum_{i=1}^n p_i f_i$ is the ambiguity-neutral density defined in (9). Since $\lambda_0 > 0$ and $\mu_h > 0$ as previously remarked, the ratio of EMW simplifies to a positive constant:

$$G(x) = \frac{\lambda_0 \bar{f}(x)}{\mu_h \bar{f}(x)} = \frac{\lambda_0}{\mu_h} \equiv \tilde{\lambda}_0 > 0. \tag{50}$$

In this case the deductible x_1 is defined as the unique solution to $H_\omega(x)|_{\omega=0}$ while the upper limit x_2 is the unique solution to $H_\omega(x)|_{\omega=1}$. Moreover, the co-insurance equation (46) simplifies to:

$$I'(x) = \frac{r_u(W_A(x))}{r_u(W_A(x)) + \frac{\psi''(I(x))}{1+\psi'(I(x))}}, \tag{51}$$

which depends on x if and only if the cost of indemnity provision ψ is not linear. Observe that in this case $K(x)$ in (37) becomes:

$$K(x) = u'(W_A - \pi - x)\tilde{\lambda}_0 - (1 + \psi'(0)). \tag{52}$$

We obtain the following result. Consider the case where both decision makers are ambiguity neutral. The optimal indemnity schedule entails a unique deductible $x_1 \in (0, \bar{x})$, defined as the zero of $K(x)$ in (52). The optimal contract (I,π) satisfies:

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ \Sigma_x^{-1}(\tilde{\lambda}_0) \in (0, x) & x \in (x_1, \bar{x}], \end{cases} \tag{53}$$

and the optimal premium $\pi \in (\underline{\pi}, \overline{\pi})$ is determined by the saturated participation constraint of the insurer:

$$\pi = \int_{x_1}^{\bar{x}} (I(x) + \psi'(I(x)))\bar{f}(x)dx, \tag{54}$$

where $\bar{f}$ is the ambiguity-neutral density defined in (9). Moreover, indemnity beyond the deductible x_1 is characterized by:

$$I'(x) = \frac{r_u(W_A - \pi - x + I(x))}{r_u(W_A - \pi - x + I(x)) + \frac{\psi''(I(x))}{1+\psi'(I(x))}}, \tag{55}$$

where $r_u(\cdot) = \frac{-u''(\cdot)}{u'(\cdot)}$ is the Arrow-Pratt absolute risk aversion.

Proof. See Subsection 6.7. ■

Observe that if the cost ψ of indemnity provision is linear (constant loading), then the differential equation characterizing co-insurance implies that $I'(x) = 1$ for all losses beyond the deductible. In other words, the contract is a straight deductible, as is obtained in Proposition 1 of Gollier (2014). Moreoever, full insurance under fair pricing can also be proven using our approach. These results are summarized in the following corollary. Suppose that the cost of indemnity provision is linear:

$$\psi(I) = mI, \quad m > 0. \tag{56}$$

Then the optimal insurance contract is a straight deductible, namely that the pair (I,π) satisfies

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ x - x_1 & x \in (x_1, \bar{x}]. \end{cases} \tag{57}$$

and

$$\pi = (1+m)\int_{x_1}^{\bar{x}} (x - x_1)\bar{f}(x)dx. \tag{58}$$

Moreover, full insurance is optimal if and only if insurance is actuarially fair, namely if and only if $m = 0$.[3]

Proof. See Subsection 6.8 ■

Hence if contracting parties are ambiguity-neutral, the introduction to ambiguity does not alter the shape of the optimal contract. In particular, the function form of the indemnity schedule is robust to the introduction of ambiguity.

[3]Note that full insurance is just a special case of a straight deductible policy with a deductible equal to zero.

4.2 Two ambiguous states

Recall from our discussion in Section 3.2 that in general the sign of G' may vary with respect to x. Nevertheless, when there are two ambiguous states, the monotone likelihood ranking hypothesis (Assumption 5) allows us to say more.

Lemma 4. *In the case of two ambiguous states $n = 2$, the monotonic behavior of G, the ratio of EMWs, is independent of the size of the loss. Furthermore, the EMWs of both DMs are increasing ($A' \geq 0$ and $P' \geq 0$) on I_x.*

Proof. See Subsection 6.9.

■

Recall that $\pi \in \{\underline{\pi}, \overline{\pi}\}$ corresponds to the boundary cases discussed in Lemma 3. If $\pi \in (\underline{\pi}, \overline{\pi})$, then we can rewrite (34) as:

$$\int_{I_x} u'(W_A(x))\langle \lambda_y, f(x)\rangle dx = \langle \lambda_z, \mathbf{1}\rangle, \tag{59}$$

where $W_A(x) = W_A - \pi - x + I(x)$ and $\mathbf{1}$ denotes the n-dimensional vector with all elements being equal to one. Since the RHS is strictly positive, we can divide both sides of (59) by this term, obtaining:

$$\int_{I_x} u'(W_A(x))G(x)\tilde{f}(x)dx = 1, \tag{60}$$

where

$$\tilde{f}(x) = \frac{\langle \lambda_z, f(x)\rangle}{\langle \lambda_z, \mathbf{1}\rangle} = \sum_{i=1}^{n} \frac{\lambda_z^i f_i(x)}{\sum_{j=1}^{n} \lambda_z^j}. \tag{61}$$

Observe that $\tilde{f}$ in (61) is strictly positive on I_x and $\int \tilde{f}(x)dx = 1$. Hence it is a density function. In particular, it is a density function that deviates from the ambiguity-neutral density by the insurer's ambiguity aversion. Specifically, if the insurer is ambiguity-neutral then $\tilde{f} \equiv \bar{f}$. If $G' \geq 0$, then K defined in (37) is strictly increasing. An optimum is either one of the corner cases discussed in Lemma 3, or consists of the pair (I, π) such that the indemnity function is of the form:

$$\begin{cases} I(x) = 0 & x \in [0, x_1], \\ I(x) \in (0, x] & x \in (x_1, \bar{x}], \end{cases} \tag{62}$$

where $x_1 \in (0, \bar{x})$, the deductible, is the unique solution to $K(x) = 0$. The associated premium $\pi \in (\underline{\pi}, \overline{\pi})$ satisfies:

$$\sum_{i=1}^{n} p_i \phi_P \left(W_P + \pi - \int_{I_x} (I(x) + \psi(I(x)) f_i(x) dx \right) = \phi_P(W_P). \tag{63}$$

Moreover if ψ is linear, then in consideration of L defined in (38), one of the following cases could occur.

- If $L(\bar{x}) \geq 0$, then there exists a unique $x_2 \in (x_1, \bar{x}]$, the smallest solution to $L(x) = 0$, such that an optimal indemnity function satisfies:

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ \Sigma_x^{-1}(G(x)) & x \in (x_1, x_2), \\ x & x \in [x_2, \bar{x}]. \end{cases} \tag{64}$$

- If $L(\bar{x}) < 0$, then an optimal indemnity function has the form:

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ \Sigma_x^{-1}(G(x)) & x \in (x_1, \bar{x}]. \end{cases} \tag{65}$$

Proof. See Subsection 6.10 ■

An immediate consequence of Proposition 4.2 is that when the insurer is ambiguity-neutral and the policyholder ambiguity-averse, an optimal insurance policy necessarily involves a deductible since in this case the change in the expected marginal welfare ratio is entirely determined by the change in the expected marginal welfare of the policyholder, i.e., $G'/G = A'/A$, which must be positive in view of Lemma 4. In the case of two ambiguous states ($n = 2$) with ambiguity-averse policyholder and ambiguity-neutral insurer, there exists a unique $x_1 \in (0, \bar{x})$, called the deductible, such that $K(x_1) = 0$. An optimum is either one of the corner cases discussed in Lemma 3, or is such that the indemnity function has a deductible x_1, namely that:

$$\begin{cases} I(x) = 0 & x \in [0, x_1], \\ I(x) \in (0, x] & x \in (x_1, \bar{x}]. \end{cases} \tag{66}$$

The associated premium $\pi \in (\underline{\pi}, \overline{\pi})$ satisfies:

$$\pi = \int_{x_1}^{\bar{x}} (I(x) + \psi(I(x)))\bar{f}(x)dx. \tag{67}$$

Moreover if ψ is linear, then one of the following forms of co-insurance beyond a deductible can occur.

- If $L(\bar{x}) \geq 0$, then there exists a unique $x_2 \in (x_1, \bar{x}]$, the smallest solution to $L(x) = 0$, such that an optimal indemnity function is of the form:

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ \Sigma_x^{-1}(A(x)/\mu_h) & x \in (x_1, x_2), \\ x & x \in [x_2, \bar{x}]. \end{cases} \tag{68}$$

- If $L(\bar{x}) < 0$, then an optimal indemnity function satisfies:

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ \Sigma_x^{-1}(A(x)/\mu_h) & x \in (x_1, \bar{x}]. \end{cases} \tag{69}$$

Moreover, the co-insurance rate is given by

$$I'(x) = \frac{r_u(W_A(x)) + \frac{A'(x)}{A(x)}}{r_u(W_A(x))}. \tag{70}$$

Proof. See Subsection 6.11. ■

Note that the existence of a threshold x_2 beyond which the deductible disappears completely (first case in Proposition 6.11) depends on the sign of $L(\bar{x})$, which in turn depends on the optimal premium and the values of the co-states λ_y and λ_z. This poses challenges to ex-ante checking whether $L(\bar{x})$ is negative. A potential solution is to perform numerical simulation, the idea of which is discussed in the conclusion.

As is well-known in the literature, a disappearing deductible kind of contract gives rise to ex-post moral hazard. In other words, policyholders may intentionally increase the loss to raise indemnity. The only possibility to rule out a disappearing deductible in this particular case is to set $A' = 0$ so that the contract becomes a straight deductible. Nevertheless this is not true in general.

To illustrate this point, consider the contract of the form (69) where insurance beyond the deductible is captured by the co-insurance rule (70). A straight deductible contract in this case would require $I'(x) = 0$ on $[0, x_1]$ and $I'(x) = 1$ on $(x_1, \bar{x}]$. Recall that

$$A'(x) = \lambda_0 \frac{p_1 p_2 \left(\phi_A'(y_1(\bar{x})) - \phi_A'(y_2(\bar{x}))\right) f_2^2(x) \ell_{12}'(x)}{\bar{f}^2(x)}, \tag{71}$$

where $\bar{f}$ again denotes the ambiguity-neutral density. Clearly $A'(x) = 0$ if and only if $y_1(\bar{x}) = y_2(\bar{x})$ or $\ell_{12}'(x) = 0$. Using integration by parts and the hypothesis $I'(x) = 1$ on $(x_1, \bar{x}]$, we have:

$$y_1(\bar{x}) - y_2(\bar{x}) = \int_0^{x_1} u'(W_A(x))(F_1(x) - F_2(x))dx. \tag{72}$$

Suppose there exists a sub-interval (i.e. a subset of positive measure) of $[0, x_1]$ such that $F_1(x) > F_2(x)$, then (72) implies $y_1(\bar{x}) > y_2(\bar{x})$ by the strict monotonicity of the utility function. In this case $A'(x) = 0$ on $(x_1, \bar{x}]$ if and only if $\ell_{12}'(x) = 0$ on this interval.[4] Intuitively, this implies that the size of the loss is not informative about the second-order state in which the loss is realized. In other words, so long as the second-order states are indistinguishable beyond the deductible (i.e., when it matters), straight deductibles would remain optimal.

At fist glance, the fact that straight deductibles are not optimal under unilateral risk and ambiguity aversion seem puzzling. Shouldn't the insurer, the party neutral to both types of uncertainty, bear all the uncertainty beyond a certain threshold, as in the case of pure risk? Some reflection suggests that the inefficiency of straight deductibles is essentially a consequence of ambiguity and the fact that the ambiguous state itself cannot be included in the contract. In other words, all that matters for the indemnity is the size of the loss realized, not the ambiguous state in which it is realized. To illustrate, take the case of health insurance in the case of contracting a rare disease, which typically involves some degree of ambiguity. The hypothesis of non-inclusion of the ambiguous state implies that it is impossible (or perhaps infeasible) to specify in the contract whether a loss x occurs to a policy holder under a rare or normal condition (e.g., severe cold versus covid). Once we allow for the contract to specify the ambiguous states, then we can write contracts in the form $(I_i(\cdot),\ \pi_i)$ for each second-order state i, then straight deductibles remain optimal under one-sided risk and ambiguity aversion. To put it differently, the incompleteness of the contract renders it impossible to efficiently allocate risk and ambiguity between the contracting parties according to their preferences.

[4] If there exists a sub-interval of $(x_1, \bar{x}]$ in which $\ell_{12}'(x) < 0$, then $A'(x) > 0$ in this interval in view of (71), implying that $I'(x) > 1$, contradicting the straight deductible hypothesis.

5 Conclusion

We set out to determine the optimal insurance policy under ambiguity and ambiguity aversion in the sense of Klibanoff et al. (2005). First, we provide a rigorous proof of the existence of an optimal insurance policy in presence of ambiguity. Next, we derive a risk-and-ambiguity sharing rule, which implies that in the case of two ambiguous states and one-sided ambiguity aversion, the optimal contract necessarily entails disappearing deductibles when the states can be ranked according to the monotone likelihood ratio property. Moreover, it is possible that the deductible disappears completely, suggesting full insurance beyond a sufficiently large loss.

There are two main limitations to our analysis. First, most concrete analytical results are restricted to the case of two ambiguous states. Second, we have proved existence but have not addressed uniqueness. Both of these issues, while challenging analytically, can be resolved numerically via the shooting method. The idea behind this method is homotopy, or continuous deformation of a well-known solution. In particular, let us first discuss the issue of two ambiguous states. Observe that the unambiguous (pure-risk) setting is simply one where the probability of one of the ambiguous states is null. Numerically solving for the contract in the two-state case involves gradually deforming the pure-risk contract by gradually increasing the prior of the additional state. A bifurcation being detected in the process might suggest a change of behavior (i.e., functional form or qualitative features) of the contract moving from the pure-risk to the ambiguous setting. Once the two-ambiguous-state contract has been solved for numerically, it can be used as the starting point to solve for three-ambiguous-state contract, and so on. In principle, this method could be used to successively determine the optimal contract for any setting beyond the case of two ambiguous states. The uniqueness issue can be addressed in a similar manner. We know from the existing literature that optimal insurance contracts are unique in unambiguous settings. If a bifurcation is detected while performing continuation on the prior probability of the additional second-order state (for any set of chosen parameters), then uniqueness is disproved. Otherwise, uniqueness might still hold. The execution of this method, however, is beyond the scope of this paper and is reserved for future research.

6 Proofs

6.1 Proof of Proposition 3.1

Let $\delta = \inf_{(J,\pi)\in\mathcal{U}\times I_\pi} g(J,\pi)$. Consider a sequence of trajectories $\{X^k(\cdot)\}_{k\in\mathbb{N}}$ associated with the sequence of admissible controls $\{J^k(\cdot)\}_{k\in\mathbb{N}}$ defined by

$$X^k(x) = \begin{pmatrix} y^k(x) \\ z^k(x) \\ \pi^k(x) \end{pmatrix} = \begin{pmatrix} \left(y_i^k(x)\right)_{i\in\mathcal{I}} \\ \left(z_i^k(x)\right)_{i\in\mathcal{I}} \\ \pi^k \end{pmatrix}, \quad x \in I_x \setminus \{0\},$$

such that $g(J^k,\pi^k) \longrightarrow \delta$ as $k \longrightarrow \infty$, where

$$\begin{aligned} y_i^k(x) &= \int_0^x u(W_A - \pi^k - t + tJ^k(t)) f_i(t)dt, \quad i \in \mathcal{I}, \\ z_i^k(x) &= \int_0^x (W_P + \pi^k - tJ^k(t) - \psi(tJ^k(t))) f_i(t)dt, \quad i \in \mathcal{I}. \end{aligned}$$

By the weak-$\star$ compactness of $\mathcal{U}$, the sequence $\{J^k(\cdot)\}_{k\in\mathbb{N}}$ converges to $J^*(\cdot) \in \mathcal{U}$ up to some subsequence, i.e., $J^k \longrightarrow J^*$. Likewise the compactness of I_π implies $\pi^k \longrightarrow \pi^* \in I_\pi$ up to some subsequence. Let $\bar{X}^*$ stand for the limiting trajectory defined by

$$\bar{X}^*(x) = \begin{pmatrix} \bar{y}^*(x) \\ \bar{z}^*(x) \\ \pi^* \end{pmatrix} = \begin{pmatrix} \left(\bar{z}_i^*(x)\right)_{i\in\mathcal{I}} \\ \left(\bar{y}_i^*(x)\right)_{i\in\mathcal{I}} \\ \pi^* \end{pmatrix}, \quad x \in I_x \setminus \{0\},$$

where

$$\begin{aligned} \bar{y}_i^*(x) &= \int_0^x u(W_A - \pi^* - t + tJ^*(t)) f_i(t)dt, \quad i \in \mathcal{I}, \\ \bar{z}_i^*(x) &= \int_0^x (W_P + \pi^* - tJ^*(t) - \psi(tJ^*(t)))\, f_i(t)dt, \quad i \in \mathcal{I}. \end{aligned}$$

The remain of the proof is completed in two steps. First, we show that the limiting trajectory is shown to satisfy the constraint. Second, we prove that this trajectory is an optimal one.

6.1.1 The limiting trajectory verifies the constraint

Let us now show that $h(J^*,\pi^*) \geq 0$. By construction $h(J^k,\pi^k) \equiv \sum_{i=1}^n p_i\phi_P(z_i^k(\bar{x})) - \bar{V} \geq 0$ for all $k \in \mathbb{N}$. For $i \in \mathcal{I}$ and $k \geq 0$, let us write

$$z_i^k(\bar{x}) = \int_{I_x} (W_P + \pi^* - xJ^k(x) - \psi(xJ^k(x))) f_i(x)dx + \Delta^k,$$

where $\Delta^k = \int_{I_x} \left(\pi^k - \pi^*\right) f_i(x)dx = \pi^k - \pi^*$. Clearly Δ^k tends to zero as k tends to infinity. Let $\Gamma_i(J^k(x)) \equiv -(W + \pi^* - xJ^k(x) - \psi(xJ^k(x))) f_i(x)$, then $\Gamma_i(J^k(x))$ is convex in $I^k(x)$ since $\psi(\cdot)$ is convex in $J^k(x)$ and $f_i(x) > 0$ by Assumption 4. Hence from Lee and Markus (1967), we have

$$\int_{I_x} \Gamma_i(J^*(x)) \geq \liminf \int_{I_x} \Gamma_i(J^k(x))dx,$$

or

$$\begin{aligned} -\bar{z}_i^*(\bar{x}) &\geq \liminf(-z_i^k(\bar{x}) - \delta^k) \\ \iff \bar{z}_i^*(\bar{x}) &\geq \limsup z_i^k(\bar{x}). \end{aligned}$$

By the continuity of $\bar{z}_i^*$, for all $\epsilon > 0$, there exists a sufficiently large positive integer K such that

$$\bar{z}_i^*(\bar{x}) \geq z_i^K(\bar{x}) - \epsilon.$$

Since ϕ_P is increasing,

$$\phi_P(\bar{z}_i^*(\bar{x})) \geq \phi_P(z_i^K(\bar{x}) - \epsilon). \tag{73}$$

By the first fundamental theorem of calculus,

$$\phi_P(z_i^K(\bar{x})) - \phi_P(z_i^K(\bar{x}) - \epsilon) = \int_a^b \phi_P'(\zeta)d\zeta,$$

where $b \equiv z_i^K(\bar{x})$ and $a = b - \epsilon$. Since ϕ_P' is bounded by Assumption 10, let $M \in \mathbb{R}_+$ be an upperbound of ϕ_P' over $[a, b]$. Then

$$\phi_P(z_i^K(\bar{x})) - \phi_P(z_i^K(\bar{x}) - \epsilon) \leq M(b - a) = M\epsilon,$$

implying

$$\phi_P(z_i^K(\bar{x}) - \epsilon) \geq \phi_P(z_i^K(\bar{x})) - M\epsilon,$$

which, together with (73) imply

$$\begin{aligned} \phi_P(\bar{z}_i^*(\bar{x})) &\geq \phi_P(z_i^K(\bar{x})) - M\epsilon \qquad (74) \\ \implies \sum_{i=1}^n p_i\phi_P(\bar{z}_i^*(\bar{x})) &\geq \sum_{i=1}^n p_i\phi_P(z_i^K(\bar{x})) - M\epsilon. \qquad (75) \end{aligned}$$

Observe that $\sum_{i=1}^n p_i\phi_P(z_i^K(\bar{x})) \geq \bar{V}$ since $\sum_{i=1}^n p_i\phi_P(z_i^k(\bar{x})) \geq \bar{V}$ for all $k \geq 0$. Thus from (75), we have

$$\sum_{i=1}^n p_i\phi_P(\bar{z}_i^*(\bar{x})) \geq \bar{V} - M\epsilon.$$

Since ϵ was arbitrary, letting $\epsilon \to 0$ completes the proof. We next show that the cost functional achieved by the limiting trajectory is optimal.

6.1.2 The optimality of the limiting trajectory

We now prove that the cost achieved by the limiting trajectory is optimal, i.e., $g(J^*, \pi^*) = \delta$, where $\delta = \inf_{(J,\pi)\in\mathcal{U}\times I_\pi} g(J, \pi)$. Since (J^*, π^*) is admissible, $g(J^*, \pi^*) \geq \delta$. It remains to show that $g(J^*, \pi^*) \leq \delta$. Let us write:

$$y_i^k(\bar{x}) = \int_{I_x} u(W_A - \pi^* - x + xJ^k(x))f_i(x)dx + \Delta_i^k,$$

where

$$\Delta_i^k \equiv \int_{I_x} \left[u(W_A + \pi^k - x + xJ^k(x)) - u(W_A + \pi^* - x + xJ^k(x))\right] f_i(x)dx. \tag{76}$$

Observe that Δ_i^k tends to zero as k tends to infinity since u is bounded and continuous, and f_i is continuous. Let $\Gamma_i(J^k(x)) \equiv -u(w - \pi^* - x + xJ^k(x))f_i(x)$. Then Γ_i is convex in J^k since $-u$ is convex and f_i is strictly positive. Again from Lee and Markus (1967),

$$\begin{aligned} \int_{I_x} \Gamma_i(J^*(x))dx &\leq \liminf \int_{I_x} \Gamma(J^k(x))dx \\ -\bar{y}_i^*(\bar{x}) &\leq \liminf(-y_i^k(\bar{x}) - \Delta_i^k) \\ \bar{y}_i^*(\bar{x}) &\geq \limsup y_i^k(\bar{x}). \end{aligned}$$

Proceed similarly to the proof of the previous lemma, we have that for all $\epsilon > 0$, there exists a sufficiently large integer K such that

$$-\sum_{i=1}^n p_i\phi_A\left(\bar{y}_i^*(\bar{x})\right) \leq -\sum_{i=1}^n p_i\phi_A\left(y_i^K(\bar{x})\right) + M\epsilon.$$

Letting ϵ tend to zero yields:

$$g(J^*, \pi^*) \leq g(J^K, \pi^K),$$

which implies that $g(J^*, \pi^*)$ is a lower bound for $g(J^K, \pi^K)$. Hence $g(J^*, \pi^*) \leq \delta$ by definition of the infimum, as desired.

To sum up, we have proved that the limiting trajectory satisfies the constraint and the cost achieved by this trajectory is the minimum cost. Thus the pair (J^*, π^*) is an optimal pair, and the associated insurance contract (I^*, π^*), where

$$I^*(x) = \begin{cases} 0 & x = 0 \\ xJ^*(x) & x \in (0, \bar{x}] \end{cases} \tag{77}$$

is an optimal one.

6.2 Proof of Lemma 1

Suppose by contradiction that $h(J, \pi) > 0$ for an optimal pair (J, π). We have

$$\begin{aligned} &\sum_{i=1}^n p_i\phi_P\left(\int_{I_x}(W_P + \pi - xJ(x) - \psi(xJ(x)))f_i(x)dx\right) - \bar{V} &> 0 \\ \iff &\sum_{i=1}^n p_i\phi_P\left(W_P + \pi - \int_{I_x}\left(xJ(x) + \psi(xJ(x))\right)f_i(x)dx\right) - \bar{V} &> 0. \end{aligned}$$

If $J(x) = 1$ for a.e. $x \in I_x$, then by the continuity of ϕ_P with respect to π there exists some positive real number $\eta > 0$ such that $h(J, \pi - \eta) > 0$. Since the cost is strictly increasing in π, lowering π reduces the cost. In this case we have:

$$h(J, \pi - \eta) > 0, \tag{78}$$
$$g(J, \pi - \eta) < g(J, \pi), \tag{79}$$

implying that the contract $(J, \pi - \eta)$ is feasible and yields a lower cost. Hence (J, π) is not optimal, a contradiction.

If J is not equal to 1 almost everywhere on I_x, then by the continuity of the mapping $J \mapsto xJ + \psi(xJ)$ and strict positivity of the conditional densities, there exists $K_x \subset I_x$ of positive measure and a sufficiently small $\epsilon_K > 0$ satisfying $0 \leq J(x) + \epsilon_K \leq 1$ for all $x \in K_x$ such that $g(\tilde{J}, \pi) \geq 0$, where $\tilde{J} : (0, \bar{x}] \to [0, 1]$ is defined by

$$\tilde{J}(x) = \begin{cases} J(x) & x \in (0, \bar{x}] \setminus K_x, \\ J(x) + \epsilon_K & x \in K_x. \end{cases}$$

Since the cost is strictly decreasing in J, the modified control $\tilde{J}$ yields a lower cost, i.e., $g(\tilde{J}, \pi) < g(J, \pi)$, contradicting the hypothesis that (J, π) is optimal.

We conclude that if (J, π) constitutes an optimal pair, then $h(J, \pi) = 0$.

6.3 Proof of Proposition 3.2

Let us first prove two lemmas.

Lemma 5. *The normal cone at $X(0)$ depends on the value of π. In particular:*

- *If $\pi \in (\underline{\pi}, \overline{\pi})$, then $N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \{0\}$;*
- *If $\pi = \underline{\pi}$, then $N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_-$;*
- *If $\pi = \overline{\pi}$, then $N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$.*

The normal cone at $X(\bar{x})$ is:

$$N_{\mathcal{M}_1}(X(\bar{x})) = \begin{pmatrix} 0 \\ -\mu_h \nabla_z h(J, \pi) \\ -\mu_h \nabla_\pi h(J, \pi) - \mu_{\underline{\pi}} + \mu_{\overline{\pi}} \end{pmatrix}. \tag{80}$$

Proof. First, consider $N_{\mathcal{M}_0}(X(0))$, where $X(0) = (0, 0, \pi)$. Let $\xi = (\xi_y, \xi_z, \xi_\pi)$ be an element in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Let M_0 be an element in $\mathcal{M}_0$, hence $M_0 = (0, 0, a)$ for some $a \in I_\pi$. The normal cone to $\mathcal{M}_0$ at $X(0)$ can be written as:

$$\begin{aligned} N_{\mathcal{M}_0}(X(0)) &= \left\{\xi \in \mathbb{R}^{2n+1} \mid \langle \xi, M_0 - X(0) \rangle \leq 0,\ \forall M_0 \in \mathcal{M}_0 \right\} \\ \implies N_{\mathcal{M}_0}(X(0)) &= \left\{\xi \in \mathbb{R}^{2n+1} \mid \xi_\pi (a - \pi) \leq 0,\ \forall a \in I_\pi \right\}. \end{aligned}$$

One of the following cases can occur.

- If $\pi \in (\underline{\pi}, \overline{\pi})$, then $\xi_\pi = 0$ since $\xi_\pi(a - \pi)$ must be negative for any a in I_π. Hence in this case,
$$N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \{0\}. \tag{81}$$
- If $\pi = \underline{\pi}$, then $a - \pi \geq 0$ for all $a \in I_\pi$. Thus $\xi_\pi(a - \pi)$ is negative for any $a \in I_\pi$ requires $\xi_\pi \leq 0$, implying that:
$$N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_-. \tag{82}$$
- If $\pi = \underline{\pi}$, then $a - \pi \leq 0$ for all $a \in I_\pi$, implying that $\xi_\pi \geq 0$ and the normal cone in this case is:
$$N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+. \tag{83}$$

Let us now compute the normal cone at the target compute $N_{\mathcal{M}_1}(X(\bar{x}))$, where $X(\bar{x}) = (y(\bar{x}), z(\bar{x}), \pi)$. Following Clarke (1990), we can write

$$N_{\mathcal{M}_1}(X(\bar{x})) = -\mu_h \nabla_X h(J,\pi) + \mu_{\underline{\pi}} \nabla_X (\underline{\pi} - \pi) + \mu_{\overline{\pi}} \nabla_X (\pi - \overline{\pi}), \tag{84}$$

where $\mu = (\mu_h, \mu_{\underline{\pi}}, \mu_{\overline{\pi}})$ satisfies the complementary slackness conditions

$$\mu_{\underline{\pi}} \geq 0, \qquad \mu_{\underline{\pi}}(\underline{\pi} - \pi) = 0, \tag{85}$$
$$\mu_{\overline{\pi}} \geq 0, \qquad \mu_{\overline{\pi}}(\pi - \overline{\pi}) = 0. \tag{86}$$

Since

$$\nabla_X h(J,\pi) = (0, \nabla_z h(J,\pi), \nabla_\pi h(J,\pi)), \tag{87}$$
$$\nabla_X (\underline{\pi} - \pi) = (0,0,-1), \tag{88}$$
$$\nabla_X (\pi - \overline{\pi}) = (0,0,1), \tag{89}$$

we can rewrite (84) as:

$$N_{\mathcal{M}_1}(X(\bar{x})) = \begin{pmatrix} 0 \\ -\mu_h \nabla_z h(J,\pi) \\ -\mu_h \nabla_\pi h(J,\pi) - \mu_{\underline{\pi}} + \mu_{\overline{\pi}} \end{pmatrix}. \tag{90}$$

■

Lemma 6. *If $\pi \in (\underline{\pi}, \overline{\pi})$, then $\lambda_\pi(0) = 0$. If $\pi = \underline{\pi}$, then $\lambda_\pi(0) \leq 0$. If $\pi = \overline{\pi}$, then $\lambda_\pi(0) \geq 0$.*

Proof. The proof follows directly from condition (28) applied to different forms of $N_{\mathcal{M}_0}(X(0))$ depending on where π takes value (at the optimum). In particular, if $\pi \in (\underline{\pi}, \overline{\pi})$, then the normal cone takes the form (81), implying that $\lambda_\pi(0) = 0$. If $\pi = \underline{\pi}$, then the normal cone takes the form (82), implying that $\lambda_\pi(0) \leq 0$. Finally if $\pi = \overline{\pi}$, then the normal cone in (83) implies that $\lambda_\pi(0) \geq 0$. ■

From the adjoint equation (25), we have that for almost every x in I_x,

$$\begin{pmatrix} \dot{\lambda}_y(x) \\ \dot{\lambda}_z(x) \\ \dot{\lambda}_\pi(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u'(W_A - \pi - x + xJ(x)) \langle \lambda_y, f(x) \rangle - \langle \lambda_z, f(x) \rangle \end{pmatrix}. \tag{91}$$

Hence $\lambda_y(x) = cons \equiv \lambda_y$ and $\lambda_z(x) = cons \equiv \lambda_z$ for all $x \in I_x$. In view of (90) we can rewrite the tranversality condition (29) as:

$$\begin{pmatrix} \lambda_y \\ \lambda_z \\ \lambda_\pi(\bar{x}) \end{pmatrix} = \begin{pmatrix} -\lambda_0 \nabla_y g(J,\pi) \\ \mu_h \nabla_z h(J,\pi) \\ \nabla_\pi \left(-\lambda_0 g(J,\pi) + \mu_h h(J,\pi)\right) + \mu_{\underline{\pi}} - \mu_{\overline{\pi}} \end{pmatrix}.$$

This yields

$$\lambda_y = \lambda_0 \left(p_i \phi_A'(y_i(\bar{x}))\right)_{i\in\mathcal{I}}, \tag{92}$$
$$\lambda_z = \mu_h \left(p_i \phi_P'(z_i(\bar{x}))\right)_{i\in\mathcal{I}}, \tag{93}$$
$$\lambda_\pi(\bar{x}) = \lambda_0 \sum_{i=1}^n p_i \phi_A'(y_i(\bar{x})) \frac{\partial y_i(\bar{x})}{\partial \pi} + \mu_h \sum_{i=1}^n p_i \phi_P'(z_i(\bar{x})) \frac{\partial z_i(\bar{x})}{\partial \pi} + \mu_{\underline{\pi}} - \mu_{\overline{\pi}}. \tag{94}$$

Let us substitute (92) and (93) into (94) to express $\lambda_\pi(\bar{x})$ more compactly as

$$\lambda_\pi(\bar{x}) = \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle + \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle + \mu_{\underline{\pi}} - \mu_{\overline{\pi}}. \tag{95}$$

Observe that

$$\begin{aligned} \int_{I_x} \dot{\lambda}_\pi(x)dx &= \int_{I_x} u'(W_A - \pi - x + xJ(x)) \langle \lambda_y, f(x) \rangle \, dx - \int_{I_x} \langle \lambda_z, f(x) \rangle \, dx \\ &= -\left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle - \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle, \end{aligned} \tag{96}$$

where $\frac{\partial z_i(\bar{x})}{\partial \pi} = 1$ for all $i \in \mathcal{I}$. Hence in view of (95) and (96)

$$\begin{aligned} \lambda_\pi(0) &= \lambda_\pi(\bar{x}) - \int_{I_x} \dot{\lambda}_\pi(x)dx, \\ &= 2\left(\left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle + \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle \right) + \mu_{\underline{\pi}} - \mu_{\overline{\pi}}. \end{aligned} \tag{97}$$

Observe that when $\pi \in (\underline{\pi}, \overline{\pi})$, we have $(\mu_{\underline{\pi}}, \mu_{\overline{\pi}}) = 0 \in \mathbb{R}^2$ thanks to the complementary slackness conditions (85) and (86). Hence for $\pi \in (\underline{\pi}, \overline{\pi})$, we have:

$$\lambda_\pi(0) = 2\left(\left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle + \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle \right) = 2\lambda_\pi(\bar{x}). \tag{98}$$

To arrive at (34), recall that $\lambda_\pi(0) = 0$ for interior values of π by Lemma 6.

6.4 Proof of Lemma 2

Suppose by contradiction that $\lambda_0 = \mu_h = 0$. Then equations (30) and (31) imply:

$$\lambda_z = \lambda_y = 0 \in \mathbb{R}^n. \tag{99}$$

Hence from (91)

$$\dot{\lambda}_\pi(x) = 0, \quad a.e. \ x \in I_x, \tag{100}$$

implying that $\lambda_\pi(x)$ is constant with respect to x. Denote

$$\lambda_\pi(x) = \lambda_\pi, \quad \forall x \in I_x. \tag{101}$$

Then from (95) we have:

$$\lambda_\pi = \mu_{\underline{\pi}} - \mu_{\overline{\pi}}. \tag{102}$$

Consider the following cases.

- If $\pi \in (\underline{\pi}, \overline{\pi})$, then $\mu_{\underline{\pi}} = \mu_{\overline{\pi}} = 0$, implying that $\lambda_\pi = 0 \in \mathbb{R}^{2n+1}$, violating the condition $(\lambda, \lambda_0) \neq 0$.
- If $\pi = \overline{\pi}$, then $\lambda_\pi = -\mu_{\overline{\pi}} \leq 0$. Non-negativity implies that $\mu_{\overline{\pi}} = 0$, which in turn implies $\lambda_\pi = 0$, again violating the condition $(\lambda, \lambda_0) \neq 0$.
- If $\pi = \underline{\pi}$, then $\lambda_\pi = \mu_{\underline{\pi}} \geq 0$. If $\mu_{\underline{\pi}} = 0$, then $\lambda_\pi = 0$ and a similar contradiction ensues. If $\mu_{\underline{\pi}} > 0$, then $\lambda_\pi(0) = \lambda_\pi > 0$, inconsistent with Lemma 6.

Thus in any case, a contradiction follows if $\mu_h = \lambda_0 = 0$, establishing the lemma.

6.5 Proof of Lemma 3

Suppose that $\lambda_0 = 0$. Then $\mu_h > 0$ by Lemma 2. By equations (30) and (31), the costate λ_y is a zero vector and the costate λ_z has strictly positive components, implying that $\langle \lambda_y, f(x) \rangle = 0$ and $\langle \lambda_z, f(x) \rangle > 0$ since the densities are strictly positive. Hence by Assumption 7,

$$H_\omega = -x(1+\psi'(x\omega)) \langle \lambda_z, f(x) \rangle < 0, \quad a.e.\ x \in I_x, \tag{103}$$
$$H_{\omega\omega} = -x^2 \psi''(x\omega) \langle \lambda_z, f(x) \rangle \leq 0, \quad a.e.\ x \in I_x, \tag{104}$$

implying that $J(x) = 0$ for a.e. $x \in I_x$. By Remark 3.1 uniformly zero insurance constitutes an optimum if and only if $\pi = \underline{\pi} = 0$.

Next, consider the case $\mu_h = 0$, which by Lemma 2 implies $\lambda_0 > 0$. Thus $\langle \lambda_y, f(x) \rangle > 0$ and $\langle \lambda_z, f(x) \rangle = 0$. Hence by Assumption 8,

$$H_\omega = xu'(W_A - \pi - x + x\omega) \langle \lambda_y, f(x) \rangle > 0, \quad a.e.\ x \in I_x, \tag{105}$$
$$H_{\omega\omega} = x^2 u''(W_A - \pi - x + x\omega) \langle \lambda_y, f(x) \rangle < 0, \quad a.e.\ x \in I_x, \tag{106}$$

implying that $J(x) = 1$ for a.e. $x \in I_x$. Again by Remark 3.1, if ϕ_P is strictly concave then the admissible set is empty, violating Proposition 3.1. This implies that if $\mu_h = 0$ at an optimum if and only if ϕ_P is identity (i.e., if the insurer is ambiguity-neutral). In this case, the admissible set is a singleton containing only the pair $(J = 1, \pi = \overline{\pi})$, which is then trivially optimal.

6.6 Proof of Proposition 3.2

Since the Hamiltonian is strictly concave in ω for a.e. $x \in I_x$, one of the following must occur:

- either $H_\omega(x)|_{\omega=0} \leq 0 \iff K(x) \leq 0$, then $J(x) = 0$;
- or $H_\omega(x)|_{\omega=1} \geq 0 \iff L(x) \geq 0$, then $J(x) = 1$;
- or $K(x) > 0$ and $L(x) < 0$, then $J(x) \in (0, 1)$ is characterized by:

$$u'(W_A(x))G(x) = 1 + \psi'(xJ(x))$$
$$u'(W_A - \pi - x + xJ(x)) = 1 + \psi'(xJ(x)), \tag{107}$$

  which can be inverted to yield:

$$xJ(x) = \Sigma_x^{-1}(G(x)). \tag{108}$$

We can then recover the optimal indemnity function I in (45) associated with the optimal coverage function J in (44) by recalling that $I(x) = 0$ for $x = 0$ and $I(x) = xJ(x)$ for $x > 0$. The coinsurance rate (46) is obtained by differentiating (107) with respect to x upon substituting $I(x) = xJ(x)$ into the expression. In particular,

$$u'(W_A - \pi - x + I(x))G(x) = 1 + \psi'(I(x)), \tag{109}$$
$$u''(W_A(x))(I'(x) - 1)G(x) + u'(W_A(x))G'(x) = -\psi''(I(x))I'(x), \tag{110}$$

implying that:

$$\underbrace{\frac{-u''(W_A(x))}{u'(W_A(x)}}_{r_u(W_A(x))}(1 - I'(x)) + \frac{G'(x)}{G(x)} = \frac{\psi''(I(x))I'(x)}{1 + \psi'(I(x))}, \tag{111}$$

which yields (46) upon gathering $I'(x)$ and simplifying.

6.7 Proof of Proposition 4.1

The special cases associated to either $\lambda_0 = 0$ or $\mu_h = 0$ are discussed in Lemma 3. In particular, the optimal contract is the trivial one ($I = 0, \pi = \underline{\pi}$) if $\lambda_0 = 0$, or the uniformly full insurance one ($I = x, \pi = \overline{\pi}$) if $\mu_h = 0$. For $\mu_h > 0$ and $\lambda_0 > 0$, the optimal premium takes an interior value $\pi \in (\underline{\pi}, \overline{\pi})$. Invoking Proposition 3.2 for $\phi'_P = \phi'_A = 1$ (the DMs are ambiguity-neutral), we can write:

$$\int_{I_x} u'(W_A - \pi - x + I(x))\tilde{\lambda}_0 \bar{f}(x)dx = 1 \tag{112}$$

Since $x \geq I(x)$ for all $x \in I_x$ and u is concave:

$$u'(W_A - \pi) \leq u'(W_A - \pi - x + I(x)), \quad \forall x \in I_x, \tag{113}$$

with strict inequality whenever $x > I(x)$. Hence

$$\begin{aligned} \tilde{\lambda}_0 \int_{I_x} u'(W_A - \pi - x + I(x))\bar{f}(x) &\geq \tilde{\lambda}_0 u'(W_A - \pi) \\ 1 &\geq u'(W_A - \pi)\tilde{\lambda}_0. \end{aligned} \tag{114}$$

Since $\psi' > 0$, in view of (52) we have:

$$K(0) = u'(W_A - \pi)\tilde{\lambda}_0 - (1 + \psi'(0)) < 0. \tag{115}$$

Note that since u and ψ are continuously differentiable, the function K in (52) is also continuously differentiable. Furthermore by the strict concavity of u,

$$K'(x) = -u''(W_A - \pi - x) > 0, \tag{116}$$

implying that K is continuous and strictly increasing on I_x. Hence K is strictly increasing and satisfies $K(0) < 0$. If $K(\bar{x}) \leq 0$, then $K(x) \leq 0$ on I_x, implying that $I(x) = 0$ for all $x \in I_x$, which is not optimal for $\pi > 0$ by Lemma 3. Hence $K(\bar{x}) > 0$. By continuity there exists $x_1 \in (0, \bar{x})$, called the deductible, such that $K(x_1) = 0$. By the strict monotonicity of K the deductible x_1 is unique. Now since K is strictly increasing, for all losses below x_1 we have $K(x) \leq 0$, or $H_\omega|_{\omega=0}(x) \leq 0$, implying that $I(x) = 0$ for all $x \leq x_1$. Likewise for all $x > x_1$, we have $H_\omega|_{\omega=0}(x) > 0$, implying that $I(x) > 0$ for for such losses. Observe that under ambiguity neutrality,

$$H_\omega|_{\omega=1}(x) \equiv L(x) = u'(W_A - \pi)\tilde{\lambda}_0 - (1 + \psi'(x)), \tag{117}$$

which is continuous and differentiable with respect to x. Since ψ' is convex,

$$L'(x) = -\psi''(x) \leq 0, \quad \forall x \in (0, \bar{x}), \tag{118}$$

with strict inequality if ψ is strictly convex. This implies $L(\bar{x}) \leq L(0)$. But $L(0) = K(0)$, which is strictly negative as previously shown, implying that $L(x) < 0$ for all $x \in (0, \bar{x})$. Hence full insurance $I(x) = x$ is never reached for losses beyond x_1. Therefore for $x \in (x_1, \bar{x}]$, we have $I(x) \in (0, x)$ satisfying

$$\begin{aligned} u'(W_A - \pi - x + I(x))\tilde{\lambda}_0 &= 1 + \psi'(I(x)) \\ I(x) &= \Sigma_x^{-1}(\tilde{\lambda}_0). \end{aligned} \tag{119}$$

Finally, the co-insurance equation (55) is obtained from (46) for $G' = 0$. This completes the proof.

6.8 Proof of Corollary 4.1

The shape of the indemnity function (57) can be obtained immediately by solving the initial value problem:

$$I'(x) = 1, \quad I(x_1) = 0. \tag{120}$$

The associated premium (58) is obtained by substituting (57) into the equality constraint (54). We now show full insurance under no loading. Without loss of generality let us normalize $\lambda_0 = 1$. In this case (112) simplifies to:

$$\int_{I_x} u'(W_A - \pi - x + I(x))\bar{f}(x)dx = \mu_h. \tag{121}$$

By Corollary 4.1, the contract in this case has the shape of a straight deductible of size x_1, which is characterized by:

$$K(x_1) = 0 \iff u'(W_A - \pi - x_1) = \mu_h(1 + L). \tag{122}$$

Equation (121) can be rewritten as:

$$\int_0^{x_1} u'(W_A - \pi - x)\bar{f}(x)dx + \int_{x_1}^{\bar{x}} u'(W_A - \pi - x_1)\bar{f}(x)dx = \mu_h. \tag{123}$$

Observe that by IBP, the first term on the LHS becomes:

$$u'(W_A - \pi - x_1)\bar{F}(x_1) + \int_0^{x_1} u''(W_A - \pi - x)\bar{F}(x)dx, \tag{124}$$

where $\bar{F}(x) \equiv \int_0^x \bar{f}(t)dt$ is the ambiguity-neutral cumulative distribution function (cdf), with $\bar{F}(0) = 0$. Likewise, using $\bar{F}(\bar{x}) = 1$, the second term on the LHS of (123) can be rewritten as:

$$(1 - \bar{F}(x_1))u'(W_A - \pi - x_1). \tag{125}$$

Combining (124) and (125) yields:

$$u'(W_A - \pi - x_1) + \int_0^{x_1} u''(W_A - \pi - x)\bar{F}(x)dx = \mu_h. \tag{126}$$

We can now use (122) to substitute out μ_h in (126), which gives:

$$\frac{m}{m+1} = \int_0^{x_1} \frac{-u''(W_A - \pi - x)}{u'(W_A - \pi - x_1)}\bar{F}(x)dx. \tag{127}$$

Since the density is strictly positive everywhere and the policyholder is strictly risk-averse, the term inside the integral on the RHS of the expression above is strictly positive. It is then immediate to see that $x_1 = 0$ if and only if $m = 0$.

6.9 Proof of Lemma 4

Observe that both A and P defined in (39) and (40), respectively, are strictly positive-valued, bounded and continuously differentiable on $(0, \bar{x})$. Differentiating with respect to x yields:

$$A'(x) = \lambda_0 \frac{\sum_{1 \le i < j \le n} p_i p_j \left(\phi'_A(y_i(\bar{x})) - \phi'_A(y_j(\bar{x}))\right) f_j^2(x)\ell'_{ij}(x)}{\langle p, f(x)\rangle^2}, \tag{128}$$

and

$$P'(x) = \mu_h \frac{\sum_{1\le i<j\le n} p_i p_j \left(\phi_P'(z_i(\bar{x})) - \phi_P'(z_j(\bar{x}))\right) f_j^2(x)\ell_{ij}'(x)}{\langle p, f(x)\rangle^2}. \tag{129}$$

Thus in general (for $n > 2$) the monotonic behavior of A and P depends on the ordering of the second-order states. In particular, a sufficient condition for A to be increasing is that $y_i(\bar{x}) \ge y_j(\bar{x})$ for all $i < j$ since ϕ_A is concave and $\ell_{ij}' \le 0$. Likewise a sufficient condition for P to be increasing is $z_i(\bar{x}) \ge z_j(\bar{x})$ for all $i < j$. By integration by parts (IBP):

$$\begin{aligned} z_i(\bar{x}) &= W_P(\bar{x}) + \int_{I_x} I'(x)\left[1 + \psi'(I(x))\right] F_i(x)dx, & (130)\\ y_i(\bar{x}) &= u(W_A(\bar{x})) + \int_{I_x} \left[1 - I'(x)\right] u'(W_A(x))F_i(x)dx, & (131)\end{aligned}$$

where

$$\begin{aligned} W_A(x) &= W_A - \pi - x + I(x), & (132)\\ W_P(x) &= W_P + \pi - I(x) - \psi(I(x)). & (133)\end{aligned}$$

Hence

$$z_i(\bar{x}) - z_j(\bar{x}) = \int_{I_x} I'(x)[1 + \psi'(I(x))](F_i(x) - F_j(x))dx, \tag{134}$$

$$y_i(\bar{x}) - y_j(\bar{x}) = \int_{I_x} [1 - I'(x)](F_i(x) - F_j(x))dx. \tag{135}$$

Since $F_i(x) \ge F_j(x)$ on I_x with strict inequality at least on a subset of positive-measured of I_x by Assumption 5, the ordering of the states depend crucially on the magnitude of I' relative to one. Observe that from Proposition 3.2,

$$I'(x) \in \left\{0, 1, \frac{r_u(W_A(x) + \frac{G'(x)}{G(x)}}{r_u(W_A(x) + \frac{\psi''(I(x))}{1+\psi'(I(x))})}\right\}. \tag{136}$$

Notice that for $n = 2$, then (128) and (129) reduce to:

$$A'(x) = \lambda_0 \frac{p_1 p_2 \left(\phi_A'(y_1(\bar{x})) - \phi_A'(y_2(\bar{x}))\right) f_2^2(x)\ell_{12}'(x)}{\bar{f}^2(x)}, \tag{137}$$

$$P'(x) = \mu_h \frac{p_1 p_2 \left(\phi_P'(z_1(\bar{x})) - \phi_P'(z_2(\bar{x}))\right) f_2^2(x)\ell_{12}'(x)}{\bar{f}^2(x)}. \tag{138}$$

It is immediate from Assumption 5 that the signs of both A' and P' (and likewise G') are independent of x. Next, using (135) and (134) for $n = 2$ yields:

$$\begin{aligned} z_1(\bar{x}) - z_2(\bar{x}) &= \int_{I_x} I'(x)\left[1 + \psi'(I(x))\right](F_1(x) - F_2(x))\,dx, & (139)\\ y_1(\bar{x}) - y_2(\bar{x}) &= \int_{I_x} \left[1 - I'(x)\right] u'(W_A(x))\,(F_1(x) - F_2(x))\,dx, & (140)\end{aligned}$$

where $W_A(x) = W_A - \pi - x + I(x)$. Recall that $F_1(x) - F_2(x) \ge 0$ on I_x is implied by Assumption 5. Since the sign of G' is independent of the size of the loss, either $G' \ge 0$ or $G' \le 0$. Let us consider two cases.

Suppose that $G' \geq 0$. In this case $I' \geq 0$ on I_x, implying that $z_1(\bar{x}) \geq z_2(\bar{x})$ in view of (139). But this implies, via (138) that $P' \geq 0$ on I_x. Observe that (47) is equivalent to

$$\frac{A'}{A} = \frac{G'}{G} + \frac{P'}{P}, \tag{141}$$

which must be positive since both terms on the RHS are positive. Hence $A' \geq 0$ since A is strictly positive.

Suppose that $G' \leq 0$. In this case $I'(x) \leq 1$ on I_x, implying via (140) that $y_1(\bar{x}) \geq y_2(\bar{x})$. Hence (137) implies $A' \geq 0$ on I_x. Again from (47) we can write:

$$\frac{P'}{P} = -\frac{G'}{G} + \frac{A'}{A}, \tag{142}$$

implying that $P' \geq 0$. This completes the proof.

6.10 Proof of Theorem 4.2

Observe that K and L defined in (37) and (38), respectively, are continuous and differentiable. We have:

$$K'(x) = -u''(W_A - \pi - x)G(x) + u'(W_A - \pi - x)G'(x), \tag{143}$$
$$L'(x) = u'(W_A - \pi)G'(x) - \psi''(x). \tag{144}$$

Notice that if $G' \geq 0$, then K' above is strictly positive since G is strictly positive and u is strictly increasing and strictly concave (Assumption 8). Moreover, the monotonicity of G also implies:

$$G(0) \leq G(x), \quad x \in I_x. \tag{145}$$

In addition, Assumption 6 and Assumption 8 imply

$$u'(W_A - \pi) \leq u'(W_A(x)), \quad x \in I_x. \tag{146}$$

Since G and u' are strictly positive, conditions (145) and (146) imply

$$u'(W_A - \pi)G(0) \leq u'(W_A(x))G(x), \quad x \in I_x. \tag{147}$$

Taking expectation with respect to the density $\tilde{f}$ on both sides yields:

$$u'(W_A - \pi)G(0) \leq \int_{I_x} u'(W_A(x))G(x)\tilde{f}(x)dx \tag{148}$$
$$u'(W_A - \pi)G(0) \leq 1, \tag{149}$$

where the second line follows from (60). Since $\psi' > 0$ (Assumption 7), this implies:

$$u'(W_A - \pi)G(0) - (1 + \psi'(0)) < 0 \tag{150}$$
$$K(0) < 0. \tag{151}$$

If $K(\bar{x}) \leq 0$, then $K(x) \leq 0$ on I_x since K is continuous, strictly increasing and $K(0) < 0$. In this case $I(x) = 0$ on I_x, which constitutes an optimum if and only if $\pi = \underline{\pi} = 0$ by Lemma 3, contradicting the hypothesis that π takes an interior value. Hence $K(\bar{x}) > 0$, implying (by continuity and strict monotonicity) that there exists a unique deductible

$x_1 \in (0, \bar{x})$ such that $K(x_1) = 0$.

For losses beyond the deductible, consider L in (38). Note that by the strict concavity of the Hamiltonian, $K(x_1) > L(x_1)$, implying that $L(x_1) < 0$. By continuity $L(x) < 0$ at least on a sufficiently small open interval to the right of x_1. Denote this interval $(x_1, x_1+\epsilon)$, then on $(x_1, x_1 + \epsilon)$ we have $K(x) > 0$ and $L(x) < 0$, implying that $I(x) \in (0, x)$ and is characterized by $I(x) = \Sigma_x^{-1}(G(x))$. Beyond $x_1 + \epsilon$ the shape of the indemnity function depends on the monotonic behavior of L. In view of (144), let us consider the following cases.

1. If ψ is strictly convex, the sign of L' is indeterministic and our analysis reaches an impasse.

2. If ψ is linear, then $L'(x) \geq 0$, implying that L is increasing. We know that $L(x_1) < 0$. Consider the following subcases.

   a. If $L(\bar{x}) < 0$ then monotonicity implies $L(x) < 0$ on $(x_1, \bar{x}]$. Therefore on $(x_1, \bar{x}]$ the indemnity function satisfies $I(x) \in (0, x)$ and is characterized by $I(x) = \Sigma_x^{-1}(G(x))$. In other words, the indemnity function has the form (65).

   b. If $L(\bar{x}) \geq 0$ then by continuity the equation $L(x) = 0$ has a solution. Denote

$$S = \{x \in (x_1, \bar{x}] \mid L(x) = 0\}. \tag{152}$$

   It is easy to see that S is closed and bounded. Then we can uniquely define x_2 as the smallest element of S. In this case since L is increasing we have $L(x) \geq L(x_2) = 0$ for all $x \in [x_2, \bar{x}]$ and $L(x) < 0$ for all $x \in (x_1, x_2)$. Hence beyond x_1, the indemnity function is characterized by:

$$I(x) = \begin{cases} \Sigma_x^{-1}(G(x)) & x \in (x_1, x_2), \\ x & x \in [x_2, \bar{x}]. \end{cases} \tag{153}$$

   Therefore in this case the indemnity function has the form (64).

6.11 Proof of Proposition 4.2

It is immediate to see that in this case

$$G(x) = \frac{A(x)}{\mu_h}, \quad x \in I_x. \tag{154}$$

Consequently $G'(x) = \frac{A'(x)}{\mu_h}$, implying that $G' \geq 0$ on I_x by Lemma 4. The rest of the proof follows that of Theorem 4.2 verbatim.

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