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Stochastic Accumulation and the Optimal Investment in Human Capital

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Abstract

This chapter examines the impact of risk and ambiguity on the op- timal level of investment in human and physical capital. Uncertainty (both in the sense of risk and of ambiguity) is introduced to the accumulation of human capital (HC) via two channels. When uncertainty is on the depreciation rate of human capital (uncertain skills obsolescence), I found that the optimal level of investment in human capital always increases, regardless of whether a risk-less physical capital (PC) is present. This response to uncertainty of an optimizing household is typically a self-insurance type of behavior. By contrast, when uncertainty is introduced to the efficiency of HC accumulation, the optimal investment in HC declines for the group of representative households with CRRA utility with relative risk aversion less than one. This response to uncertainty is typical of a household who views the investment as an asset with risky return instead of an insurance alternative.

Key words. Stochastic human capital accumulation, smooth ambiguity aversion

JEL classification. D81, E24

1 Introduction

Never before has the concern regarding developing human capital become so pressing as nowadays. Human capital becomes a crucial issue because for certain nations especially those deprived of natural resources such as Israel, Japan or Singapore, it is human capital that brings about the economic miracles. Accordingly, investment in education and training is of utmost importance in these countries. Nevertheless, especially in developing countries, there is evidence of co-existence of over-education in some sectors (such as in management and finance), and severe lack of skill in others, especially in highly specialized technical fields. One reason for this imbalance is that while technical expertise consumes much more resource to develop, its market value is highly susceptible to uncertainty. The technicians who are in demand today might see their skills become obsolete tomorrow at the arrival of a new technology. Without sufficient job security and investment in lifelong learning from the private and public sectors, there might not be enough incentive to specialize in technical areas.

Uzawa (1965) emphasizes that technological progresses or knowledge should not be viewed as *manna from heaven*. This idea does not seem to receive much attention until the late 80s and early 90s, when several contributions are devoted to understanding the role of human capital as a source of endogenous growth, such as Romer (1986) and Lucas (1988, 1990). Mankiw et al. (1992) defend the explanatory power of the Solow-Swan model by taking into account human capital.[1] Starting with the seminal contribution of Becker (1964), it is now consensus that the main building blocks of human capital are health and education, both of which are at risk at the current pace of technological progress and environmental complications. Climate change and epidemics with unpredictable severity and probability of occurrence can have damaging consequences on public health, which in turn might lead to economic crises. The *Information Age* that began in the late 1950s-1970s has dramatically altered the way we work. Low-skilled and routine jobs are being replaced by machines in both developing countries and advanced economies. In fact, even skilled-jobs are not immune to obsolescence, as suggested by the results from the European skills and jobs survey 2014 illustrated in Figure 1. Overall, about 45 percent of the respondents indicated that many of their skills were likely to become obsolete in the following five years. Interestingly, when asked whether they thought this scenario would be *very likely*, this statistic dropped to about 20 percent. This implies a high degree of

[1]Their model is often referred to as the *augmented Solow* model. Since the diminishing return hypothesis still holds, this model also predicts no long-run growth.

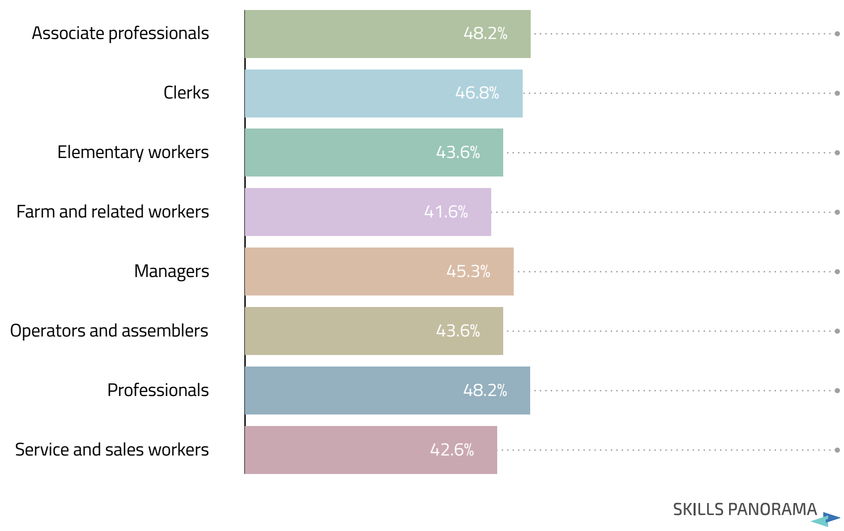

Figure 1: Subjective skills obsolescence in a five-year horizon by occupations in Europe. *Source:* European skills and jobs survey (Cedefop, 2014).

subjective uncertainty.

Uncertainty in human capital accumulation has become more relevant than ever before, posing challenges to both policymakers and private corporations. On the policy side, the World Bank has recognized that countries are underinvesting in human capital, which might broaden the skills gap even further between rich and poor countries. To address this pressing issue, the group created the Human Capital Project in 2018. Notably, a human capital index (HCI) based on different measures of health and education is being computed for each country (see Avitabile et al. (2020)). According to Collin and Weil (2018), better health not only enhances productivity but also enables us to enjoy life better. In other words, good health is desirable both instrumentally and intrinsically. On the business side, Nalbantian (2017) calls for a distinction between human capital risk and ambiguity in order to address them with proper measures. Deloitte (2018) highlights the importance of sufficiently investing in the workforce to pre-arm for *the forth industrial revolution*, the age of artificial intelligence.

For these reasons a more general approach to modeling human capital, allowing for uncertainty in the form of risk as well as ambiguity in its accumulating process is desirable. Interestingly, in the field of management, Chauhan and Chauhan (2008) documented "superiors' attitude" as a crucial determinant of skills obsolescence perceived by managers. Needless to say, this factor is highly ambiguous by nature. In a more recent article, Nalbantian (2017) calls for an a distinction between risk and ambiguity[2] in addressing issues related to human capital since they have distinct implications on a company's responses to uncertain skill obsolescence.

The first works on introducing stochastic elements into the law of motion of human capital date back to the 70s. This line of research was initiated by Levhari and Weiss (1974), who argued for the need of relaxing the assumption of perfect foresight in human capital accumulation, which had been maintained heretofore in the seminal treatises of Becker (1964) and Schultz (1971). In a two-period model, the authors categorized human capital risks as "input" and "output". The former includes factors concerning the *production* of human capital, such as learning abilities of the individuals, or schooling quality. The latter reflects the market conditions (supply and demand of the type of labor produced) in the post-production period, determining whether the skill produced is valued by the market. In the direction of this approach, Williams (1979) allowed for risky depreciation and net productivity (both log-normally distributed) in a model where human capital is accumulated linearly with respect to the level of investment in education. Although not explicitly stated, the so-called "net productivity" parameter in Williams (1979) can be placed under "input" as the production of human capital occurs at the beginning of the period. The risky depreciation rate, on the other hand, could be categorized as "output": how robust is the produced skill/human capital to the market conditions. If the economy is caught by a technological or organization shock, some types of skill might become obsolete very quickly.

In this paper, I investigate the impact of uncertain human capital accumulation from two different viewpoints. First, in a spirit similar to Krebs (2003), the depreciation rate of human capital is viewed as a random variable to capture the so-called uncertain obsolescence of skills phenomenon. Then, I consider the case where the random variable is effectiveness (or net productivity) of human capital accumulation rather than its depreciation rate. In each case, I study the effect uncertainty, both in the form of risk (measurable uncertainty) and ambiguity (immeasurable uncertainty), on the optimal level of investment in human capital. The two views result in completely different implications.

2 Statement of the problem and assumptions

Consider the discrete time analog of the Ben-Porath model Ben-Porath (1967) with no learning time and ambiguous stochastic depreciation rate of human capital. For simplicity

[2]In the terms of in Knight (1921), this is the difference between measurable and immeasurable uncertainty.

let us examine a two-period economy without physical capital. The program faced by a representative agent is:

$$\max_{c_0\geq 0, e\geq 0} \quad u(c_0) + \beta\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))) \tag{1}$$

$$s.t. \quad c_0 + e = y_0, \tag{2}$$

$$\tilde{c}_{1\theta} = \tilde{y}_{1\theta}, \quad \theta \in \Theta, \tag{3}$$

$$y_t = AF(h_t), \quad t \in \{0,1\}, \tag{4}$$

$$\tilde{h}_{1\theta} = h_0(Be^\alpha + 1 - \tilde{\delta}_\theta), \quad \alpha \in (0,1], \ \theta \in \Theta, \tag{5}$$

$$A > 0, \ B > 0, \ h_0 > 0 \ \ given. \tag{6}$$

In this program c_t, y_t, h_t denote consumption, output, and human capital in period t, respectively, for $t \in \{0,1\}$. The investment in human capital (control variable in the initial period) is e. In addition, the parameters A and B stand for the total factor of output and human capital productivity, respectively. Ambiguity enters through the unknown scenario θ, which belongs to the scenario space Θ. The representative agent has perfect knowledge of Θ and the conditional distribution of $\tilde{\delta}_\theta$, for each scenario $\theta \in \Theta$, but faces uncertainty regarding which scenario is going to occur. This is in contrast to the unambiguous stochastic setting, where the distribution of the stochastic variable is assumed to be objectively known. In other words, in absence of ambiguity, the agent knows exactly which scenario will occur.

The attitude towards ambiguity of the decision maker is modeled in the smooth sense of Klibanoff et al. (2005) and Klibanoff et al. (2009), via the functional ϕ (also called second-order utility functional). This functional being concave, linear or convex corresponds to a decision maker that is ambiguity-averse, ambiguity-neutral or ambiguity-seeking, respectively. Note that the maximin expected utility (MEU) of Gilboa and Schmeidler (1989) is a special case of smooth ambiguity. In particular, Klibanoff et al. (2005) proved that the MEU representation is achieved when the decision-maker has infinite absolute ambiguity aversion in the smooth sense.

Assumption 1 (Finite scenario space)**.** *Let the scenario space be* $\Theta = \{1, 2, \cdots, n\}$*, where* n *is a positive integer.*

Hence associated to each scenario θ is a random variable $\tilde{\delta}_\theta$ whose distribution is perfectly known. We shall assume that all the scenario-conditional distributions $\tilde{\delta}_\theta$ have a common support.

Assumption 2 (Finite common support)**.** *Assume that all scenario-conditional random variables*

$\tilde{\delta}_\theta$'s have a common finite support $\chi = \{\delta_1, \cdots, \delta_m\}$, where $\delta_j \in (0,1)$ for all $j = 1, \ldots, m$, for some positive integer m.

In other words, the scenarios do not shift the support of the random variable, which is crucial. Under this assumption, we can rewrite constraints (3) and (5) in the agent's program as:

$$c_1(\delta_j) = y_1(\delta_j), \tag{7}$$

$$h_1(\delta_j) = h_0(Be^\alpha + 1 - \delta_j), \tag{8}$$

for all $\delta_j \in \chi$.

Assumption 3 (Smooth ambiguity aversion)**.** *The functional $\phi : \mathbb{R} \to \mathbb{R}$, the second order utility function, is strictly increasing, concave and continuously differentiable.*

This assumption reflects a large body of empirical evidence since Ellsberg (1961) that decision makers are ambiguity-averse. The next assumption is also standard.

Assumption 4 (Risk aversion)**.** *The vNM utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, concave and satisfies the Inada conditions:*

$$\lim_{w \to 0} u'(w) = +\infty, \tag{9}$$

$$\lim_{w \to +\infty} u'(w) = 0. \tag{10}$$

Assumption 5 (Production technology)**.** *Assume that the production function $F : \mathbb{R}^n_+ \to \mathbb{R}_+$ is strictly increasing and concave with respect to each factor of production and satisfies the Inada conditions:*

$$\lim_{x_j \to 0} F'_j(\cdot) = +\infty, \tag{11}$$

$$\lim_{x_j \to +\infty} F'_j(\cdot) = 0, \tag{12}$$

where $F'_j(\cdot)$ denotes the partial derivative of $F(\cdot)$ with respect to the j^{th} factor of production x_j, $j = 1, \ldots, n$.

Remark 1. *The conditional expectation $\mathbb{E}_\theta(\cdot)$ is the expectation with respect to each scenario-conditional distribution of $\tilde{\delta}$. As a consequence of ambiguity, $\mathbb{E}_\theta[u(\tilde{c}_{1\theta})]$ is a random variable depending on θ. On the other hand, the outer expectation $\mathbb{E}(\cdot)$ is taken over the priors in the scenario space. For example let $(q_1, \cdots, q_n)$ be priors on scenarios satisfying $\sum_{\theta=1}^{n} q_\theta = 1$ and $q_\theta \geq 0$ for* *all $\theta \in \Theta$. Let $p_\theta(\delta_j)$ be the probability that the depreciation rate takes value δ_j under scenario θ. Then the objective function could be written explicitly as:*

$$u(c_0) + \beta\phi^{-1}\left(\sum_{\theta=1}^{n} q_\theta \phi\left(\sum_{j=1}^{m} p_\theta(\delta_j) u(c_1(\delta_j))\right)\right). \tag{13}$$

3 Optimal investment in human capital in absence of physical capital

Let us rewrite the objective function in (1) as:

$$V(e) = u(y_0 - e) + \beta\phi^{-1}\left(\mathbb{E}\phi\left(\mathbb{E}_\theta u\left(y_0(Be^\alpha + 1 - \tilde{\delta}_\theta)^\mu\right)\right)\right). \tag{14}$$

To simplify notations, let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the mapping defined by:

$$f(x) = u'(y_0 x^\mu) x^{\mu-1}. \tag{15}$$

Lemma 1. *Under Assumption 4, the function f defined by (15) is strictly decreasing. If moreover the utility function satisfies $u''' \geq 0$, then f is strictly convex.*

Proof. Observe that f is twice differentiable, hence:

$$\begin{aligned} f'(x) &= \mu y_0 u''(x^\mu) x^{1(\mu-1)} + (\mu - 1)u'(x^\mu)x^{\mu-2} \\ &= -u'(x^\mu)x^{\mu-2}\left(\mu y_0 R_u + 1 - \mu\right), \end{aligned} \tag{16}$$

where

$$R_u := -\frac{x^\mu u''(x^\mu)}{u'(x^\mu)} \tag{17}$$

is the Arrow-Pratt measure of relative risk aversion. Since $\mu \in (0, 1)$ and u satisfies Assumption 4, it is clear that $f' > 0$. Moreover,

$$\begin{aligned} f''(x) &= (\mu y_0)^2 u'''(x^\mu)x^{3\mu-3} + 3\mu y_0(\mu - 1)u''(x^\mu)x^{2\mu-3} + (1 - \mu)(2 - \mu)u'(x^\mu)x^{\mu-3} \\ &= u'(x^\mu)x^{\mu-3}\left[(\mu y_0)^2 P_u R_u + 3(\mu y_0)(1 - \mu)R_u + (1 - \mu)(2 - \mu)\right], \end{aligned} \tag{18}$$

where

$$P_u := -\frac{x^\mu u'''(x^\mu)}{u''(x^\mu)} \tag{19}$$

is the degree of relative prudence in the sense of Kimball (1990a). Since $\mu \in (0, 1)$ and

Assumption 4 holds, a sufficient condition for f to be convex in x is $P_u \geq 0$, which is equivalent to $u''' \geq 0$. ■

Observe that

$$V'(e) = -u'(y_0 - e) + \zeta y_0 e^{\alpha-1} \frac{\mathbb{E}\phi'\left(\mathbb{E}_\theta u(\tilde{c}_{1\theta})\right)\mathbb{E}_\theta f(\tilde{X}_\theta)}{\phi'\left(\phi^{-1}\left(\mathbb{E}\phi\left(\mathbb{E}_\theta u(\tilde{c}_{1\theta})\right)\right)\right)}, \tag{20}$$

where

$$\zeta \equiv B\beta\alpha\mu, \tag{21}$$

and

$$\tilde{X}_\theta = Be^\alpha + 1 - \tilde{\delta}_\theta. \tag{22}$$

Note that V is not necessarily concave in e when ϕ is strictly concave since ϕ^{-1} is not concave. Nevertheless, if absolute ambiguity tolerance (the inverse of absolute ambiguity aversion) is concave, then it can be shown that we indeed have a concave problem.

Assumption 6 (Concave absolute ambiguity tolerance)**.** *Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be the absolute ambiguity tolerance function defined by $T(u) = -\frac{\phi'(u)}{\phi''(u)}$. Then T is concave in $\mathbb{R}$.*

Remark 2. *A very popular class of utility functions is the class of hyperbolic absolute risk aversion (HARA), which has linear absolute risk tolerance (in wealth).*[3] *In the same vein, ambiguity preferences that belong of the hyperbolic absolute ambiguity aversion (HAAA) class have linear absolute ambiguity tolerance (in utility), thus satisfying Assumption 6. HAAA ambiguity preferences include those that satisfy DAAA (decreasing absolute ambiguity aversion), CAAA (constant absolute ambiguity aversion), or IAAA (increasing absolute ambiguity aversion).*

Lemma 2. *Assumption 6 is sufficient for the objective function V defined in (14) to be strictly concave in e.*

Proof. See subsection 7.1. ■

3.1 Deterministic depreciation rate of human capital

Suppose first that there is perfect foresight, so that the depreciation rate is known with certainty to be δ. Define:

$$X \equiv Be^\alpha + 1 - \delta. \tag{23}$$

[3] See Chapter 4 of Lengwiler (2004) for a detailed discussion on HARA.

Denote by V_1 the objective function under perfect foresight. We have:

$$V_1'(e) = -u'(c_0) + \zeta y_0 e^{\alpha-1} f(X). \tag{24}$$

Since u is strictly concave and the positive constants α and μ are less than one, it is easy to see that V_1 is strictly concave in e. Hence the first order condition (FOC) is both necessary and sufficient for an unique optimal. Let e_1 denote the optimal level of investment in human capital in the deterministic case, we have:

$$V_1'(e_1) = 0. \tag{25}$$

3.2 Unambiguous stochastic depreciation of human capital

Under no ambiguity, we know exactly which scenario occurs. Let $\tilde{\delta}$ be the stochastic depreciation rate associated to this scenario and suppose that this uncertainty adds a zero-mean risk to the deterministic rate δ. In particular,

$$\tilde{\delta} = \delta + \tilde{\varepsilon}, \quad \mathbb{E}\tilde{\varepsilon} = 0, \tag{26}$$

where $\tilde{\varepsilon}$ is a zero-mean risk. Then X defined earlier becomes a random variable, which we shall denote by $\tilde{X}$ to mean:

$$\tilde{X} = Be^{\alpha} + 1 - \tilde{\delta}. \tag{27}$$

Clearly (26) implies

$$\mathbb{E}\tilde{X} = X. \tag{28}$$

Let the objective function under pure risk be V_2. We have:

$$V_2'(e) = -u'(y_0 - e) + \zeta y_0 e^{\alpha-1} \mathbb{E} f(\tilde{X}), \tag{29}$$

where f is the mapped defined in (15). It is easy to see that V_2 is strictly concave in e since u is strictly concave and f is strictly decreasing in e. Hence the FOC is also sufficient for an unique solution. Denote by e_2 the level of optimal investment in human capital under pure risk, then:

$$V_2'(e_2) = 0. \tag{30}$$

Assumption 7 (Risk prudence)**.** *The decision maker is prudent in the sense of Kimball (1990b). In particular,*

$$u' > 0,\ u'' < 0,\ u''' > 0. \tag{31}$$

Observe that monotonicity and risk aversion are already embedded in Assumption 4. Prudence adds a third order requirement to reflect the agent's aversion to fluctuation in marginal utilities.

Proposition 1. *If risk preference satisfies prudence (Assumption 7), then the introduction of a zero-mean risk to the depreciation rate of human capital raises the optimal level of investment in human capital.*

Proof. See subsection 7.2. ∎

Remark 3. *Observe that the increased saving (investment in human capital) in response to an increase in risk comes from the convexity of f. In view of (18), this property is attributed to two sources: prudence (Assumption 7) and the concavity of the production function $(1-\mu)$. Clearly, if the production function were linear $(\mu = 1)$, then the introduction of risk raises savings if and only if the DM is prudent. On the other hand, if the DM is imprudent in the sense that $u''' = 0$ (for example if he has quadratic utility), then the strict concavity of the production function (Assumption 5 is necessary and sufficient for a rise in savings.*

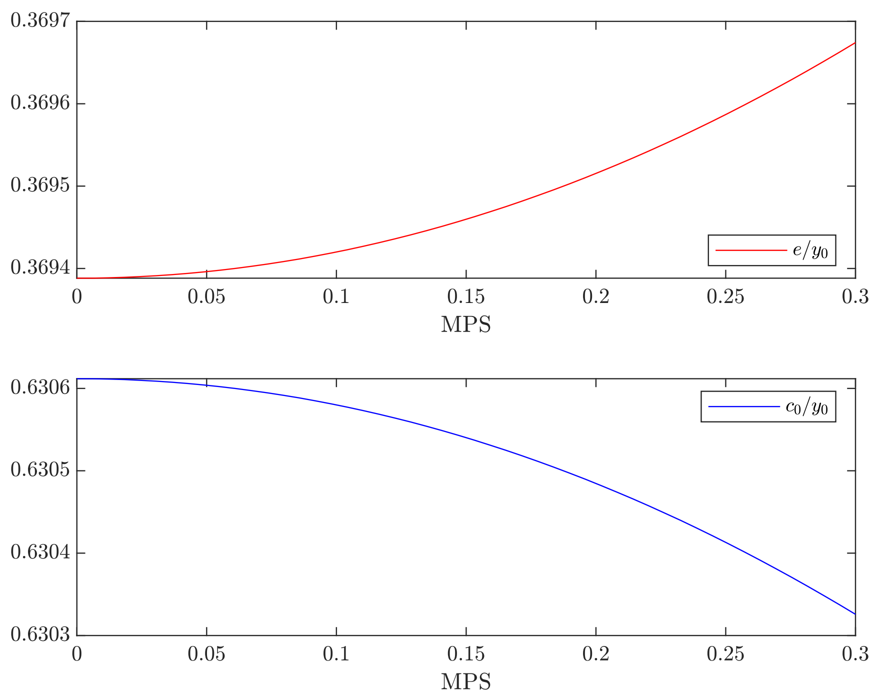

Figure 2: The effect of increasing risk on the depreciation rate of HC (increasing mean-preserving spreads (MPS)) on the optimal consumption and investment in HC

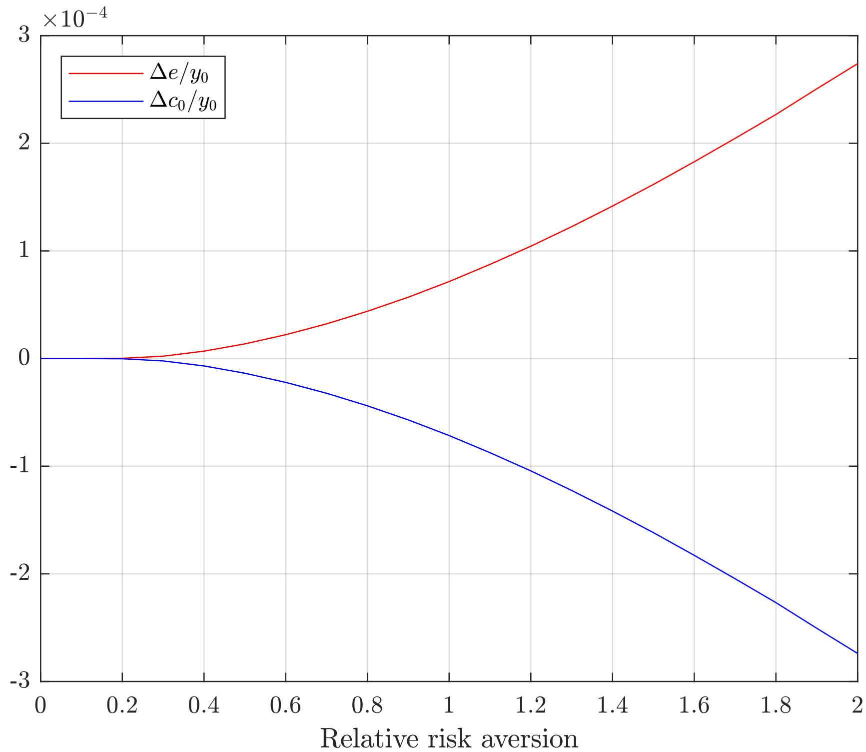

Figure 3: The effect of increasing relative risk aversion on the optimal consumption and investment in HC

3.3 Ambiguous depreciation rate of human capital accumulation

We now move to the ambiguous setting. Recall that the law of motion for human capital accumulation in this case is:

$$\tilde{h}_{1\theta} = h_0(Be^{\alpha} + 1 - \tilde{\delta}_{\theta}), \quad \theta \in \Theta. \tag{32}$$

Observe that each $\tilde{\delta}_{\theta}$ is a random variable taking values in the common support χ (defined in Assumption 2) for all θ in the scenario space Θ. We now consider two subcases: one where the decision maker is ambiguity-neutral (linear ϕ), and the other where she is ambiguity-averse (strictly concave ϕ).

3.3.1 Ambiguity-neutral agent

Let V_3 be the objective function in this case. Ambiguity neutrality implies that ϕ is linear, so that (20) can be simplified to:

$$V_3'(e) = -u'(y_0 - e) + \zeta y_0 e^{\alpha-1} \mathbb{E}\mathbb{E}_\theta f(\tilde{X}_\theta), \tag{33}$$

where f is the map defined in (15) and $\tilde{X}_\theta$ defined in (22). We assume that ambiguity enters in the following manner.

Assumption 8 (SSD ordering of scenarios)**.** *Suppose that the scenario-conditional distributions could be ranked according to the sense of second order stochastic dominance (SSD). In particular,*

$$\tilde{\delta}_\theta = \delta + \sum_{j=1}^{\theta} \tilde{\epsilon}_j, \quad \forall \theta \in \Theta, \tag{34}$$

where $\{\tilde{\epsilon}_j\}_{j\in\Theta}$ are white noises, i.e.,

$$\mathbb{E}\tilde{\epsilon}_j = 0, \quad \forall j \in \Theta. \tag{35}$$

This structure is essentially a sequence of mean preserving spreads (MPS), with the higher value of θ associated to an increase in risk (or a deterioration in SSD) in the sense of Rothschild and Stiglitz (1970). In fact Assumption 8 is the mildest ranking criterion of the conditional distributions in order to generate a differential effect on the level of optimal saving. As will be shown later, any ranking criterion stronger than SSD dominance will push optimal investment in human capital in the same direction.

Proposition 2. *Under risk prudence, adding ambiguity as a sequence of MPSs described under (Assumption 8) induces the ambiguity-neutral agent to raise investment in human capital relative to the deterministic case.*

Proof. See subsection 7.3. ∎

3.3.2 Ambiguity-averse agent

Finally, we examine the impact of ambiguity on the optimal choice of an ambiguity-averse agent. The structure of ambiguity remains unchanged. We also maintain the risk prudence assumption. Let us rewrite (20) as:

$$V'(e) = -u'(y_0 - e) + \zeta y_0^{1-\gamma} e^{\alpha-1} \left(J + K \times \mathbb{E}\mathbb{E}_\theta f(\tilde{X}_\theta) \right), \tag{36}$$

where f is defined in (15), and

$$J := \frac{Cov\left(\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})), \mathbb{E}_\theta f(\tilde{X}_\theta)\right)}{\phi'\left(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))\right)}, \tag{37}$$

$$K := \frac{\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))}{\phi'\left(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))\right)}. \tag{38}$$

Remark 4. *It is noteworthy that J and K defined above are two fundamental effects of ambiguity aversion. On the one hand, the impact from J is due to pessimism according to Gollier (2011), in the sense of over-weighting worst scenarios. Naturally, J is nil under ambiguity neutrality. On the other hand, the impact from K results from preference for the timing of resolution of uncertainty according to Strzalecki (2013).*

Assumption 9 (Ambiguity prudence)**.** *The agent is ambiguity prudent. In particular,*

$$\phi' > 0,\ \phi'' < 0,\ \phi''' > 0. \tag{39}$$

Proposition 3. *When the scenario-conditional distributions can be ranked according to SSD described under Assumption 8, the ambiguity-averse representative agent with risk preference satisfying prudence (Assumption 7) raises investment in human capital relative to the ambiguity-neutral agent if his ambiguity preference satisfies either constant absolute ambiguity aversion (CAAA) or decreasing absolute ambiguity aversion (DAAA). The impact of ambiguity aversion is ambiguous under increasing absolute ambiguity aversion (IAAA).*

Proof. Let us first examine the impact of pessimism by proving the following lemma.

Lemma 3. *Pessimism under ambiguity aversion induces the agent to raise investment in human capital: $J > 0$.*

Proof. Recall that a worse scenario in the pure ambiguity structure previously defined means a deterioration in second order stochastic dominance (SSD). Since f is strictly decreasing and convex ($-f$ strictly increasing and concave), an SSD deterioration raises $\mathbb{E}_\theta f(\tilde{X}_\theta)$. By contrast, since u is increasing and concave in $\tilde{X}_\theta$, this deterioration reduces $\mathbb{E}_\theta u(\tilde{c}_{1\theta})$, thus raising $\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))$ by the concavity of ϕ. Hence the two random variables in the covariance term move in the same direction, implying that $J > 0$. ■

Next, we prove that the preference for timing of resolution of uncertainty has the following properties.

Lemma 4. *K defined in* (38) *manifests:*

- *preference for early resolution of uncertainty ($K > 1$) if ϕ satisfies DAAA;*
- *indifference to the timing of resolution of uncertainty ($K = 1$) if ϕ satisfies CAAA;*
- *preference for late resolution of uncertainty ($K < 1$) if ϕ satisfies IAAA.*

Proof. Note that all the three ambiguity preferences satisfying DAAA, CAAA or IAAA also satisfy ambiguity prudence. Following Berger (2011), by ambiguity aversion and ambiguity prudence there exist an utility ambiguity premium $\pi(e) \geq 0$ and an utility ambiguity precautionary premium $\psi(e) \geq 0$ implicitly defined by:

$$\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) = \phi(\mathbb{E}\mathbb{E}_\theta u(\tilde{c}_{1\theta}) - \pi(e)), \tag{40}$$

$$\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) = \phi'(\mathbb{E}\mathbb{E}_\theta u(\tilde{c}_{1\theta}) - \psi(e)). \tag{41}$$

Thus

$$K = \frac{\phi'(\mathbb{E}\mathbb{E}_\theta u(\tilde{c}_{1\theta}) - \psi(e))}{\phi'(\mathbb{E}\mathbb{E}_\theta u(\tilde{c}_{1\theta}) - \pi(e))}. \tag{42}$$

It is easy to see that a necessary and sufficient condition for ambiguity preference to satisfying DAAA is that $-\phi'$ is more concave than ϕ in the sense of Arrow-Pratt.[4] By definition, this implies that ceteri paribus, the ambiguity premium associated to $-\phi'$ is greater than that associated to ϕ. This is equivalent to saying that the ambiguity precautionary premium is greater than the ambiguity premium, implying that $K \geq 1$ under DAAA. The arguments can be repeated for the CAAA and IAAA cases. ∎

When $K \geq 1$, the preference for early resolution of uncertainty acts in the same direction as pessimism, inducing the agent to raise e since in this case,

$$V'(e) \geq V_3'(e), \tag{43}$$

implying $V'(e_3) \geq 0$. Let e_4 be the unique solution to $V'(e) = 0$, then by the concavity of V proven in Lemma 2, we conclude that $e_4 \geq e_3$.

When $K \leq 1$, the two effects act in opposite directions, rendering the final impact on e ambiguous. ∎

What happens to Proposition 3 if the conditional distributions are ranked by a stronger notion of stochastic dominance? Consider an improvement in first order stochastic dominance (FSD) of the distribution of $\tilde{X}_\theta$ when θ decreases. This is equivalent to an FSD

[4]See, for example, Chapter 2.5 of Gollier (2001) for a proof.

deterioration in the distribution of $\tilde{\delta}_\theta$. In other words, the distribution of $\tilde{\delta}_\theta$ dominates that of $\tilde{\delta}_{\theta+1}$ in FSD for each $\theta \in \Theta$. Mathematically, this means

$$Pr(\tilde{\delta}_\theta \leq \delta_j) \leq Pr(\tilde{\delta}_{\theta+1} \leq \delta_j), \quad \forall \delta_j \in \chi. \tag{44}$$

Since f is decreasing in $\tilde{X}_\theta$, the FSD improvement reduces its expectation. By contrast, since $u(\tilde{c}_{1\theta}) = u(y_0 \tilde{X}_\theta^\mu)$ increasing in $\tilde{X}_\theta$, the FSD improvement raises $\mathbb{E}_\theta u(\tilde{c}_{1\theta})$, hence reducing $\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))$. Consequently J is also positive . In other words, if we allow for the mean of each conditional distribution to be lower or greater than δ, but the ambiguity-neutral mean satisfies $\mathbb{E}\mathbb{E}_\theta \tilde{\delta}_\theta = \delta$, then Proposition 3 still holds. The same reasoning holds for stronger notion of stochastic dominance than FSD. [5]

Corollary 1. *The result of Proposition 3 still holds if ambiguity enters as a series of conditional distributions that can be ranked according to the FSD order as in (44), or stronger, so long as* $\mathbb{E}\mathbb{E}_\theta \tilde{\delta}_\theta = \delta$.

3.4 Comparative statics of increasing ambiguity aversion

Does increasing ambiguity aversion raise optimal investment in human capital? Let us focus on the CAAA and DAAA cases.

Consider first the impact on the preference for timing of ambiguity resolution. Clearly it is nil under CAAA since the ambiguity premium and ambiguity precautionary premium are always equal. Under DAAA, which belongs to the class of hyperbolic absolute ambiguity aversion (HAAA) second order utility functions, we can write the measure of absolute ambiguity prudence $P(\cdot)$ as:

$$P(U) = \left(1 + \frac{1}{\sigma}\right) A(U), \quad \sigma > 0, \tag{45}$$

where $A(z) > 0$ is the measure of absolute ambiguity aversion defined by

$$A(U) = \left(a + \frac{U}{\sigma}\right)^{-1}. \tag{46}$$

If $a = 0$, then ϕ satisfies CRAA with constant relative ambiguity aversion equal to σ.[6]

[5] For example, it holds also if the conditional distributions are ranked according to the probability ratio (PR), the hazard rate (HR), or the likelihood ratio (LR). See, for example, the Appendix of Krishna (2009), or Levy (2015) for further discussion.

[6] Gollier (2001) discusses the properties of the utility functions belonging to the hyperbolic absolute risk aver-

Denote:

$$\tilde{U}(e) = \mathbb{E}_\theta u(\tilde{c}_{1\theta}) \equiv \mathbb{E}_\theta u(\tilde{c}_{1\theta}(e)). \tag{47}$$

Define the ambiguity-neutral expected utility:

$$U(e) = \mathbb{E}\tilde{U}(e). \tag{48}$$

Assumption 10 (Pure ambiguity)**.** *Assume that ambiguity is a zero-mean risk added to $U(e)$. Specifically,*

$$\tilde{U}(e) = U(e) + \tilde{\eta}, \quad \mathbb{E}\tilde{\eta} = \sum_\theta q_\theta \tilde{\eta}(\theta) = 0. \tag{49}$$

Then the utility ambiguity premium $\pi(e)$ and the utility ambiguity precautionary premium $\psi(e)$ satisfy:

$$\mathbb{E}\phi(U(e) + \tilde{\eta}) = \phi(U(e) - \pi(e)), \tag{50}$$

$$\mathbb{E}\phi'(U(e) + \tilde{\eta}) = \phi'(U(e) - \psi(e)). \tag{51}$$

Lemma 5. *Under Assumption 10 and DAAA, the preference for early resolution of ambiguity manifest by K in* (38) *is increasing in ambiguity aversion.*

Proof. Following Pratt (1964) and Arrow (1965), a Taylor approximation around $U(e)$ yields:

$$\begin{aligned} \mathbb{E}\phi(U(e) + \tilde{\eta}) &\approx \phi(U(e)) + \phi'(U(e))\mathbb{E}\tilde{\eta} + 0.5\phi''(U(e))\mathbb{E}\tilde{\eta}^2 \\ &= \phi(U(e)) + 0.5\phi''(U(e))\mathbb{E}\tilde{\eta}^2, \end{aligned} \tag{52}$$

where the second line follows from $\tilde{\eta}$ being a zero-mean risk. Clearly, the smaller the risk, the more precise is the approximation. Similarly, a first order approximation around $U(e)$ for the RHS of (50) gives:

$$\phi(U(e) - \pi(e)) \approx \phi(U(e)) + U(e)\phi'(U(e)). \tag{53}$$

Equating (52) and (53), we arrive at the familiar Arrow-Pratt's approximation:

$$\pi(e) \approx 0.5A(U(e))V(\tilde{\eta}), \tag{54}$$

where $V(\tilde{\eta}) = \mathbb{E}\tilde{\eta}^2$ denotes the variance of $\tilde{\eta}$. The larger the the variance of the priors

sion (HARA) class. By analogy, a HAAA second order utility function can be written as $\phi(U) = \xi\left(a + \frac{U}{\sigma}\right)^{1-\sigma}$, where $a + \frac{U}{\sigma} > 0$. Monotonicity and ambiguity aversion require $\xi\frac{1-\sigma}{\sigma} > 0$.

(increasing ambiguity), the larger is the agent's willingness to pay to eliminate ambiguity. Likewise for the utility precautionary ambiguity premium:

$$\begin{aligned}\psi(e) &\approx 0.5P(U(e))V(\tilde{\eta}),\\ &= 0.5\left(1+\frac{1}{\sigma}\right)A(U(e))V(\tilde{\eta}),\end{aligned} \tag{55}$$

where the second line results from (45). Hence for each each fixed e, the elasticity of K with respect to $A(U(e))$ is:

$$\begin{aligned}\frac{\partial K/\partial A(U(e))}{K} &= A(U(e)-\psi(e))\frac{\partial \psi(e)}{\partial A(U(e))} - A(U(e)-\pi(e))\frac{\partial \pi(e)}{\partial A(U(e))}\\ &= 0.5V(\tilde{\eta})\frac{\left[A(U(e)-\psi(e)) - A(U(e)-\pi(e))\right]}{\sigma}.\end{aligned} \tag{56}$$

Since ϕ satisfies DAAA, the term in the square bracket on the RHS of (56) is positive. ■

Thus under DAAA, the higher the degree of ambiguity aversion, the higher the preference for early resolution of ambiguity, as illustrated in Figure 4 for a DM with CRAA and logarithmic utility. In other words, increasing ambiguity aversion has a positive impact on the HC investment. The higher the degree of ambiguity (the larger the variance of the priors), the larger this effect.

How about the impact coming from pessimism? Observe that (37) can be rewritten as:

$$J = \frac{Cov\left(\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})), \mathbb{E}_\theta f(\tilde{X}_\theta)\right)}{\phi'(\mathbb{E}\mathbb{E}_\theta u(\tilde{c}_{1\theta}) - \pi(e))}. \tag{57}$$

For any fixed e, increasing ambiguity aversion raises the utility ambiguity premium $\pi(e)$. This, however, does not imply a reduction in the denominator of J, since the curvature of ϕ is also varying. Likewise the direction of change of the numerator is also ambiguous. We thus resort to a numerical exercise to understand the impact of increasing ambiguity aversion on J. Figure 5 shows that although increasing ambiguity aversion has a non-monotone effect on the numerator of J, its impact on the denominator dominates, resulting in a net positive impact on the HC investment.

Since both effects of ambiguity aversion (preference for timing of uncertainty resolution and pessimism) point in the same direction, it is no surprise that the final impact of increasing ambiguity aversion on the investment in human capital is positive, as shown in Figure 6. The intuition is that increasing ambiguity aversion essentially reduces the utility

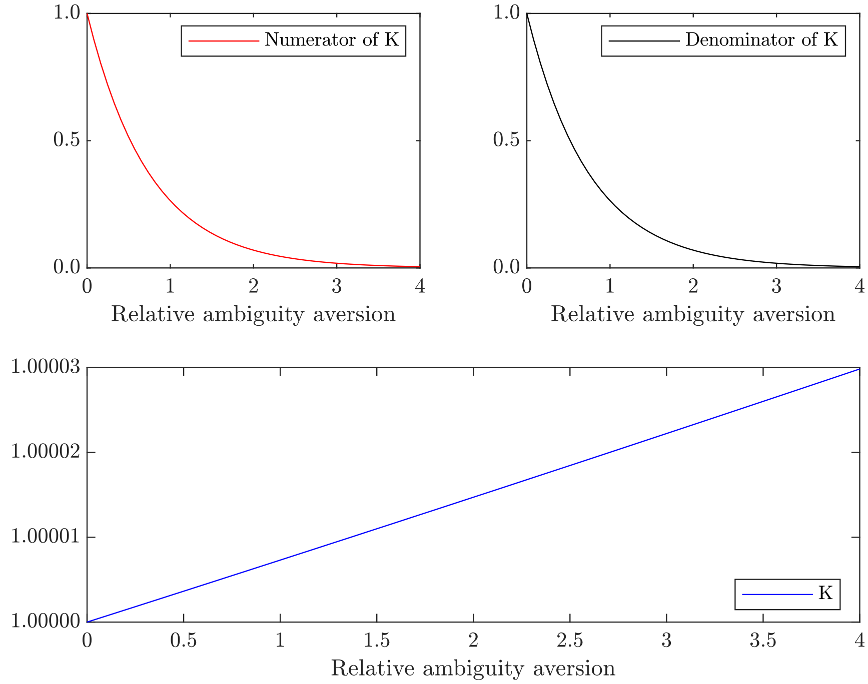

Figure 4: Impact of increasing ambiguity aversion on the preference of timing for resolution of ambiguity

ambiguity-equivalent of the next period's uncertain income, which raises this period's savings in order to smooth consumption (across scenarios). The preference for smoothing across scenarios is stronger the higher the degree of ambiguity aversion.

4 Optimal investment in human capital in presence of physical capital

We saw in the previous section that the introduction of risk or ambiguity raises the optimal level of investment in human capital. We might suspect that this is a consequence of having only one type of capital, that in presence of a risk-free physical capital, the opposite would hold. Let us now consider a more general model with physical capital. In particular, the

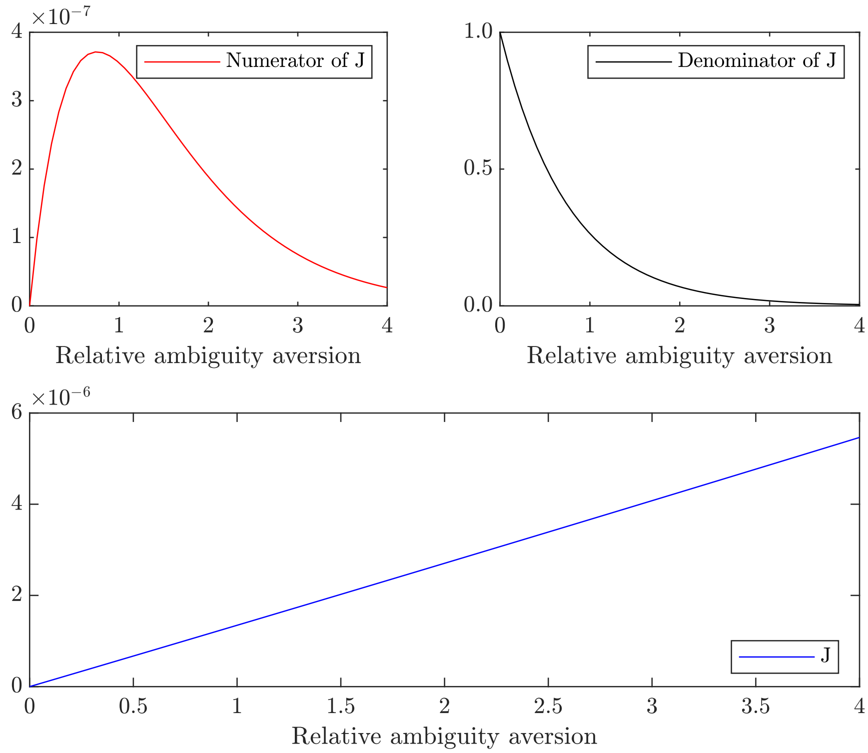

Figure 5: Pessimism due to increasing ambiguity aversion

representative household faces the following problem:

$$\max_{c_0\geq 0, e\geq 0, s\geq 0} \quad u(c_0) + \beta\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))) \tag{58}$$

$$s.t. \quad c_0 + e + s = y_0, \tag{59}$$

$$\tilde{c}_{1\theta} = \tilde{y}_{1\theta}, \quad \theta \in \Theta, \tag{60}$$

$$y_t = F(k_t, h_t), \quad t \in \{0, 1\}, \tag{61}$$

$$k_1 = (1 - \delta_k)k_0 + s, \quad \delta_k \in [0, 1], \tag{62}$$

$$\tilde{h}_{1\theta} = h_0(e^\alpha + 1 - \tilde{\delta}_\theta), \quad \alpha \in (0, 1), \ \theta \in \Theta, \tag{63}$$

$$k_0 > 0, \ h_0 > 0 \ \ given, \tag{64}$$

where δ_k is the depreciation rate of physical capital, which is assumed to be deterministic. We maintain that the production function is Cobb-Douglass:

$$F(k_t, h_t) = k_t^{1-\mu} h_t^{\mu}, \quad \mu \in (0, 1). \tag{65}$$

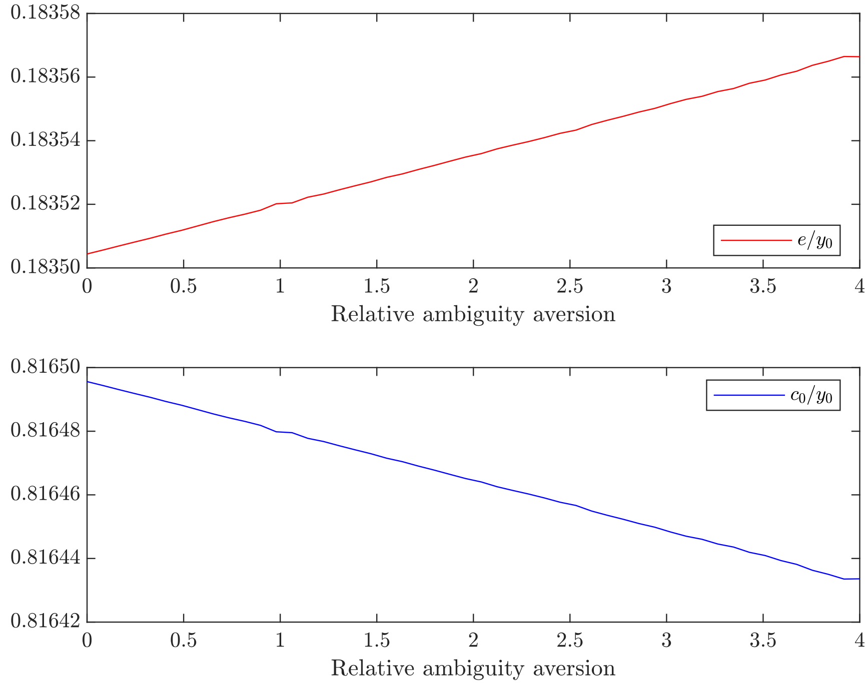

Figure 6: Increasing ambiguity and optimal saving and consumption

Note that for simplicity we have set $A = B \equiv 1$. Our new objective function is:

$$V(e,s) = u(y_0 - s - e) + \beta\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))), \tag{66}$$

where

$$\tilde{c}_{1\theta} = k_1^{1-\mu}\tilde{h}_{1\theta}^{\mu} = k_1^{1-\mu}h_0^{\mu}(e^{\alpha} + 1 - \tilde{\delta}_\theta)^{\mu}. \tag{67}$$

Denote $\frac{\partial V(e,s)}{\partial e} \equiv V_e(e,s)$ and $\frac{\partial V(e,s)}{\partial s} \equiv V_s(e,s)$. We have:

$$V_e(e,s) = -u'(c_0) + \beta\mu\alpha k_1^{1-\mu}h_0 e^{\alpha-1}\frac{\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))\mathbb{E}_\theta u'(\tilde{c}_{1\theta})\tilde{h}_{1\theta}^{\mu-1}}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}, \tag{68}$$

$$V_s(e,s) = -u'(c_0) + \beta(1-\mu)k_1^{-\mu}\frac{\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))\mathbb{E}_\theta u'(\tilde{c}_{1\theta})\tilde{h}_{1\theta}^{\mu}}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}. \tag{69}$$

Note that in this case the strict concavity of the objective function with respect to each argument (which is implied by Assumption 6) does not guarantee that it is jointly concave in both. For simplicity, we assume that the utility function is logarithmic. With log utility, the FOCs are indeed sufficient.

Lemma 6. *If the utility function is logarithmic and the second-order utility function satisfies linear* *absolute ambiguity tolerance (Assumption 6), then the objective function (66) is jointly concave in both arguments.*

Proof. See subsection 7.1. ■

Denote

$$U(s,e) \equiv \mathbb{E}\mathbb{E}_\theta u(\tilde{c}_{1\theta}). \tag{70}$$

Let $\pi(s,e)$ and $\psi(s,e)$ be the utility ambiguity premium and the utility ambiguity precautionary premium, respectively defined by

$$\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) = \phi(U(s,e) - \pi(s,e)), \tag{71}$$

$$\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) = \phi'(U(s,e) - \psi(s,e)). \tag{72}$$

Observe that under log utility, we can rewrite (68) and (69) as:

$$V_e(s,e) = -u'(c_0) + \beta\alpha\mu e^{\alpha-1}\left(L + M\mathbb{E}\mathbb{E}_\theta(\tilde{X}_\theta^{-1})\right), \tag{73}$$

$$V_s(s,e) = -u'(c_0) + \beta(1-\mu)k_1^{-1}M, \tag{74}$$

where $\tilde{X}_\theta \equiv e^\alpha + 1 - \tilde{\delta}_\theta$ as before and

$$L = \frac{Cov(\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})), \mathbb{E}_\theta(\tilde{X}_\theta^{-1}))}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}, \tag{75}$$

$$M = \frac{\phi'(U(s,e) - \psi(s,e))}{\phi'(U(s,e) - \pi(s,e))}. \tag{76}$$

4.1 Deterministic depreciation

In this case the depreciation rate of human capital is known to be δ. Observe that in this case $L = 0$ and $M = 1$. Denote by $V^1(s,e)$ the objective function under perfect foresight. We have:

$$V_e^1(s,e) = -u'(c_0) + \beta\alpha\mu e^{\alpha-1}X^{-1}, \tag{77}$$

$$V_s^1(s,e) = -u'(c_0) + \beta(1-\mu)k_1^{-1}. \tag{78}$$

Let (e_1, s_1) be optimal for the deterministic problem. Denote

$$Q(s,e) = \frac{(1-\delta_k)k_0 + s}{y_0 - s - e}. \tag{79}$$

Observe that Q so defined is strictly increasing in both of its arguments. Mathematically

$$\frac{\partial Q(s,e)}{\partial j} \equiv Q_j(s,e) > 0, \quad j \in \{s,e\}. \tag{80}$$

Then from (78):

$$Q(s_1, e_1) = \beta(1-\mu). \tag{81}$$

4.2 Risky depreciation

Proposition 4. *Under logarithmic utility, the introduction of a zero-mean risk to the depreciation rate of human capital raises the optimal level of investment in human capital and reduces the optimal investment in physical capital.*

Proof. Notice that under risk, it still holds that $L = 0$ and $M = 1$. Denote by $V^2(s,e)$ the objective function in this case. The FOC with respect to s rests unchanged, while the FOC with respect to e becomes:

$$V_e^2(s,e) = u'(c_0) + \beta\mu\alpha e^{\alpha-1}\mathbb{E}\tilde{X}^{-1}. \tag{82}$$

From the FOC with respect to s:

$$Q(s_2, e_2) = \beta(1-\mu) = Q(s_1, e_1), \tag{83}$$

implying, in view of (80) that

$$(e_2 - e_1)(s_2 - s_1) \leq 0. \tag{84}$$

On the other hand, by the convexity of the map $h \mapsto h^{-1}$, we have by Jensen inequality that:

$$\mathbb{E}\tilde{X}^{-1} > (\mathbb{E}\tilde{X})^{-1} = X^{-1} \equiv (1-\delta+e^{\alpha})^{-1}, \quad \forall e. \tag{85}$$

By optimality $V_s^2(s_2, e_2) = V_e^2(s_2, e_2) = 0$, which implies:

$$\begin{aligned} \frac{1}{y_0 - s_2 - e_2} = \beta(1-\mu)k_1^{-1} &= \beta\mu\alpha e_2^{\alpha-1}\mathbb{E}\tilde{X}^{-1} \\ &> \beta\mu\alpha e_2^{\alpha-1}(1-\delta+e_2^{\alpha})^{-1}, \end{aligned} \tag{86}$$

where the second line results from (85). Let ξ be the map defined by

$$\xi(s,e) = \frac{e + (1-\delta)e^{1-\alpha}}{(1-\delta_k)k_0 + s}. \tag{87}$$

Then (86) is equivalent to:

$$\xi(s_2, e_2) > \frac{\alpha\mu}{1-\mu} = \xi(s_1, e_1), \tag{88}$$

where the equality follows from the optimality of (s_1, e_1) in the deterministic case. Observe that from (84), either of the followings must hold:

$$e_2 \geq e_1, s_2 \leq s_1, \tag{89}$$

$$e_2 \leq e_1, s_2 \geq s_1. \tag{90}$$

Since ξ defined in (87) is strictly increasing in e and strictly decreasing in s, the case (90) would imply $\xi(s_2, e_2) \leq \xi(s_1, e_1)$, a contradiction to (88). Combining (89) and (87) yields:

$$e_2 > e_1,\ s_2 < s_1, \tag{91}$$

completing the proof. ∎

Remark 5. *The result of Proposition 4 is illustrated in Figure 7. Intuitively, increasing risk reduces the certainty equivalent of the next period's uncertain income (output) due to both risk aversion and the concavity of the production function. This raises savings due to a precautionary motive (to smooth consumption across states in the next period) and consequently reduces this period's consumption. Although aggregate saving is higher, the allocation to each type of capital moves in opposite direction. The increase in the HC investment has a self-insurance motive. This effect is stronger the more risk averse and/or concave the production function is with respect to HC (the closer μ is to zero). The reduction in the PC investment would reduce the variance of the uncertain income, which is preferred by a risk-averse agent. In other words, among the pairs (s, e) that yield the same expected next period's output, a risk-averse agent would always prefer to allocate as much as possible to the investment in human capital.*

4.3 Ambiguous depreciation

4.3.1 Ambiguity-neutral agent

Let us first consider the case of an ambiguity-neutral representative agent. Denote by V^3 the objective function in this case. Observe that since ϕ is linear under ambiguity neutrality, we have in this case that $L = 0$ and $M = 1$. Let (s_3, e_3) be optimal for V^3, then

$$V^3(s_3, e_3) = 0. \tag{92}$$

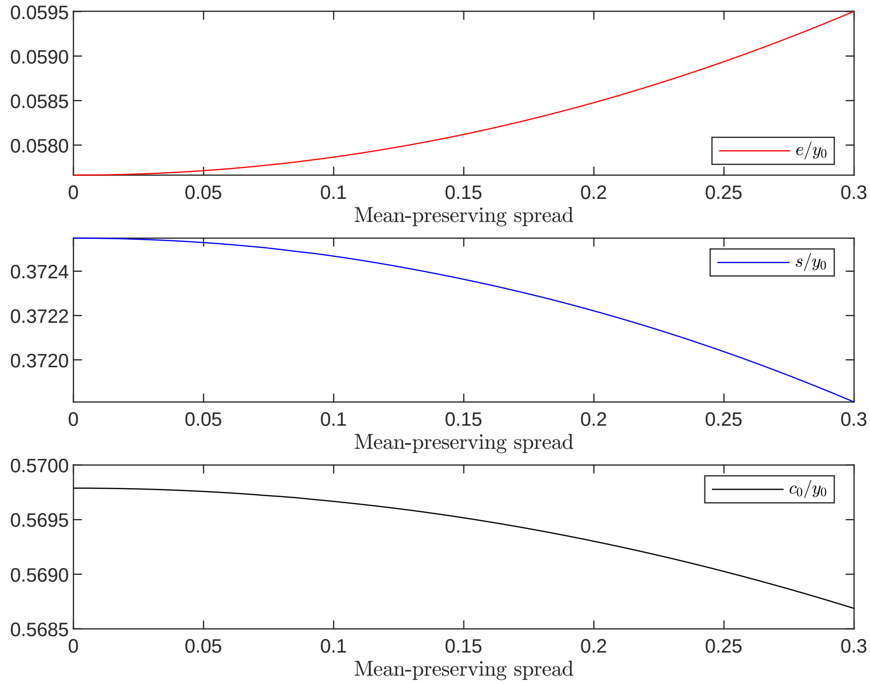

Figure 7: The impact of increasing risk (MPS) on investment and consumption. The deterministic case corresponds to MPS being equal to zero.

The objective is to examine the impact of ambiguity aversion on the optimal levels of investment in each type of capital.

4.3.2 Ambiguity-averse agent

Let V^4 be the objective function under ambiguity aversion and (s_4, e_4) be the optimal solution. As in the previous section, we need to examine two effects: one from pessimism (the sign of L in (75)) and the other from the preference for the timing of uncertainty resolution (the magnitude of M in (76)). Recall that

$$L = \frac{Cov(\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})), \mathbb{E}_\theta(\tilde{X}_\theta^{-1}))}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}. \tag{93}$$

Since the map $h \mapsto h^{-1}$ is strictly decreasing and convex, a higher value of θ (deterioration in SSD) would raise $\mathbb{E}_\theta \tilde{h}_{1\theta}^{-1}$. On the other hand since u is strictly increasing an concave, an SSD deterioration would lower $\mathbb{E}_\theta u(\tilde{c}_{1\theta})$, thus increasing $\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))$ since ϕ is strictly concave. Hence $L > 0$. As proven in the previous section, the magnitude of M depends on the property of the ambiguity preference. In particular, it is greater than one under DAAA and equal to one under CAAA.

Proposition 5. *When the scenario-conditional distributions can be ranked according to SSD (Assumption 8), the ambiguity-averse representative agent with logarithmic utility raises investment in human capital relative to the ambiguity-neutral agent if his ambiguity preference satisfies CAAA.*

Proof. Recall that under CAAA, the impact from the preference for timing of resolution of uncertainty is null, i.e., $M = 1$. Observe that the FOC with respect to s is unchanged compared to the previous cases since physical capital is always risk-free, hence $Q(s_4, e_4) = Q(s_3, e_3)$, which implies:

$$(e_4 - e_3)(s_4 - s_3) \leq 0. \tag{94}$$

Hence either of the followings must hold:

$$e_4 \geq e_3, s_4 \leq s_3, \tag{95}$$

$$e_4 \leq e_3, s_4 \geq s_3. \tag{96}$$

On the other hand, the FOCs imply:

$$\frac{\alpha\mu}{1-\mu} = \frac{e_4^{1-\alpha}}{(L + \mathbb{E}\mathbb{E}_\theta \tilde{X}_\theta^{-1})[(1-\delta_k)k_0 + s_4]} < \frac{e_4^{1-\alpha}}{\mathbb{E}\mathbb{E}_\theta \tilde{X}_\theta^{-1}[(1-\delta_k)k_0 + s_4]}, \tag{97}$$

where the third inequality comes from L being strictly positive. Let ξ^* be the map defined by:

$$\xi^*(s, e) = \frac{e^{1-\alpha}}{\mathbb{E}\mathbb{E}_\theta \tilde{X}_\theta^{-1}[(1-\delta_k)k_0 + s]}. \tag{98}$$

Observe that ξ^* is strictly increasing in e and strictly decreasing in s. Furthermore, from (97),

$$\xi^*(s_4, e_4) > \frac{\alpha\mu}{1-\mu} = \xi^*(s_3, e_3). \tag{99}$$

Clearly (96) cannot occur since this would yield a contradiction to (99). Combining (95) and (99) yields

$$e_4 > e_3,\ s_4 < s_3, \tag{100}$$

as desired. ∎

The result of Proposition 5 is illustrated in Figure 8. The direction of change is less clear analytically for the DAAA case since the effect of M on the investment in PC is no longer silent. Indeed, it is not difficult to see that in this case (94) does not necessarily hold although we still have (99). Thus the only conclusion we can draw, due to the monotonic behavior of the map $\xi^*(\cdot, \cdot)$ is that either the HC investment rises, or the PC investment falls (relative

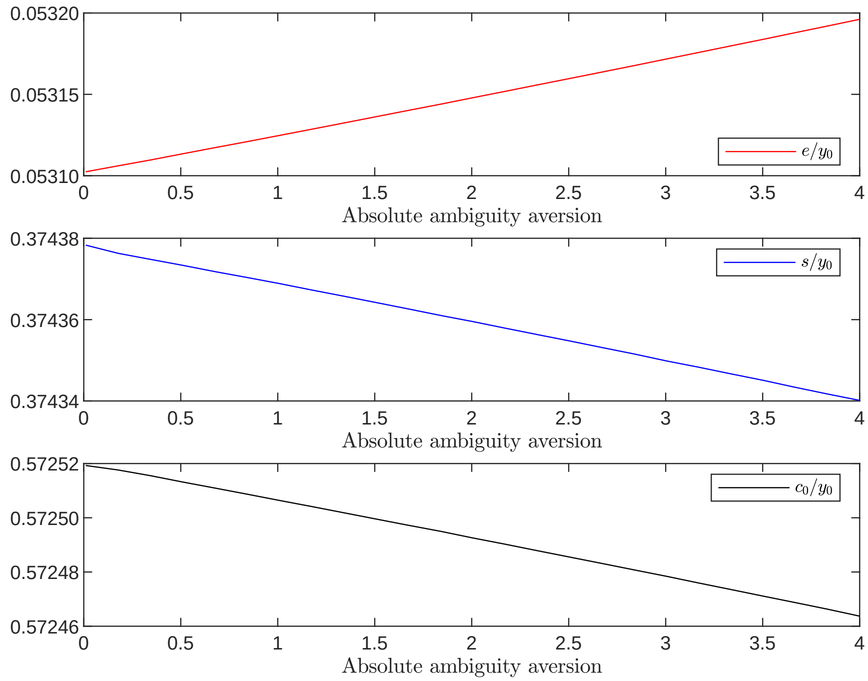

Figure 8: The impact of increasing ambiguity aversion on investment and consumption for a CAAA agent with logarithmic utility.

to the ambiguity-neutral case), but not necessarily both. Numerically, Figure 9 suggests that increasing ambiguity aversion for a CRAA agent (a particular case of DAAA) has a similar effect on consumption and investment. Observe that relative ambiguity aversion $\sigma = 0$ corresponds to an ambiguity-neutral agent.

5 Impact of uncertain effectiveness of human capital accumulation

Up to this point, it seems that the introduction of uncertainty into the simplified Ben-Porath model always leads to a rise in investment in human capital under DAAA and CAAA (the ambiguity preference most endorsed by empirical evidence). In this section, we wonder what happens if uncertainty is introduced to the model via the parameter B, the effectiveness of human capital accumulation. In particular, this means that the evolution of human capital (5) becomes:

$$\tilde{h}_{1\theta} = h_0(\tilde{B}_\theta e^\alpha + 1 - \delta), \tag{101}$$

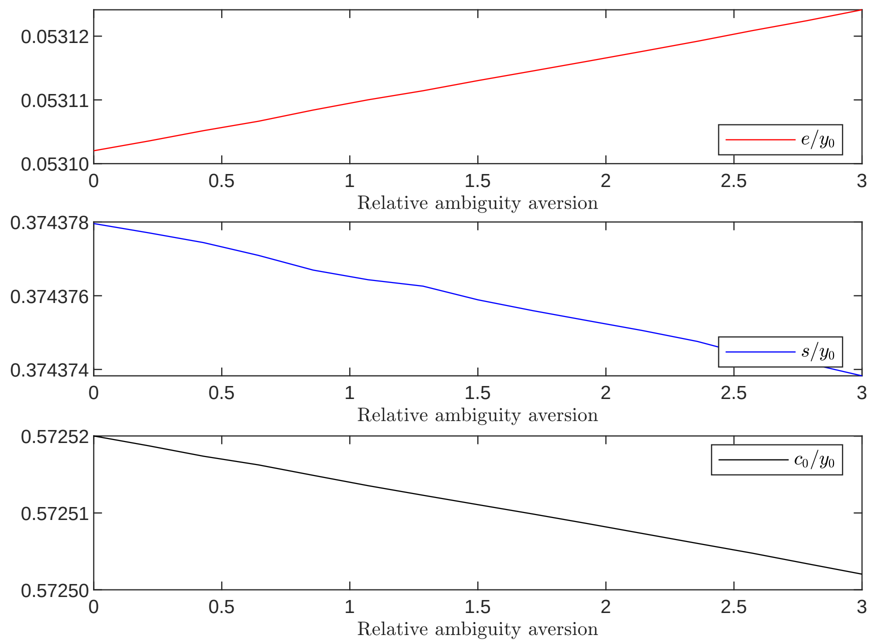

Figure 9: The impact of increasing ambiguity aversion on investment and consumption for a CRAA agent with logarithmic utility.

where $\delta \in [0, 1]$ is the deterministic depreciation rate of human capital. Would the main results of this paper up to now still hold? It turns out that for CRRA risk preference with constant relative risk aversion less than one, the direction of change will be reverse.

Proposition 6. *Under CRRA risk preference of degree $\gamma \leq 1$, the introduction of pure risk* lowers *investment in human capital.*[7] *The introduction of ambiguity as a series of MPSs around B also reduces investment in human capital for an ambiguity-neutral agent. An ambiguity-averse agent reduces investment under CAAA and IAAA; the direction of change is ambiguous under DAAA.*

Proof. See subsection 7.4. ■

The intuition behind this difference is that when uncertainty is in the depreciation parameter, raising the investment in human capital is essentially investing in self insurance. As is well-known from existing insurance literature, increasing risk raises the demand for self-insurance. By contrast, when uncertainty enters through B, the optimal choice of e is viewed as optimal investment in an uncertain asset. Typically, there are two effects acting in opposite directions in this case. According to Eeckhoudt et al. (2011), while the pure

[7] Taking into account the preference towards ambiguity, the degree of relative risk aversion being less than one is supported by a number of experiments, among which Chakravarty and Roy (2009) and Berger and Bosetti (2016).

increase in risk makes a risk-averse agent less interested in the investment (second-order effect), the sufficiently prudent agent is still induced to raise the investment via the precautionary channel (third-order effect). For CRRA risk preference with linear accumulation of human capital, for instance, it is easy to verify that "sufficient prudence" means that the degree of relative prudence, which is defined by the map $x \mapsto -xu'''(x)/u''(x)$, is greater than two. It is straightforward to see that this translates to the degree of relative risk preference being greater than one for CRRA preferences. This problem treats a non-linear human capital law of motion, and thus it is much more complicated to obtain a clear-cut rule for "sufficient prudence".

6 Conclusion

The co-existence of over-education in some sectors and skills shortage in others can be explained by the sources of uncertainty faced by different types of individuals.

On the one hand, if uncertainty is on the net productivity of human capital accumulation, then the investment is viewed as one with increasingly uncertain return, making it less attractive to a decision maker who is uncertainty-averse (risk averse or ambiguity averse). This might be the culprit behind the lack of skills in technical sectors in developing countries, where the quality of training is highly questionable due to the lack of infrastructure. One case in point is Vietnam. Highly skilled workers and technicians are in great demand, but the quality of vocational schools across the countries is hardly consistent. Thus even if the expected return on investment in vocational training for households remain high, the highly uncertain outcome makes it much less attractive. To address this issue, public policies need to work on improving the quality of training as well as to communicating this improvement to the groups of interest. This would raise expected return *and* reduce uncertainty on the quality of vocational education, rendering investment in it more attractive.

On the other hand, if uncertainty is on the obsolescence parameter, then the investment in human capital also serves as a type of insurance against labor income fluctuations, which is assumed to be nonexistent due to market incompleteness. In this case, individuals facing idiosyncratic uncertainty are induced to invest more, leading to ove-reducation. [8]

There is yet another implication of Proposition 3 and/or Proposition 1 on over-education. Individuals who do not have the means (being constrained by the first period's wage) to

[8] Recall that investment in human capital always increases under DAAA and CAAA ambiguity preferences. The experimental evidence of Berger and Bosetti (2016) is in favor of these types of preferences under CRRA utilities.

raise investment to the optimal level might opt for career choices that are less subject to obsolescence risks. Typically, they might accept jobs that pay less, where they are overqualified or over-educated in exchange for more security. This also causes a problem since numerous research has shown that overqualified workers are more likely to be dissatisfied at the workplace, leading to lower productivity. The policy response to this issue must also be multidimensional. Clearly, there remains the uncertainty-reducing role of the government by providing more precise data on the labor market conditions. Companies that work in sectors highly susceptible to uncertain obsolescence must also take a proactive approach in investing in their human resource. This should encourage uncertainty-averse individuals to be more willing to accept offers in these sectors, rather than migrating to where they are overqualified.

In a simple two-period framework, this work represents the first attempt to address ambiguous stochastic human capital accumulation, an issue that is increasingly relevant in the modern economy. In fact, the model is general enough to allow for an analysis of optimal investment in physical or financial capital. An abrupt technological change could render all existing machines obsolete. The burst of a financial bubble could wipe out the value of financial assets in a blink of an eye. Indeed, the evolution of any types of capital are ridden with uncertainty nowadays.

The model has at least two short-comings. First, the potential welfare-enhancing role of social security is neglected and the welfare analysis thereof. Second, the model is static, leaving the question on long-run growth open. At least in these dimensions can future research extend.

7 Appendix of Proofs

7.1 Proof of Lemma 2

Let us define the function $G : \mathbb{R}^n \to \mathbb{R}$ by

$$G(U) \equiv G(U_1, \ldots, U_n) = \phi^{-1}\left(\sum_{\theta} q_\theta \phi(U_\theta)\right), \tag{102}$$

where $U : \mathbb{R}_+ \to \mathbb{R}^n$ is the vector-valued function defined by:

$$U(e) = (U_1(e), \ldots, U_n(e)), \tag{103}$$

where $U_\theta(e) \equiv \mathbb{E}_\theta u(\tilde{c}_{1\theta}(e))$, for each $\theta \in \Theta$. Notice that G is increasing in U since ϕ is increasing (thus so is its inverse ϕ^{-1}). Also, by Lemma 8 of Gollier (2001), the function G is concave in $\mathbb{R}^n$ under Assumption 6. Our goal is to show that the composite function $G \circ U : \mathbb{R}_+ \to \mathbb{R}$ is concave in $\mathbb{R}_+$. It is easy to see that $U : \mathbb{R} \to \mathbb{R}^n$ is concave in $\mathbb{R}_+$ since U_θ is concave in e for all θ. In particular, following Dattorro (2018), this means that for positive scalars e^1 and e^2, and any $\lambda \in (0,1)$

$$U(\lambda e^1 + (1-\lambda)e^2) \geq \lambda U(e^1) + (1-\lambda)U(e^2), \tag{104}$$

where the notation $\geq$ denotes an element-wise inequality. By the monotonicity of G,

$$G\left(U(\lambda e^1 + (1-\lambda)e^2)\right) \geq G\left(\lambda U(e^1) + (1-\lambda)U(e^2)\right). \tag{105}$$

Furthermore, by the concavity of G in $\mathbb{R}^n$,

$$G\left(\lambda U(e^1) + (1-\lambda)U(e^2)\right) \geq \lambda G(U(e^1)) + (1-\lambda)(U(e^2)). \tag{106}$$

From (105) and (106), we conclude that $G \circ U$ is concave in $\mathbb{R}_+$. Since u is strictly concave in e and $\beta > 0$, the objective function is the sum of two concave functions, so it is indeed strictly concave in e, as desired.

7.2 Proof of Proposition 1

Since f is convex by Lemma 1, by Jensen inequality

$$\mathbb{E}u'(\tilde{c}_1)\tilde{X}^{\mu-1} \equiv \mathbb{E}f(\tilde{X}) > f(\mathbb{E}\tilde{X}) = u'(y_0X^\mu)X^{\mu-1}, \tag{107}$$

where the last equality has used the assumption of zero-mean risk via (26). Thus if e_2 is optimal for V_2, we have shown that

$$V_2'(e_2) = 0 \implies V_2'(e_1) > 0, \tag{108}$$

implying $e_2 > e_1$ by the strict concavity of V_2.

7.3 Proof of Proposition 2

As shown in the proof of Proposition 1, the map f is strictly convex under risk prudence. By Jensen inequality,

$$\mathbb{E}f(\tilde{X}_\theta) \geq f(\mathbb{E}\tilde{X}_\theta) = f(X), \quad \forall\theta \in \Theta, \tag{109}$$

where the equality comes from Assumption 8. Hence if e_3 is optimal for V_3, we have shown that:

$$V_3'(e_3) = 0 \implies V_3'(e_1) > 0, \tag{110}$$

implying $e_3 > e_1$ by the concavity of V_3.

7.4 Proof of Proposition 6

Under stochastic effectiveness of investment in human capital, the law of motion governing the accumulation of human capital (constraint (5)) becomes:

$$\tilde{h}_{1\theta} = h_0\tilde{Z}_\theta, \text{ where } \tilde{Z}_\theta \equiv \tilde{B}_\theta e^\alpha + 1 - \delta, \tag{111}$$

and $\delta \in [0, 1]$ is the deterministic depreciation rate of human capital. Equation (20) becomes:

$$V'(e) = -u'(y_0 - e) + \upsilon y_0 e^{\alpha-1} \frac{\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))\mathbb{E}_\theta u'(\tilde{c}_{1\theta})\tilde{B}_\theta\tilde{Z}_\theta^{\mu-1}}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}, \tag{112}$$

where $\upsilon \equiv \beta\mu\alpha$. Thus under CRRA risk preference

$$V'(e) = -u'(c_0) + \upsilon y_0^{1-\gamma} e^{\alpha-1} \frac{\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))\mathbb{E}_\theta\tilde{B}_\theta f(\tilde{Z}_\theta)}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}, \tag{113}$$

where f is the mapping defined in (15). Let V_1, V_2, V_3, and V denote the objective function in the deterministic, unambiguous stochastic (pure risk), ambiguous stochastic with ambiguity neutral agent, and ambiguous stochastic with ambiguity averse agent case, respectively. Observe that Lemma 2 still applies.

As before, we study three different settings: pure risk, ambiguity in the sense of a sequence of MPSs around B with ambiguity-neutral agents, then with ambiguity-averse agents.

7.4.1 Optimal investment under pure risk

When B is deterministic, we have:

$$V_1'(e) = -u'(c_0) + \upsilon y_0^{1-\gamma} e^{\alpha-1} B f(Z), \tag{114}$$

where Z denotes the deterministic value of $\tilde{Z}_\theta$. Let $\tilde{Z} = \tilde{B}e^\alpha + 1 - \delta$. Then (113) simplifies to:

$$V_2'(e) = -u'(c_0) + \upsilon y_0^{1-\gamma} e^{\alpha-1} \mathbb{E} g(\tilde{B}), \tag{115}$$

where g is the mapping defined by:

$$g(B) = Bf(Z) \equiv B(Be^\alpha + 1 - \delta)^\rho, \tag{116}$$

where $\rho = \mu(1-\gamma) - 1 < 0$ for all $\gamma \geq 0$ and $\mu \in (0,1)$. We now show that g is increasing and concave for $\gamma \leq 1$. Indeed, in this case $\rho \in [-1, 0)$, so:

$$g'(B) = Z^{\rho-1}[Be^\alpha(1+\rho) + 1 - \delta] > 0, \tag{117}$$

and

$$g''(B) = e^\alpha Z^{\rho-2} \rho \left[Be^\alpha(1+\rho) + 2(1-\delta)\right] < 0. \tag{118}$$

Hence by Jensen inequality,

$$\mathbb{E}\tilde{B}f(\tilde{Z}) \equiv \mathbb{E}g(\tilde{B}) < g(\mathbb{E}\tilde{B}) = Bf(Z), \tag{119}$$

implying

$$V_2'(e_2) = 0 \implies V_2'(e_1) < 0, \tag{120}$$

which in turn implies $e_2 < e_1$ by the strict concavity of V_2.

Notice that for $\gamma > 1$, the sign of g'' is in general ambiguous, but the higher is B, the more likely is g'' to be positive, inducing *more* investment in human capital. This is intuitive since higher B raises the mean of the risky investment.

7.4.2 Optimal investment under ambiguity neutrality

The first order derivative now reads:

$$\begin{aligned} V_3'(e) &= u'(c_0) + \upsilon y_0^{1-\gamma} e^{\alpha-1} \mathbb{E}\mathbb{E}_\theta \tilde{B}_\theta f(\tilde{Z}_\theta). \\ &= u'(c_0) + \upsilon y_0^{1-\gamma} e^{\alpha-1} \mathbb{E}\mathbb{E}_\theta g(\tilde{B}_\theta) \end{aligned} \tag{121}$$

The proof is completed by recognizing that g is strictly increasing and concave for $\gamma \leq 1$.

7.4.3 Optimal investment under ambiguity aversion

In this case we also have two effects from pessimism and preference for early resolution of uncertainty. In particular,

$$V'(e) = -u'(c_0) + v y_0^{1-\gamma} e^{\alpha-1} \left(J + K \times \mathbb{E}\mathbb{E}_\theta g(\tilde{Z}_\theta) \right), \tag{122}$$

where

$$J = \frac{Cov(\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})), \mathbb{E}_\theta g(\tilde{B}_\theta))}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}, \tag{123}$$

and

$$K = \frac{\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}. \tag{124}$$

The rest of the proof is almost identical, except that now the concave function g takes place of the convex function f in (37) and (38).

We remark also as in the case of stochastic depreciation, the result of Proposition 6 is robust to any ranking criterion stronger than SSD. This is a direct consequence of the fact that in this case the function g is increasing.

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