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Jean-Pierre Drugeon and Thai Ha-Huy

CEPS
Centre for Economics at Paris-Saclay

école normale supérieure paris—saclay
université evry val-d'essonne
université PARIS-SACLAY
université PARIS-SACLAY
Maison des Sciences de l'Homme PARIS-SACLAY

On Multiple Time-Varying Discount Rates with Recursive Time-Dependent Orders*

Jean-Pierre Drugeon† & Thai Ha-Huy‡

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†Paris School of Economics and Centre National de la Recherche Scientifique. Email: jean-pierre.drugeon@cnrs.fr

‡Université Paris-Saclay, Univ Evry, EPEE, 91025, Evry-Courcouronnes, France.

Abstract

This study addresses time-dependent orders that lead to recursive representations based on a Max-Min configuration. The article introduces and analyzes a structure that is combined with a time-varying multiple discounts. This setup contributes to the understanding of the much discussed present biases. A representation result for robust orders is also presented.

Keywords: Axiomatization, Time-Dependent Orders, Time-Varying Multiple Discounts, Multiple Present Biases.

JEL Classification: D11, D15, D90.

Contents

1. Introduction

1.1 Motivation and Results

As a result of the evidence documented by numerous laboratory experiments, a large body of recent literature on the evaluation of utility streams has focused on the time inconsistencies and present biases. Naturally, this leads to a consideration of time-dependencies in the preferences order of an economic agent.

In this spirit, we here assume that the economic agent is characterized by a set of *temporal evaluation* orders $(\succeq_T)_{T=0}^{\infty}$, with $\succeq_T$ denoting the decision maker's preference after T days, assuming he or she behaves in a time-consistent manner, in the sense that the evaluation at time T never results in a disagreement with its present and future if these two are in agreement with one another.

The analysis begins by considering a list of standard axioms in the temporal discounting literature. The preferences orders can be represented by index functions that are constantly additive and homogeneous of degree one. From this point onward, the article follows two directions. The first direction tries to find an alternative to classical representations and to clarify the conditions under which such preferences are temporally consistent. The second direction is to examine whether *temporal bias* preferences can be combined with *multiple discount rates* configurations that have gained increasing popularity in the recent literature[1].

By assuming that the temporal orders do not depend on the head of the sequences and, crucially, that they satisfy a *consistency* property that indicates some agreement among them—in other words, a strong monotonicity property—, the temporal evaluation can be represented as a recursive convex sum between the utility level at that date and the evaluation of the utility stream at the subsequence date. Interestingly, the weighting parameters of this convex sum are not constant and depend on the very nature of that stream. In other words, there is a possibility of multiple choices for the discount rate that is used to evaluate the utility stream. In this context, two configurations emerge; a configuration in which the economic agent is more affected by the losses[2] than by the

[1]See Chambers and Echenique (2018) or Drugeon et al (2020).

[2]Usually called *averse at loss* behaviour.

gains and, conversely, a configuration in which the joy of a gain is greater than the harm of a loss.

It is to be stressed that the possibility of multiple choices for discount rates does not lead to contradiction with *temporal consistency*. Indeed, a generalization of that property may be obtained by adding a *sationarity* condition, with the direct consequence that the set of possible discount rates does not change over time and does not affect the presence or the absence of loss aversion behavior.

Considering the scope for temporal biases, some behaviour shall be called as *present biased* when the *temporal distance* between two successive dates is decreasing over time. This means that the optimal discount factor is increasing. This is a consequence of an axiom that constrains the range of admissible time-dependent orders. In other words, the temporal distance that is perceived between two successive dates in an immediate future is larger than the distance that is perceived between two successive dates in a more remote future. The study concludes by providing a representation result, *i.e.*, a characterization of the set of possible discount rates being used to evaluate inter-temporal utility streams, be it with or without temporal biases.

1.2 Related Literature

To the best of our knowledge, the work that is closest to this work is the study by to Wakai (2007). That article follows a decision theory approach based on multiple priors Gilboa and Schmeidler (1989) applied to infinite utility streams. The author examines on temporal orders and streams of lotteries. The core analysis is based on a *time-variability aversion* condition that can be considered an extension of an *ambiguity aversion* property to an inter-temporal context. He provides an insightful account of smoothing behaviors in which the optimal discount is defined in an maxmin recursive representation.

Also related to the current study with an approach based on the set of bounded real sequences, Chambers and Echenique (2018) put forth an axiomatic approach to multiple discounts. Recently, this approach was revisited and extended by Drugeon and Ha-Huy (2023) in which, based on an alternative system of axioms, an axiomatization of α-Maxmin criteria with multiple discounts is built on the set of unanimous pre-orders.

As for the present bias dimension of this study, the most influential works on temporal inconsistencies under the so-called *quasi-hyperbolic discounting* dates back to Phelps and Pollack (1968) and, more recently, the contributions of Laibson (1997) and Frederick et al. (2002). Numerous experiments have supported the accuracy of this formulation.

Montiel Olea and Strzalecki (2014) have proposed an axiomatic approach to the quasi-hyperbolic discounting representation and, more generally, to present-biased preferences. They suppose that, for any two equivalent future sequences, a patient one and an impatient one, pushing both toward the present, will distort the preferences towards the impatient choice. It is to be emphasized that, by contrast, this article assumes the present bias notion for every given date and not only for the initial one. The index functions at that date are further determined by a set of multiple discount rates. The present bias notation of this article incorporates thus two separate parts; the first one relating to the upper bound of discount rates and the second one relating to the lower bound of discount rates.

Finally, Chakraborty (2017) presented a generalized notion of present biases within the Fishburn and Rubinstein (1982) approach, in which preferences are defined based on the realization of a single outcome at a given date. Even though this was based on an different approach from ours, his *weak present bias* axiom **A4** shares some similarities with the decreasing temporal distance axiom **B1** in our article. Recently, Bach et al. (2024) extended the axiomatic approach of Chambers and Echenique (2018) to a MaxMin representation that encompasses the quasi-hyperbolic discounting in the literature. Following an alternative system of axioms, Drugeon and Ha-Huy (2023) provide an α-Maxmin presentation of the scope for present biases.

1.3 Contents

This article is organized as follows. Section 2 details how the introduction of time-dependencies in the preference order will result in a recursive representation. Section 3 shows that adding structure may provide a new picture for multiple temporal biases. Section 4, considering the robust orders, equips the analysis with a representation result for time-dependent orders. Proofs are given in the Appendix.

2. Basic Axioms and a Recursive Min-Max Representation

2.1 Fundamentals, Basic Axioms, and the Construction of Index Functions

This article introduces an axiomatic approach to the evaluation of infinite utility streams in a discrete time configuration. In order to avoid confusion, the letters x, y, z will be used for the sequences (of utilities) $(x_s)_{s=0}^{\infty}$, $(y_s)_{s=0}^{\infty}$, and $(z_s)_{s=0}^{\infty}$ with values in $\mathbb{R}$. The notations $c\mathbf{1}$, $c'\mathbf{1}$ will be used for the constant sequences $(c, c, c, \ldots)$ and $(c', c', c', \ldots)$. The notation $\mathbf{1}$ is simply the constant unitary sequence $(1, 1, 1, \ldots)$.

The space ℓ_∞ is defined as the set of real sequences $(x_s)_{s=0}^{\infty}$ such that $\sup_{s\geq 0} |x_s| < +\infty$. For every $x \in \ell_\infty$ and $T \geq 0$, let $x_{[0,T]} = (x_0, x_1, \ldots, x_T)$ denote its first $T+1$ components; and $x_{[T+1,\infty)} = (x_{T+1}, x_{T+2}, \ldots)$ its *tail* starting from date $T+1$, and $(z_{[0,T]}, x_{[T+1,+\infty)}) = (z_0, z_1, \ldots, z_T, x_{T+1}, x_{T+2}, \ldots)$. By convention, if $T = -1$, let $(z_{[0,T]}, x_{[T+1,\infty)}) = x$.

As we will see in this article, a sequence $\left(0\mathbf{1}_{[0,T]}, \mathbf{1}\right)$ represents x such that $x_0 = x_1 = \ldots = x_T = 0$ and $x_{T+s} = 1$ for every $s \geq 1$. The sequence $\left(\mathbf{1}_{[0,T]}, 0\mathbf{1}\right)$ represents y such that $y_0 = y_1 = \ldots = y_T = 1$ and $y_{T+s} = 0$ for every $s \geq 1$. Similarly, the sequence $\left(0\mathbf{1}_{[0,T]}, -\mathbf{1}\right)$ represents z such that $z_0 = z_1 = \ldots = z_T = 0$ and $z_{T+s} = -1$ for every $s \geq 1$.

The preferences of the economic agent are characterized by a sequence of temporal orders $(\succeq_T)_{T=0}^{\infty}$, being defined on the set of real bounded sequences ℓ_∞. The order $\succeq_T$ evaluates utility sequences from time T. More precisely, given $x, y \in \ell_\infty$, the order T makes a comparison between them while disregarding their heads $x_0, x_1, \ldots x_{T-1}$ and $y_0, y_1, \ldots, y_{T-1}$. Such a comparison is independent of what happens before period T. The main properties and the relation between temporal orders are presented in the following fundamental axiom.

Axiom **F.** For every $T \geq 0$, the order $\succeq_T$ satisfies the following properties:

(i) *Completeness* and *transitivity:* For every $x, y \in \ell_\infty$, either $x \succeq_T y$ or $y \succeq_T x$. If $x \succeq_T y$ and $y \succeq_T z$, then $x \succeq_T z$. Denote as $x \sim_T y$ the case in which $x \succeq_T y$ and

$y \succeq_T x$. Denote as $x \succ_T y$ the case in which $x \succeq_T y$ and $y \not\succeq_T x$.

(ii) *Monotonicity:* If $x, y \in \ell_\infty$ and $x_s \geq y_s$ for every $s \geq T$, then $x \succeq_T y$.

(iii) *Archimedeanity:* For $x \in \ell_\infty$, and constants c, c' such that $c\mathbf{1} \succ_T x \succ_T c'\mathbf{1}$, there are $0 < \lambda, \mu < 1$ such that

$$(1-\lambda)c\mathbf{1} + \lambda c'\mathbf{1} \succ_T x \succ_T (1-\mu)c\mathbf{1} + \mu c'\mathbf{1}.$$

(iv) *Constant additivity:* For every $x, y \in \ell_\infty$, constant c and $0 < \lambda < 1$,

$$x \succeq_T y \Leftrightarrow (1-\lambda)x + \lambda c\mathbf{1} \succeq_T (1-\lambda)y + \lambda c\mathbf{1}.$$

(v) *Head-insensitivity:* For $T \geq 1$, $x, y, z, z' \in \ell_\infty$,

$$x \succeq_T y \text{ if and only if } (z_{[0,T-1]}, x_{[T,\infty)}) \succeq_T (z'_{[0,T-1]}, y_{[T,\infty)}).$$

(vi) *Coherency:* For every $x, y \in \ell_\infty$, if $x_T = y_T$ *and* $x \succeq_{T+1} y$, then $x \succeq_T y$.

Conditions (i) to (iv) are commonly used in the temporal axiomatization literature. Curious readers may find a careful analysis and comments on these in Chambers and Echenique (2018). Combined with the non-triviality condition (there exist x and y such that $x \succ_T y$), they ensure the existence of an index function I_T representing the order $\succeq_T$. Such an index function is furthermore positively homogeneous and constantly additive.

Observe that we do not exclude the possibility that, for some T, the temporal order $\succeq_T$ is trivial.[3] Such a generalization is aimed at encompassing situations in which the economic agent cares only about what happens before some fixed date but is indifferent afterward. See de Andrade et al. (2021) for an interesting discussion about this type of behavior. Throughout the article and when needed, we will make precise the non-triviality property.

Condition (v), head-insensitivity, characterizes a core property of temporal orders. The comparison by order $\succeq_T$ between x and y is independent of what happens before the date T. In other words, the values of $x_0, x_1, \ldots, x_{T-1}$ and $y_0, y_1, \ldots, y_{T-1}$ have no effect on the comparison between x and y.

Condition (vi), consistency, is the most important one and establishes some *agreement* between the temporal orders. It states that, an order $\succeq_T$ never leads to a disagreement

[3] For every $x, y \in \ell_\infty$, $x \sim_T y$.

with its evaluations of the present and the future, if these two are to agree with each other. This may also be considered a generalization of the monotonicity property. It is interesting to remark that this intuitive and almost obvious condition will play a key role in obtaining of a recursive representation with multiple discount rates.

We will first present in Lemma 2.1 the main properties of the index functions. If the order $\succeq_T$ is non-trivial, it can be represented by an index function I_T in the sense that, $x \succeq_T y$ if and only if $I_T(x) \geq I_T(y)$. More precisely, this index function is defined as

$$I_T(x) = \sup\Big\{c \in \mathbb{R} \text{ such that } x \succeq_T c\mathbf{1}\Big\}. \tag{1}$$

The proof of parts (i) and (ii) of Lemma 2.1 can be found in Drugeon and Ha-Huy (2022), Lemma 2.1. As to part (iii), by the head insensitivity property, it is obvious that the value of $I_T(x)$ does not depend on $x_0, x_1, \ldots, x_{T-1}$.

Lemma **2.1.** *Assume axiom* **F** *and that the order $\succeq_T$ is not trivial. Then this order can be represented by index function I_T in (1). This function satisfies the* positive homogeneity*,* constantly additive *and* head-insensitivity *properties:*

(i) $I_T(\lambda x) = \lambda I_T(x)$*, for every* $\lambda \geq 0$.

(ii) $I_T(x + c\mathbf{1}) = I_T(x) + c$*, for every constant* $c \in \mathbb{R}$.

(iii) *For every* $T \geq 0$, $x, z \in \ell_\infty$,

$$I_T(z_{[0,T-1]}, x_{[T,\infty)}) = I_T(x).$$

For the sake of simplicity, by convention, in the case in which the order $\succeq_T$ is trivial, the temporal index function I_T will be defined as: $I_T(x) = 0$ for every $x \in \ell_\infty$. To end this subsection, as a remark, if the order $\succeq_T$ is no trivial, from the monotonicity property, for every $x \in \ell_\infty$ and a constant c, we have $\inf_{s\geq 0} x_s \leq I_T(x) \leq \sup_{s\geq 0} x_s$ and $I_T(c\mathbf{1}) = c$.

2.2 A Recursive Representation

2.2.1 Asymmetry between Gains and Losses

Let χ_g^T be the evaluation of a constant gain in the future when considered from time $T+1$ onward, and let χ_ℓ^T denote the evaluation of a constant loss in the future when considered from time $T+1$ onward. More precisely,

$$\chi_g^T = I_T\Big(0\mathbf{1}_{[0,T]}, \mathbf{1}\Big),$$
$$\chi_\ell^T = -I_T(0\mathbf{1}_{[0,T]}, -\mathbf{1}).$$

It is obvious that both χ_g^T and χ_ℓ^T belong to the interval $[0,1]$. Two configurations naturally come under consideration, namely those that correspond to the cases $\chi_g^T \leq \chi_\ell^T$ and $\chi_g^T \geq \chi_\ell^T$. The first case represents *loss aversion* behavior, in which the loss affects the economic agent more than the gain. The second case represents the opposite behaviour.

Lemma 2.2 is a crucial step in developing a recursive formula with multiple discount rates. The evaluation at period T is a recursive convex combination of the utility level at that time and the evaluation at period $T+1$. It's important to remember that the convex parameter isn't a constant; it depends on the sequence at hand. If the future from period $T+1$ onwards is better than the utility level at time T, the convex parameter used is χ_g^T, which corresponds to a constant gain. Conversely, if the future from period $T+1$ onwards is worse, the parameter used is χ_ℓ^T, which corresponds to a constant loss.

LEMMA **2.2.** *Assume axiom* **F**. *Assume also that the order* $\succeq_T$ *is non-trivial. For every* $x \in \ell_\infty$,

(i) *If* $x_T \leq I_{T+1}(x)$, *then*

$$I_T(x) = (1-\chi_g^T)x_T + \chi_g^T I_{T+1}(x).$$

(ii) *If* $x_T \geq I_{T+1}(x)$, *then*

$$I_T(x) = (1-\chi_\ell^T)x_T + \chi_\ell^T I_{T+1}(x).$$

Corresponding to these two configurations, two recursive operators, namely min and max, emerge. Proposition 2.1 presents one of the main results of this article. It introduces the two recursive operators that, at each period T, chose the optimal discount rates as a function of the utility streams.

PROPOSITION **2.1.** *Assume axiom* **F**. *Assume also that the order* $\succeq_T$ *is no trivial.*

Let $\underline{\delta}_T = \min\{\chi_g^T, \chi_\ell^T\}$ *and* $\overline{\delta}_T = \max\{\chi_g^T, \chi_\ell^T\}$.

(i) *If* $\underline{\delta}_T = \chi_g^T$ *and* $\overline{\delta}_T = \chi_\ell^T$*, then:*

$$I_T\Big(x\Big) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1-\delta)x_T + \delta I_{T+1}(x)\right].$$

(ii) *If* $\underline{\delta}_T = \chi_l^T$ *and* $\overline{\delta}_T = \chi_g^T$*, then:*

$$I_T\Big(x\Big) = \max_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1-\delta)x_T + \delta I_{T+1}(x)\right].$$

At each date, the evaluation of a utility stream is based on a recursive convex sum between the utility level at that date, and the evaluation at the subsequent date of this stream. Hence, a multitude of choices is possible for the weighting parameters of this convex sum.

2.2.2 An asymmetry between Head Insensitivity and Tail Insensitivity

With respect to earlier formulations in the literature, it is of interest to emphasize the specificity of the scope for separability between time T and the past dates, which is central to this study. Indeed, both the classical approach of Koopmans (1960) and the more recent axiomatization of quasi-hyperbolic discounting due to Montiel Olea and Strzalecki (2014) assume that the first or the first and second components of two utility streams can be compared independently of their future components. Together with stationarity or quasi-stationarity postulates on the preferences ordering, these imply the existence of a unique discount rate for every day or every generation, such a discount rate being constant for any $T \geqslant 0$ with stationarity postulate, or constant from $T = 1$ onward with a quasi-stationarity postulate.

In contrast, the approach of this article postulates that the components of two utility streams starting from a given date can be compared independently of their earlier past components, which gives rise to the possibility of multiple discount rates.

The following example proves that, for multiple discount rates, neither the *independence* nor the *extended independence* properties of Koopmans (1960) is satisfied. Hence, the two approaches differ, be it in their formulation or in their predictions.

Example **2.1.** Consider a configuration in which for any $T \geq 0$, $\underline{\delta}_T = 0.5$, $\overline{\delta}_T = 0.8$ and the operator is min. For any T, the order $\succeq_T$ is represented by

$$I_T\Big(x\Big) = \min_{0.5 \leq \delta \leq 0.8} \left[(1-\delta)x_T + \delta I_{T+1}(x)\right].$$

Consider the following two utility streams $x = (1, 0, 0, 0, \ldots)$ and $y = (0.5, 0.5, 0, 0, \ldots)$. Obviously, $I_1\big(x\big) = 0$, and hence, $I_0(x) = (1 - 0.8) \times 1 + 0.8 \times 0 = 0.2$. Similarly, since $I_2(y) = 0$, we have $I_1\big(y\big) = (1 - 0.8) \times 0.5 + 0.8 \times 0 = 0.1$. This implies $I_0\big(y\big) = (1 - 0.8) \times 0.5 + 0.8 \times 0.1 = 0.18$. Hence, $x \succ_0 y$.

Consider now $x' = (1, 0, 0.5, 0.5, \ldots)$ and $y' = (0.5, 0.5, 0.5, 0.5, \ldots)$. The two sequences x and y are changed by keeping the first two components intact. Since y' is a constant sequence, $I_0(y') = 0.5$. For the same reason, x' is constant from $T = 2$, so $I_2(x') = 0.5$. Calculus give $I_1\big(x'\big) = (1-0.5)\times 0+0.5\times 0.5 = 0.25$ and $I_0(x') = (1-0.8)\times 1+0.8\times 0.25 = 0.4$. Hence, $y' \succ_0 x'$. The extended stationarity property of Koopmans (1960) is therefore not satisfied.

2.2.3 Time-Variability Aversion and MaxMin Representation

It is to be stressed that, relying on a system of axioms based on *time-variability aversion*, *i.e.*, a generalization to an inter-temporal context of the well-known *ambiguity aversion* of Gilboa and Schmeidler (1989), Wakai (2007) provided an insightful account of smoothing behaviors with *gain/loss asymmetry* that explicitly builds on a related recursive representation with multiple discount rates and a min operator for every time T. Combined with a stationarity property, the sets of discount rates over time were proven by the author to have the same lower and upper bounds.

This application of an ambiguity aversion property leads to a maxmin representation, corresponding to a loss aversion configuration, in which the losses affect the economic agent more than his or her gains. Based on a simpler set than the set of lotteries but using only the consistency property, this article presents a more general recursive representation with multiple discount rates, which encompasses the configuration in Wakai (2007) as a special case.

Axiom S 1. For any constant $c \in \mathbb{R}$, and utility streams $x, y \in \ell_\infty$,

$$(c, x) \succeq_0 (c, y) \text{ if and only if } x \succeq_0 y.$$

Corollary 2.1. *Assume axioms* **F** *and* **S1**. *Suppose that the order $\succeq_0$ is non-trivial. Then:*

(i) *For every T, $\underline{\delta}_T = \underline{\delta}_0$ and $\overline{\delta}_T = \overline{\delta}_0$. Let $\underline{\delta}$ and $\overline{\delta}$ be respectively the former and the* *later values of the discount rate. Either, for every T, the corresponding operator is* min*,*

$$I_T(x) = \min_{\underline{\delta} \le \delta \le \overline{\delta}} \Big[(1-\delta)x_T + \delta I_{T+1}(x)\Big],$$

or for every T, the corresponding operator is max*,*

$$I_T(x) = \max_{\underline{\delta} \le \delta \le \overline{\delta}} \Big[(1-\delta)x_T + \delta I_{T+1}(x)\Big].$$

(ii) *Assume the* convexity *condition: if $x \sim_0 y$, one has $(1/2)x + (1/2)y \succeq_0 x$. Then for every T,*

$$I_T(x) = \min_{\underline{\delta} \le \delta \le \overline{\delta}} \Big[(1-\delta)x_T + \delta I_{T+1}(x)\Big].$$

In Wakai (2007), it is proven that a stationarity condition is sufficient to ensure the time-independency property of these rates. We remark that this condition is imposed in (i) only on the present order $\succeq_0$ corresponding to $T = 0$. Under an additional convexity property and for (ii), we obtain a maxmin representation for every period or, in other words, the economic agent becomes averse to losses.

2.3 The equivalence with α-maxmin criteria

In decision theory, under the context of uncertainty, it is well known that without assuming convexity or concavity to the preferences, with some intuitif additional conditions, we may obtain easily an α-maxmin representation. This raises naturally the question whether in this article, we can come up to a similar structure. We will prove that such representation is equivalent to a *max* our *min* operators presented in this section with new under and upper bound of discount rates. Precisely, assume the existence of $0 \le \alpha_T \le 1$ such that for every stream x,

$$I_T(x) = \alpha_T \max_{\underline{\delta}_T \le \delta \le \overline{\delta}_T} \Big[(1-\delta)x_T + \delta I_{T+1}(x)\Big] + (1-\alpha_T) \min_{\underline{\delta}_T \le \delta \le \overline{\delta}_T} \Big[(1-\delta)x_T + \delta I_{T+1}(x)\Big].$$

Consider the case $x_T \le I_{T+1}(x)$. We obtain

$$\begin{aligned} I_T(x) &= \alpha_T\Big[(1-\overline{\delta}_T)x_T + \overline{\delta}_T I_{T+1}(x)\Big] + (1-\alpha_T)\Big[(1-\underline{\delta}_T)x_T + \underline{\delta}_T I_{T+1}(x)\Big] \\ &= \Big[1 - \Big(\alpha_T\overline{\delta}_T + (1-\alpha_T)\underline{\delta}_T\Big)\Big]x_T + \Big[\alpha_T\overline{\delta}_T + (1-\alpha_T)\underline{\delta}_T\Big]I_{T+1}(x). \end{aligned}$$

Similarly, in the case $x_T \ge I_{T+1}(x)$, we have

$$\begin{aligned} I_T(x) &= (1-\alpha_T)\Big[(1-\overline{\delta}_T)x_T + \overline{\delta} I_{T+1}(x)\Big] + \alpha_T\Big[(1-\underline{\delta}_T)x_T + \underline{\delta}_T I_{T+1}(x)\Big] \\ &= \Big[1 - \Big((1-\alpha_T)\overline{\delta}_T + \alpha_T\underline{\delta}_T\Big)\Big]x_T + \Big[(1-\alpha_T)\overline{\delta}_T + \alpha_T\underline{\delta}_T\Big]I_{T+1}(x). \end{aligned}$$

Let

$$\delta^* = \min\Big\{\alpha_T\overline{\delta}_T + (1-\alpha_T)\underline{\delta}_T, (1-\alpha_T)\overline{\delta}_T + \alpha_T\underline{\delta}_T\Big\},$$
$$\delta^{**} = \max\Big\{\alpha_T\overline{\delta}_T + (1-\alpha_T)\underline{\delta}_T, (1-\alpha_T)\overline{\delta}_T + \alpha_T\underline{\delta}_T\Big\}.$$

If $\alpha \leq \frac{1}{2}$, we have $\delta^* = \alpha_T\overline{\delta}_T + (1-\alpha_T)\underline{\delta}_T$ and $\delta^{**} = (1-\alpha_T)\overline{\delta}_T + \alpha_T\underline{\delta}_T$. It is easy to verify that in that case,

$$I_T(x) = \min_{\delta^* \leq \delta \leq \delta^{**}} \Big[(1-\delta)x_T + \delta I_{T+1}(x)\Big].$$

Otherwise, if $\alpha \geq \frac{1}{2}$, we have $\delta^* = (1-\alpha_T)\overline{\delta}_T + \alpha_T\underline{\delta}_T$, $\delta^{**} = \alpha_T\overline{\delta}_T + (1-\alpha_T)\underline{\delta}_T$ and obtain the following representation:

$$I_T(x) = \max_{\delta^* \leq \delta \leq \delta^{**}} \Big[(1-\delta)x_T + \delta I_{T+1}(x)\Big].$$

Hence, the α-maxmin configuration is equivalent to the convex representation with appropriate choice of underbound and upperbound of discount rates. It is interesting to observe that the *max* and *min* configurations can be considered different α-maxmin representations corresponding to parameter α is bigger or smaller than $\frac{1}{2}$.

Remains the question: how an α-maxmin representation can be simply considered a *max* or *min* criterion? One of possible explanation, in our opinion, is that the set of possibles choices (in the case of this article, the set of discount rates) is a convex hull of a finite number of possibilities.

It is also worth noting the difference between this article and recent contributions in multiple discount rates representation, to name someone, such as Chambers and Echenique (2018), Drugeon and Ha-Huy (2023), and Bach et al. (2024).

3. A Multiple Discount Formulation for Present-Biased Preferences

3.1 An Alternative Understanding of Multiple Present Biases

In the literature, present bias is commonly understood as a behavior in which, an event that happens today affects the decision maker more than the same event some day in the future. A gain (loss) today causes more happiness (unhappiness) than the same gain or loss in the future. This is one of the main sources of time inconsistencies: the decision maker may prefer some small amount of money (or consumption good) today to a larger amount tomorrow, but that same small amount tomorrow is less enjoyable than the same larger amount on the day after tomorrow.

This section examines such a phenomenon within the current multiple discount setup. The following axiom is a move in that direction. As a result of the asymmetric nature of gain and loss, an axiom consists of two separate parts. The first part says that the delay of a perpetual gain to the next day and at time T diminishes a decision maker's happiness more than it would do at time $T+1$ or for other dates in the future of T. The second part introduces another behavior: delaying a perpetual loss at date T makes the decision maker happier than delaying the same loss in the future of T.

Axiom **B 1.** For any $T \geq 1$ and $0 < c < 1$,

(i) If $(0\mathbf{1}_{[0,T]}, \mathbf{1}) \sim_T c\mathbf{1}$, then $(0\mathbf{1}_{[0,T+1]}, \mathbf{1}) \succeq_{T+1} c\mathbf{1}$.

(ii) If $(-c)\mathbf{1} \sim_T (0\mathbf{1}_{[0,T]}, -\mathbf{1})$, then $(-c)\mathbf{1} \succeq_{T+1} (0\mathbf{1}_{[0,T+1]}, -\mathbf{1})$.

The supremum values—the greatest of the minorants—of the parameter c in parts (i) and (ii) can be used to determine the *perception of the temporal distance* between date T and date $T+1$. These extremum values are determined by the evaluation at date T of the two sequences $(0\mathbf{1}_{[0,T]}, \mathbf{1})$ and $(0\mathbf{1}_{[0,T]}, -\mathbf{1})$. Axiom **B1** means that this temporal distance is decreasing as a function of T.[4]

Delaying gain and loss affects the decision maker more at time T than at time $T+1$. Indeed, at time T, delaying the gain for one day diminishes the welfare value from 1 to c. Delaying the same gain at time $T+1$ will diminish the welfare from 1 to some value $c' \geq c$. Similarly, delaying a loss at time T increases the welfare value from -1 to $-c$, which is a higher welfare increases than that obtained by delaying the same loss at time $T+1$, namely from -1 to some value $(-c')$ smaller than $-c$.

In other words, the temporal distance that is perceived between dates T and $T+1$ is larger than the one that is perceived between dates $T+1$ and $T+2$: at date T, the

[4]Axiom 10 in Montiel Olea and Strzalecki (2014) corresponds to the second part of axiom **B1**.

evaluation of a constant sequence from tomorrow onward is lower than its corresponding evaluation at date $T+1$. This intuition is detailed in the following statement:

Proposition **3.1.** *Assume axioms* **F** *and* **B1**. *Assume also that $\succeq_T$ is non-trivial for every T. Then the two sequences $\left\{\underline{\delta}_T\right\}_{T=0}^{\infty}$ and $\left\{\overline{\delta}_T\right\}_{t=0}^{\infty}$ are increasing: $\underline{\delta}_T \leq \underline{\delta}_{T+1}$ and $\overline{\delta}_T \leq \overline{\delta}_{T+1}$.*

The orders $\{\succeq_T\}_{T=0}^{\infty}$ are hence to be understood as being present biased.

3.2 A Multiple Discount Acception for Generalized Quasi-Hyperbolic Preferences

The following *quasi-stationarity* axiom, which is similar to axiom 4 in Montiel Olea and Strzalecki (2014), implies a generalization of quasi-hyperbolic discounting in that the preferences satisfy a stationarity axiom for every $T \geq 1$.

Axiom **B 2.** For any constants $c, c' \in \mathbb{R}$ and utility streams $x, y \in \ell_\infty$,

$$(c, c', x) \succeq_0 (c, c', y) \text{ if and only if } (c, x) \succeq_0 (c, y).$$

Under axiom **B2**, one can establish a multiple discount rates version of quasi-hyperbolic discounting.

Proposition **3.2.** *Assume axioms* **F** *and* **B2**. *Assume also that, for every T, the order $\succeq_T$ is non-trivial.*

(i) *For any $T \geq 1$, $\underline{\delta}_T = \underline{\delta}_1$ and $\overline{\delta}_T = \overline{\delta}_1$.*

(ii) *Adding axiom* **B1**, *one obtains $\underline{\delta}_0 \leq \underline{\delta}_1$ and $\overline{\delta}_0 \leq \overline{\delta}_1$.*

Although for each date T, there exists a set of possible discount rates, the quasi-stationarity axiom **B2** ensures that these sets are the same for any date $T \geq 1$. As this is clarified in Proposition 3.2(ii), combined with axiom **B1**, the set of discount rates associated with date $T = 0$ has smaller lower and upper bounds that the sets associated with $T \geq 1$.

Remark **3.1.** This result provides an interesting generalization of the quasi-hyperbolic discounting of Phelps and Pollack (1968) and Laibson (1997). Consider the case in which for any T, $\underline{\delta}_T = \overline{\delta}_T = \delta_T$ with $\delta_0 \leq \delta_1 = \delta$.[5] The comparison between two inter-temporal streams becomes: $x \succeq_0 y$ if and only if

$$(1-\delta_0)x_0 + \delta_0 \left(\sum_{s=0}^{\infty} (1-\delta)\delta^s x_{1+s} \right) \geq (1-\delta_0)y_0 + \delta_0 \left(\sum_{s=0}^{\infty} (1-\delta)\delta^s y_{1+s} \right),$$

which is equivalent to

$$x_0 + \beta \left(\sum_{s=1}^{\infty} \delta^s x_s \right) \geq y_0 + \beta \left(\sum_{s=1}^{\infty} \delta^s y_s \right),$$

for $\beta = [(1-\delta_0)\delta]^{-1}\delta_0(1-\delta) \leq 1$.

Bach et al. (2024) also propose a multiple quasi-hyperbolic discounts and a MaxMin representation of the index function, with a similar set of possible discount rates $(\delta_0, \delta) \in [\underline{\delta}_0, \overline{\delta}_0] \times [\underline{\delta}, \overline{\delta}]$. Relying on a different approach and an alternative axiomatical system, Drugeon and Ha-Huy (2023) present an $\alpha-$Maxmin representation with discount rates satisfying a temporal stationarity property from a certain date in the future.

The core difference from the current work is that, whereas in Bach et al. (2024) and Drugeon and Ha-Huy (2023), the optimal discount rates are chosen at the beginning of the evaluation, in this article and as a result of the recursive representation, they are chosen in *each* period, by comparing the utility values of the present with those of the future. Moreover, in this article, a present bias property is present, with the lower and upper bounds of possible sets increasing (or at least, not decreasing) over time.

4. The Robust Temporal Pre-orders $\succeq_T^*$

In decision theory, the classical contribution of Gilboa and Schmeidler (1989) opened the way for a huge literature in which Savage's famous *sure-thing* principle is not satisfied. Without this property, the decision of an economic agent is regarded as being based on a set of probabilities on the possible set of states of nature. Recently, this idea has been developed in the temporal discounting literature. To cite some contributions, the work of Chambers and Echenique (2018) characterizes conditions under which there are multiple possible discount rates. Bach et al. (2024) and Drugeon and Ha-Huy (2023), using different approaches, extended to a situations encompassing temporal bias phenomena.

[5]This property can be obtained by adding the following assertion: $x \succeq y$ if and only if $x + z \succeq y + z$ for any $z \in \ell_\infty$. In decision theory, this is the famous *independence* property that rules out ambiguity.

The purpose of this section is to follow the same idea and to give a characterisation of the set of possibles discount rates being used to evaluate inter-temporal utility streams.

In the same spirit as Ghirardato et al. (2004), we define the robust time-dependent order $\succeq^*_{\mathrm{T}}$ as follows: $x \succeq^*_{\mathrm{T}} y$ if and only if, for any z, $x + z \succeq^*_{\mathrm{T}} y + z$. Proposition 4.1 will then provide a characterization of the weight set Ω_{T} that represents the robustness order $\succeq^*_{\mathrm{T}}$. A utility sequence x is considered robustly better than another one y if under every time discounting evaluation belonging to Ω_T, the value of x is greater than that of y.

As a preparation step, we present the following axiom.

Axiom A 1. *Tail-insensitivity*: For any $x, y, z \in \ell_\infty$, $\epsilon > 0$, there exists T_0 and s_0 such that, for any $T \geq T_0$, $S \geq s_0$,

$$\left(x_{[0,T+s]}, y_{[T+s+1,\infty)}\right) \succeq_T \left(x_{[0,T+s]}, z_{[T+s+1,\infty)}\right) - \epsilon \mathbf{1}.$$

The *tail-insensitivity* condition implies that for any $x, y \in \ell_\infty$,

$$\lim_{s \to \infty} I_T\left(x_{[0,T+s]}, y_{[T+s+1,\infty)}\right) = I_T(x).$$

The usual conditions in the literature typically assume that the effect of the tail utilities converges to zero, for example the *continuity at infinity* of Chambers and Echenique (2018). Under the tail-insensitivity property, every temporal weight system belongs to the set ℓ_1.[6]

Proposition 4.1 then equips the analysis with a representation of the weight set Ω_{T}.

Proposition **4.1.** *Assume axiom* **F**. *Assume that either for every T, the corresponding operateur is* min, *or for every T, the corresponding operator is* max. *Then, if the order $\succeq_T$ is non-trivial, the weights set Ω_{T} is the convex hull of the set*

$$\left\{\left(1-\delta_{\mathrm{T}}, \delta_{\mathrm{T}}(1-\delta_{\mathrm{T+1}}), \delta_{\mathrm{T}}\delta_{\mathrm{T+1}}(1-\delta_{\mathrm{T+2}}), \ldots, \delta_{\mathrm{T}}\delta_{\mathrm{T+1}}\ldots\delta_{\mathrm{T}+s}(1-\delta_{\mathrm{T}+s+1}), \ldots\right)\right\},$$

where $\delta_{\mathrm{T}+s} \in \left\{\underline{\delta}_{\mathrm{T}+s}, \overline{\delta}_{\mathrm{T}+s}\right\}$ for any s.

It is well known in the literature that, besides the initial order $\succeq_T$, there exists a robust or unanimous pre-order, defined on a set of *linear* index functions.[7] A given utility stream

[6]As an illustration, consider the order represented by the index function $I(x) = (1-\delta)\sum_{s=0}^{\infty}\delta^s x_s$, with some $0 < \delta < 1$.

[7]See Ghirardato et al. (2004) and Chambers and Echenique (2018) for details.

is robustly better than another one if such a comparison is unanimous among a set of linear orders associated with the initial order. Proposition 4.1 provides a clear and precise description of this set.

A. Proofs for Section 2

A.1 Proof of Lemma 2.1

Suppose that the order $\succeq_T$ is *non-trivial*.

(i)-(ii). A proof for the existence of index function I_T with properties described in the statement of this proposition can be found in **?**, proof of Lemma (2.1).

Part (iii) is a direct consequence of head-insensitivity condition. □

A.2 Proof pf Lemma 2.2

Fix $x \in \ell_\infty$. To simplify the exposition, let $c = I_{T+1}\Big(x\Big)$. By consistency property, we have $x \sim_T (x_{[0,T]}, c\mathbf{1})$. Equivalently, $I_T\Big(x\Big) = I_T\Big(x_{[0,T]}, c\mathbf{1}\Big)$. We recall that $\chi_g^T = I_T(0\mathbf{1}_{[0,T]}, \mathbf{1})$. From the constant additive property,

$$I_T(0\mathbf{1}_{[0,T]}, -\mathbf{1}) + I_T(\mathbf{1}) = I_T(\mathbf{1}_{[0,T]}, 0).$$

Since the order $\succeq_T$ is non-trivial, $I_T(\mathbf{1}) = 1$. This implies

$$\chi_\ell^T = -I_T(0\mathbf{1}_{[0,T]}, -\mathbf{1}) = 1 - I_T(\mathbf{1}_{[0,T]}, 0).$$

Consider the case $x_T \leq c$. From head-insensitivity property,

$$\begin{aligned}
I_T\Big(x\Big) &= I_T\Big(x_{[0,T]}, c\mathbf{1}\Big) \\
&= I_T\Big(x_T\mathbf{1}_{[0,T-1]}, x_T, c\mathbf{1}\Big) \\
&= x_T + I_T\Big(0\mathbf{1}_{[0,T]}, (c - x_T)\mathbf{1}\Big) \\
&= x_T + (c - x_T)I_T(0\mathbf{1}_{[0,T]}, \mathbf{1}) \\
&= \Big(1 - I_T(0\mathbf{1}_{[0,T]}, \mathbf{1})\Big)x_T + I_T(0\mathbf{1}_{[0,T]}, \mathbf{1})c \\
&= (1 - \chi_g^T)x_T + \chi_g^T c.
\end{aligned}$$

In the case where $x_T \geq c$, with head-insensitivity property:

$$\begin{aligned}
I_T\Big(x\Big) &= I_T\Big(x_{[0,T]}, c\mathbf{1}\Big) \\
&= I_T\Big((x_T - c)\mathbf{1}_{[0,T-1]}, x_T - c, 0\mathbf{1}\Big) + c \\
&= \Big(x_T - c\Big) I_T(\mathbf{1}_{[0,T]}, 0\mathbf{1}) + c \\
&= I_T(\mathbf{1}_{[0,T]}, 0) x_T + \Big(1 - I_T(\mathbf{1}_{[0,T]}, 0\mathbf{1})\Big) c \\
&= (1 - \chi_\ell^T) x_T + \chi_\ell^T c.
\end{aligned}$$

□

A.3 Proof of Proposition 2.1

Suppose that $\chi_g^T \leq \chi_\ell^T$, or equivalently, $I_T(0\mathbf{1}_{[0,T]}, \mathbf{1}) + I_T(\mathbf{1}_{[0,T]}, 0\mathbf{1}) \leq 1$. We have

$$\begin{aligned}
\underline{\delta}_T &= \chi_g^T = I_T(0\mathbf{1}_{[0,T]}, \mathbf{1}), \\
\overline{\delta}_T &= \chi_\ell^T = 1 - I_T(\mathbf{1}_{[0,T]}, 0\mathbf{1}).
\end{aligned}$$

Using Lemma 2.2, in the case $x_T \leq c$,

$$I_T\Big(x\Big) = (1 - \underline{\delta}_T) x_T + \underline{\delta}_T I_{T+1}(x).$$

Since $x_T \leq c$, it is easy to verify that for every $\delta \geq \underline{\delta}_T$,

$$(1 - \underline{\delta}_T) x_T + \underline{\delta}_T c \leq (1 - \delta) x_T + \delta c.$$

Hence,

$$I_T(x) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1 - \delta) x_T + \delta I_{T+1}(x)\right].$$

Consider the case $x_T \geq c$. Always by Lemma 2.2, we have

$$I_T\Big(x\Big) = (1 - \overline{\delta}_T) x_T + \overline{\delta}_T I_{T+1}(x).$$

Since $x_T \geq c$, it is easy to verify that for every $\delta \leq \overline{\delta}_T$,

$$(1 - \overline{\delta}_T) x_T + \overline{\delta} c \leq (1 - \delta) x_T + \delta c.$$

Hence,

$$I_T(x) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1 - \delta) x_T + \delta I_{T+1}(x)\right].$$

Now, suppose that $\chi_g^T \geq \chi_\ell^T$, or equivalently, $I_T(0\mathbf{1}_{[0,T]}, \mathbf{1}) + I_T(\mathbf{1}_{[0,T]}, 0\mathbf{1}) \geq 1$. We have

$$\begin{aligned}\underline{\delta}_T &= \chi_\ell^T = 1 - I_T(\mathbf{1}_{[0,T]}, 0\mathbf{1}),\\ \overline{\delta}_T &= \chi_g^T = I_T(0\mathbf{1}_{[0,T]}, \mathbf{1}).\end{aligned}$$

Using the same arguments as in the first part of the proof, we obtain:

$$I_T(x) = \max_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \Big[(1-\delta)x_T + \delta I_{T+1}(x)\Big],$$

which establishes the statement. □

A.4 Proof of Corollary 2.1

(i) Fix any $T \geq 0$. Suppose that $(0\mathbf{1}_{[0,T+1]}, \mathbf{1}) \sim_{T+1} c\mathbf{1}$. By head-insensitivy property, we have $(0\mathbf{1}_{[0,T+1]}, \mathbf{1}) \sim_{T+1} (0\mathbf{1}_{[0,T]}, c\mathbf{1})$.

Hence, by consistency property, $(0\mathbf{1}_{[0,T+1]}, \mathbf{1}) \sim_0 (0\mathbf{1}_{[0,T]}, c\mathbf{1})$. By axiom **S1**, this implies $(0\mathbf{1}_{[0,T]}, \mathbf{1}) \sim_0 (0\mathbf{1}_{[0,T-1]}, c\mathbf{1})$, which, by *head-insensitity* property, is equivalent to $(0\mathbf{1}_{[0,T]}, \mathbf{1}) \sim_T c\mathbf{1}$. Hence $\chi_g^T = \chi_g^{T+1}$.

Using the same arguments with c such that $(0\mathbf{1}_{[0,T+1]}, -\mathbf{1}) \sim_{T+1} (-c)\mathbf{1}$, we have $\chi_\ell^T = \chi_\ell^{T+1}$.

The sequences $\{\chi_g^T\}_{T=0}^\infty$ and $\{\chi_\ell^T\}_{T=0}^\infty$ being constants through time, for any $T \geq 0$, $\underline{\delta}_T = \underline{\delta}_0$ and $\overline{\delta}_T = \overline{\delta}_0$. Moreover, this implies that either for every T, $\chi_g^T \leq \chi_\ell^T$, or for every T, $\chi_g^T \geq \chi_\ell^T$. The statement in Corollary 2.1 is proved.

(ii) This is a direct consequence of the convexity property, which implies that the operator corresponding to $T = 0$ is min. The details of the argument can be found in Gilboa and Schmeidler (1989) or Wakai (2007). The rest follows part (i) of this Corollary.

B. Proofs for Section 3

B.1 Proof of Proposition 3.1

Recall that, for any T,

$$\begin{aligned}\underline{\delta}_T &= \min\Big\{I_T(0\mathbf{1}_{[0,T]}, \mathbf{1}), 1 - I_T(\mathbf{1}_{[0,T]}, 0\mathbf{1})\Big\},\\ \overline{\delta}_T &= \max\Big\{I_T(0\mathbf{1}_{[0,T]}, \mathbf{1}), 1 - I_T(\mathbf{1}_{[0,T]}, 0\mathbf{1})\Big\}.\end{aligned}$$

From axiom **B1**, both of the two sequences $\{I_T(0\mathbf{1}_{[0,T]},\mathbf{1})\}_{T=0}^{\infty}$ and $\{1-I_T(\mathbf{1}_{[0,T]},0\mathbf{1})\}_{T=0}^{\infty}$ are increasing. This in its turn implies that the two sequences $\{\underline{\delta}_T\}_{T=0}^{\infty}$ and $\{\overline{\delta}_T\}_{T=0}^{\infty}$ are also increasing.

B.2 Proof of Proposition 3.2

(i) Fix any $T \geq 1$. Suppose that $(0\mathbf{1}_{[0,T+1]},\mathbf{1}) \sim_{T+1} c\mathbf{1}$. By head-insensitivy property, we have $(0\mathbf{1}_{[0,T+1]},\mathbf{1}) \sim_{T+1} (0\mathbf{1}_{[0,T]},c\mathbf{1})$.

Hence, by *consistency* property, $(0\mathbf{1}_{[0,T+1]},\mathbf{1}) \sim_0 (0\mathbf{1}_{[0,T]},c\mathbf{1})$. By axiom **B2**, this implies $(0\mathbf{1}_{[0,T]},\mathbf{1}) \sim_0 (0\mathbf{1}_{[0,T-1]},c\mathbf{1})$, which, by head-insensitity property, is equivalent to $(0\mathbf{1}_{[0,T]},\mathbf{1}) \sim_T c\mathbf{1}$. Hence $\chi_g^T = \chi_g^{T+1}$.

Use the same arguments, one gets $\chi_\ell^T = \chi_\ell^{T+1}$. Then for any $T \geq 1$, $\underline{\delta}_T = \underline{\delta}_1$ and $\overline{\delta}_T = \overline{\delta}_1$.

(ii) The second part is a direct consequence of the present bias property. □

C. Proofs for Section 4

C.1 Proof of Proposition 4.1

First, recall that the dual space of ℓ_∞ , *i.e.*, the set of bouneded real sequences can be decomposed into the direct sum of two subspaces, ℓ_1 and ℓ_1^d: $(\ell_\infty)^* = \ell_1 \oplus \ell_1^d$. The subspace ℓ_1 satisfies σ-additivity property. The subspace ℓ_1^d, the disjoint complement of ℓ_1, is the one of finitely additive measures defined on $\mathbb{N}$. More precisely, for each measure $\phi \in \ell_1^d$, for any $x \in \ell_\infty$, the value of $\phi \cdot x$ depends only on the distant behaviour of x, and does not change if there are only a change in a finite number of values x_s, $s \in \mathbb{N}$.

Define $\mathcal{P}_T^*$ as the positive polar cone of $\mathcal{P}_T = \Big\{x \in \ell_\infty \text{ such that } x \succeq_T^* 0\mathbf{1}\Big\}$ in the dual space $\Big(\ell_\infty\Big)^*$:

$$\mathcal{P}^* = \Big\{\mathrm{P} \in (\ell_\infty)^* \text{ such that } \mathrm{P}\cdot x \geq 0 \text{ for every } x \succeq_T^* 0\mathbf{1}\Big\}.$$

Observe that by the very definition of the order $\succeq^*$, $\mathcal{P}$ is convex and separable by the vector $-\mathbf{1}$, the cone $\mathcal{P}^*$ does not degenerate to $\{0\mathbf{1}\}$. We have $x \succeq_T^* y$ if and only if $P\cdot x \geq P \cdot y$ for every $P \in \mathcal{P}_T$.

For each $\mathrm{P} \in \mathcal{P}^*$, define

$$\pi(\mathrm{P}) = \frac{1}{\mathrm{P}\cdot\mathbf{1}}\mathrm{P}.$$

Since $x \succeq_T^* 0\mathbf{1}$ for every $x \in \ell_\infty$ satisfying $x_s \geq 0$ for all s, it follows that $\mathrm{P} \cdot x \geq 0$ for every x such that $x_s \geq 0$ for every s.

Now, we begin the main part of the proof.

Without loss of generalisation, we have only to prove Proposition 4.1 for $T = 0$. Consider the case where every operator is *min*. For $T \geq 0$,

$$I_T\Big(x\Big) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1-\delta)x_T + \delta I_{T+1}(x)\right].$$

We have

$$I_0(x) = \min_{\underline{\delta}_0 \leq \delta_0 \leq \overline{\delta}_0, \ldots, \underline{\delta}_T \leq \delta_T \leq \overline{\delta}_T} \Big\{(1-\delta_0)x_0 + \delta_0(1-\delta_1)x_1 + \ldots + \delta_0\delta_1 \ldots \delta_{T-1}(1-\delta_T)x_T + \delta_0\delta_1 \ldots \delta_T I_{T+1}(x)\Big\}. \tag{2}$$

Observe that

$$\lim_{T \to \infty} \overline{\delta}_0\overline{\delta}_1 \ldots \overline{\delta}_T = 0.$$

Indeed, consider some value $c > 0$. Using (2), it is easy to verify that

$$I_0\Big(0\mathbf{1}_{[0,T]}, (-c)\mathbf{1}\Big) = (\overline{\delta}_0\overline{\delta}_1 \ldots \overline{\delta}_T) \times (-c).$$

By tail-insensitivity property, one has

$$\lim_{T \to \infty} I_0\Big(0\mathbf{1}_{[0,T]}, (-c)\mathbf{1}\Big) = 0,$$

which implies that

$$\lim_{T \to \infty} \overline{\delta}_0\overline{\delta}_1 \ldots \overline{\delta}_T = 0.$$

Let Ω_{T} is the convex hull of the set

$$\Big\{\Big(1-\delta_{\mathrm{T}}, \delta_{\mathrm{T}}(1-\delta_{\mathrm{T}+1}), \delta_{\mathrm{T}}\delta_{\mathrm{T}+1}(1-\delta_{\mathrm{T}+2}), \ldots, \delta_{\mathrm{T}}\delta_{\mathrm{T}+1} \ldots \delta_{\mathrm{T}+s}(1-\delta_{\mathrm{T}+s+1}), \ldots\Big)\Big\},$$

where $\delta_{\mathrm{T}+s} \in \Big\{\underline{\delta}_{\mathrm{T}+s}, \overline{\delta}_{\mathrm{T}+s}\Big\}$ for any s.

Since $\lim_{T \to \infty} \overline{\delta}_0\overline{\delta}_1 \cdots \overline{\delta}_T = 0$, we verify that $\Omega_0 \subset \ell_1$.

Now, consider some sequence $x \in \ell_\infty$. First, we prove that

$$I_0(x) = \inf_{\omega \in \Omega_0} (\omega \cdot x).$$

Denote by $\{\delta_T^*\}_{T=0}^\infty$ the sequence of discount rates such that for every $T \geq 0$,

$$I_T\Big(x\Big) = (1-\delta_T^*)x_T + \delta_T^* I_{T+1}(x).$$

Recall that

$$I_0(x) = (1-\delta_0^*)x_0 + \delta_0^*(1-\delta_1^*)x_1 + \ldots + \delta_0^*\delta_1^*\ldots\delta_{T-1}^*(1-\delta_T^*)x_T + \delta_0^*\delta_1^*\ldots\delta_T^* I_{T+1}(x).$$

Let T converges to infinity, since $\delta_0^*\delta_1^*\ldots\delta_T^*$ converges to zero, we have

$$I_0(x) = \lim_{T\to\infty}\Big((1-\delta_0^*)x_0 + \delta_0^*(1-\delta_1^*)x_1 + \ldots + \delta_0^*\delta_1^*\ldots\delta_{T-1}^*(1-\delta_T^*)x_T\Big).$$

Assume that $I(x) > \inf_{\omega\in\Omega_0}(\omega\cdot x)$. Then there exists a sequence $\{\delta_T\}_{T=0}^{\infty}$ such that for every T, $\underline{\delta}_T \le \delta_T \le \overline{\delta}_T$, and

$$\begin{aligned} I_0(x) &= \lim_{T\to\infty}\Big((1-\delta_0^*)x_0 + \delta_0^*(1-\delta_1^*)x_1 + \ldots + \delta_0^*\delta_1^*\ldots\delta_{T-1}^*(1-\delta_T^*)x_T\Big) \\ &> \lim_{T\to\infty}\Big((1-\delta)x_0 + \delta_0(1-\delta_1)x_1 + \ldots + \delta_0\delta_1\ldots\delta_{T-1}(1-\delta_T)x_T\Big). \end{aligned}$$

Hence, for T sufficiently large, one gets

$$\begin{aligned} &(1-\delta_0^*)x_0 + \delta_0^*(1-\delta_1^*)x_1 + \ldots + \delta_0^*\delta_1^*\ldots\delta_{T-1}^*(1-\delta_T^*)x_T + \delta_0^*\delta_1^*\ldots\delta_T^* I_{T+1}(x) \\ &> (1-\delta)x_0 + \delta_0(1-\delta_1)x_1 + \ldots + \delta_0\delta_1\ldots\delta_{T-1}(1-\delta_T)x_T + \delta_0\delta_1\ldots\delta_T I_{T+1}(x), \end{aligned}$$

a contradiction with (2).

Let $P_0 = \pi(\mathcal{P}_0)$, with $\mathcal{P}_0$ is defined in the first part of the proof. The set P_0 represents the weights set corresponding to the robuste order $\succeq^*$. We have to prove that $P_0 = \Omega_0$.

If Ω_0 is not a subset of P_0, then there exists $x \succeq^* 0\mathbf{1}$ such that $\omega\cdot x < 0$, for some $\omega\in\Omega_0$. This implies $I(x) < 0$: a contradiction. Hence, $\Omega_0 \subset P_0$.

Now, assume that for $x, y \in \ell_\infty$, we have $\omega\cdot x \ge \omega\cdot y$ for every $\omega\in\Omega_0$. It is easy to verify that for any $z\in\ell_\infty$,

$$\begin{aligned} I_0(x+z) &= \inf_{\Omega_0}\omega\cdot(x+z) \\ &\ge \inf_{\Omega_0}\omega\cdot(y+z) \\ &= I_0(y+z), \end{aligned}$$

which implies $x \succeq^* y$, by the definition of the robuste order $\succeq^*$. Hence, $P_0 \subset \Omega_0$.

Consider the case where every operator is max. For $T\ge 0$,

$$I_T\Big(x\Big) = \max_{\underline{\delta}_T\le\delta\le\overline{\delta}_T}\left[(1-\delta)x_T + \delta I_{T+1}(x)\right].$$

Define the order $\hat{\succeq}_T$ as: $x\hat{\succeq}_T y$ if and only if $(-y) \succeq_T (-x)$. We can verify that the sequence of orders satisfies Axiom **F**. For every $T \geq 0$, the order $\hat{\succeq}_T$ can be represented by and index function $\hat{I}_T$. Moreover,

$$\hat{I}_T\Big(x\Big) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \Big[(1-\delta)x_T + \delta \hat{I}_{T+1}(x)\Big].$$

Applying the same arguments as in the first part of the proof, the claim of this Proposition is proved.

D. Declaration of interest

We, Jean-Pierre Drugeon and Thai Ha-Huy, declare that we have no relevant or material financial interests that relate to the research described in this paper.

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