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Revisiting the Marxian conjecture of the Falling Rate of Profit

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Abstract

The law of Falling Rate of Profit is a fundamental conjecture of Marxian economics and a pillar of Marxist theory with significant political and historical effects.

To understand its scope, we consider a simple economy with many capital goods and a single wage good, as well labour times across these sectors. Technology is represented by a Leontief matrix.

While finding that the profit rate is strictly less than the exploitation rate, a classic result of Marx, we prove also a version of the Fundamental Marxian Theorem: these rates are equal if and only if they are zero. Interestingly, we provide necessary and sufficient conditions for a specific Transformation Problem of values into prices, that are exogenous, involving only technologies and labor times; and we show that the profit rate is inversely proportional to the total unit cost of production.

Under a constant exploitation rate, the law of FRP is a mere tautology. Importantly, without this empirically questionable assumption, we prove in a new way that the law can be violated: the key of our reasoning is the transformation of values into prices in the spirit of Morishima [9].

Keywords: Values, labor times, Leontief technology, exploitation rate, profit rate, transformation of values into production prices, the law of Falling Rate of Profit.

JEL Classification: C62, C67, D57, D46.

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1 Introduction

In neoclassical theory, price and value are synonymous: given endowments, technologies and preferences, prices emerge in market economies from the interaction between rational agents. In the Marxian tradition, value is distinct from price, and only labor creates value. The Marxian transformation theory studies how the labor value is translated into prices.

The Falling Rate of Profit (FRP thereinafter) is understood as a downward trend with possible temporary fluctuations. Although considered an empirical possibility by the Classics (namely, Smith and Ricardo), it was later theorized as a law by Marx in *Capital* (vol. 3, ch. 13), and identified as a fundamental mechanism of capitalism. It has gained increasing importance ever since.

Unlike Ricardo, Marx did not believe in the possibility of a stabilization of capitalism, that is a happy ending. This is the foundation of his political view: the fall in the rate of profit would lead to economic crises and social struggles that would overthrow capitalism and replace it with a new economic and social order. This view is deliberately more political than scientific, but, to begin with, the law of FRP was considered a scientific result. We think that the scientific aspects of this law deserve to be studied more rigorously.

The literature on Marx's economics is huge.

According to Marx [5], the rate of profit π depends on the rate of exploitation e and on capital composition C/V (the ratio between constant and variable capital):

$$\pi = \frac{e}{1 + C/V} \tag{1}$$

Sweezy [18] writes "to the capitalist the crucial ratio is the rate of profit, in other words, the ratio of surplus value to total capital outlay" while assuming a provisional constant rate of exploitation. In contrast, Robinson [12] acknowledges that "Marx's law of the falling tendency of profits then consists simply in the tautology: when the rate of exploitation is constant, the rate of profit falls as capital per man increases". Marx himself considered the rise of exploitation as a possible counteracting force. This increase is supported by data.[1]

Since then, the debate has focused on the determinants of the exploitation rate: the very issue is the validity of the FRP law when this rate is endogenous.

In the spirit of Marx [5], Mandel [4] argues that the decline in the rate of profit is real, but not based on a constant rate of exploitation. It occurs through historical cycles shaped by dialectical forces. Foley [2] writes that a rise in the rate of surplus value may offset the FRP: "changes in the value of labor-power in response to changes in labor productivity are not automatic and involve substantial social and economic conflicts".

There is also a mathematical tradition of studies on the Marxian Transformation Problem. Our article falls within this class of models and draws on Parts 1 to 3 of Morishima's *Marx's Economy* [9]. Okishio [11] first and Morishima [9]

[1] According to Rotta and Kumar [13], the global rate of exploitation increased between 2000 and 2008, but stagnated after the financial crisis.

ten years later prove the Fundamental Marxian Theorem: $\pi > 0 \Rightarrow e > 0$ (no profit without exploitation). Before applying Morishima's formal transformation of values into prices to understand the scope of the FRP law, let us mention a few authors from the mathematical tradition.

Okishio [10] questions whether the FRP law is consistent with the behavior of price-taking capitalists and answers in the negative. His theorem states that, given a constant real wage, any rational technological change increases the equilibrium rate of profit if universally adopted.

Roemer [14] generalizes Okishio [10]: the fall in the rate of profit depends on the effects of technological change on the price and the value rate of profit: these two rates can move in opposite directions due to innovation. Roemer [16] even goes so far as to conclude that Okishio's theorem is "the end of the classical story".

Roemer [15] also bridges Marxian and Sraffian conceptions through a general equilibrium model where exploitation is necessary and sufficient to sustain positive profits under some technological restrictions. Duménil and Foley [1] show the possibility of exploitation in a market economy with capitalist competition.

Skillman [17] does not believe in the Roemer's end of the story. By following an alternative game-theoretic approach, he proves that the FRP law is consistent with an equilibrium where wages are determined by sequential bargaining within a stationary matching process.

Yoshihara [19] reviews the theory of exploitation in light of the Fundamental Marxian Theorem, while Mohun and Veneziani [7] and Monte-Rojas [8] propose general frameworks for unifying existing transformation models. In the latter, the rate of exploitation is an endogenous variable determined by values and prices of production systems.

Thus, in this literature, endogenous necessary and sufficient conditions are provided to transform values into prices: Marx himself [5] introduces some of these conditions (total surplus = total profits, uniform profit rate); as does Morishima [9] (constant capital composition across sectors). These conditions are "endogenous" because they involve values and prices.

Conversely, in our work, we give necessary and sufficient exogenous conditions to translate values into prices. They are "exogenous" because they only involve technological coefficients and labor times.

The transformation of values into prices is key to revisit the Marxian conjecture of the FRP.

Before addressing the nature of the exploitation rate, equation (1) deserves two comments.

The exploitation rate and the capital composition are computed in terms of values, while the profit rate is defined in terms of prices. Therefore, to study the FRP, we must first convert the values into prices.

The profit rate is uniform across all sectors. Since the exploitation rate is the same for the whole economy, the capital composition is identical across all sectors.

Now, there are two interpretations of (1) depending on whether we consider e exogenous or endogenous.

(1) If e is constant, then, trivially, an increase in C/V (due to capital accumulation) will lead to a fall in the rate of profit.[2] In this case, Robinson [12] is right: the law is a mere tautology.

(2) Other authors consider the rate of exploitation to be endogenous. However, some, such as Sweezy [18], Mandel [4] and Skillman [17], following Marx himself [5], still defend the FRP law under certain assumptions, while other, such as Okishio [10] and Roemer [14] do not exclude the possibility of an increasing profit rate.

We generalize the expression for the rate of profit to the case of many capital goods and demonstrate that this rate is inversely proportional to the total unit cost of production and can increase over time. We define an alternative rate of exploitation by price and wage (which we call the price-wage rate of exploitation). We show that it is always lower than the rate of exploitation defined by Marx. An economy can make profits with a zero price-wage rate of exploitation, whereas the Marxian rate of exploitation is positive.

Our results are coherent with those of Okishio [10], since a decrease in costs through a reduction in the coefficients of technical inputs leads to an increase in the rate of profit; they are also in line with Roemer [14], although we show that the composition of capital increases with technological improvement, while Roemer [14] does not.

The novelty of our approach lies in the reinterpretation of the FRP conjecture by transforming values into prices in the spirit of Morishima [9], leading to our main results: (1) the uniform rate of profit π decreases if and only if the total value of capital and labor inputs increases; (2) in a counterexample, while the capital composition C/V and the exploitation rate e increase, the rate of profit π also increases.

The rest of the article is organized as follows. We introduce the notation in Section 2. In Section 3, we present the economy, while, in Section 4, we define the values and assume a productive technology matrix to have positive values. In Sections 5 and 6, we compare the rate of exploitation and the rate of profit, while, in Section 7, we address the Transformation Problem. The critique of the FRP law is finally treated in Section 8. Section 9 concludes. All the proofs are gathered in the Appendix.

2 Notation

Linear algebra is ideal for representing Marxian economies with Leontief technologies. For the sake of clarity, we must introduce now all the vectors and matrices we will use later to transform values into prices. Throughout this article, vectors are rows and positive means strictly positive. More precisely, we will say that a vector x is positive (non-negative) if all its components are pos-

[2] Marx [5] states that "the gradual growth of constant capital in relation to variable capital must necessarily lead to a gradual fall of the general rate of profit, so long as the rate of surplus-value, or the intensity of exploitation of labour by capital, remain the same" (*Capital*, vol. 3, ch. 13).

itive (non-negative): $x > 0 \Leftrightarrow x_i > 0$ for any i ($x \geq 0 \Leftrightarrow x_i \geq 0$ for any i, respectively). A positive (non-negative) matrix is defined by the same token.

System of labor times and augmented system:

$$l \equiv \begin{bmatrix} l_1 & \cdots & l_n \end{bmatrix} \text{ and } \hat{l} \equiv \begin{bmatrix} l_1 & \cdots & l_n & l_{n+1} \end{bmatrix} \tag{2}$$

System of values and augmented system:

$$\lambda \equiv \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix} \text{ and } \hat{\lambda} \equiv \begin{bmatrix} \lambda_1 & \cdots & \lambda_n & \lambda_{n+1} \end{bmatrix} \tag{3}$$

System of prices and augmented system:

$$p \equiv \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} \text{ and } \hat{p} \equiv \begin{bmatrix} p_1 & \cdots & p_n & p_{n+1} \end{bmatrix} \tag{4}$$

Matrix of technical coefficients and augmented matrix:

$$A \equiv \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \text{ and } \hat{A} \equiv \begin{bmatrix} a_{11} & \cdots & a_{1,n+1} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,n+1} \end{bmatrix} \tag{5}$$

We will denote the jth column of matrix $\hat{A}$ by A_j:

$$A_j \equiv \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix} \tag{6}$$

with $j = 1, \ldots, n+1$.

All these variables will be precisely defined later.

3 Technology

We consider a closed economy with $n+1$ commodities. Commodities $1, \ldots, n$ are capital goods (means of production), while commodity $n+1$ is the wage good.

We assume that each sector produces a single good using a single technology, always relying on capital goods and labor; but that there are no other primary factors of production besides labor. Formally, we introduce a linear technology.

Assumption 1 (Leontief technology) *To produce one unit of good j with $j = 1, \ldots, n+1$, the firm uses a_{ij} units of capital good i with $i = 1, \ldots, n$, and l_j hours of labor, that is a vector of inputs $\begin{bmatrix} a_{1j} & \ldots & a_{nj} & l_j \end{bmatrix}$.*

4 Values

Marx believes that values are determined by technology alone and are not influenced by market changes in wages and prices, as long as the production methods remain unchanged: "All that these things now tell us is, that human labour power has been expended in their production, that human labour is embodied in them. When looked at as crystals of this social substance, common to them all, they are values." (*Capital*, vol. 1, ch. 1).[3]

In the spirit of Morishima [9], let λ_j denote the value of commodity j which is defined as the total amount of labour (in terms of labor-time) embodied (or materialized) in one unit of commodity j, with $j = 1, \ldots, n+1$. Therefore, the total labour embodied in a commodity j is given by $\lambda_j = \lambda_1 a_{1j} + \ldots + \lambda_n a_{nj} + l_j$. In matrix terms, we find

$$\begin{aligned} \lambda &= \lambda A + l \qquad (7) \\ \lambda_{n+1} &= \lambda A_{n+1} + l_{n+1} \end{aligned}$$

where l and λ are given by (2) and (3), and A and A_{n+1} by (5) and (6).

Assumption 2 (matrix positiveness) *A is a positive matrix ($a_{ij} > 0$ for any $i, j = 1, \ldots, n$).*

It should be noted that Marx did not establish the positivity of values. He took it for granted.

Definition 1 (values positiveness) *The matrix A is said to be productive if there exists a positive row vector $p \in \mathbb{R}^n_{++}$ such that $p > pA$.*

Proposition 2 (productive equivalence) *The following statements are equivalent.*

(1) The matrix A is productive.

(2) There exists a row vector $p > 0$ such that $p > pA$ (the row vector $p - pA$ has positive components).

(3) The matrix $(I - A)^{-1}$ exists and has positive elements.

(4) For any $x \geq 0$, there exists a non-negative solution to the equation $p = pA + x$.

Assumption 3 (productive matrix) *A is productive.*

Corollary 3 (existence, uniqueness and positivity) *Let Assumptions 2 and 3 hold and $l_j > 0$ for $j = 1, \ldots, n+1$. Then the values $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ exist, are unique and positive.*

5 Rate of exploitation

Let b be the worker's daily means of subsistence and T the usual length of the working day. The labor-time to obtain b is $b\lambda_{n+1}$.

We can now introduce the Marxian definition of the rate of exploitation:

$$e \equiv \frac{\text{unpaid labor}}{\text{paid labor}} = \frac{T - b\lambda_{n+1}}{b\lambda_{n+1}} = \frac{1 - b\omega\lambda_{n+1}}{b\omega\lambda_{n+1}} \qquad (8)$$

[3] Marx [5] introduces a different notion in the same chapter of *Capital*: "We see then that which determines the magnitude of the value of any article is the amount of labour socially necessary, or the labour time socially necessary for its production." (*Capital*, vol.1, ch. 1). Morishima [9] explains why the two notions of value coincide.

where $\omega \equiv 1/T$.

From a Marxian perspective, the rate of exploitation e is always positive: $T > b\lambda_{n+1}$. This assumption is at the heart of the theory of exploitation.[4]

In the following, a weaker assumption is sufficient.

Assumption 4 (exploitation) *The rate of exploitation is non-negative: $e \geq 0$, that is $b\omega\lambda_{n+1} \leq 1$.*

6 Rate of profit

Definition 4 (efficient economy) *An economy is efficient if there exists a positive price vector p and a wage per unit of labor time $w > 0$ such that*

$$p > pA + wl \tag{9}$$

where l, p and A are given by (2), (4) and (5).

Proposition 5 (efficiency and productiveness) *The economy is efficient if and only if, the matrix A is productive.*

Lemma 6 (values and prices) *Suppose that the economy is efficient and that sector $n+1$ makes no negative profit with a price system $\hat{p}$ and a wage w. Then, $p_i/w \geq \lambda_i$ for any $i = 1, \ldots, n+1$.*

Remark 7 *We can write $p_i \geq w\lambda_i$ for any i. Since λ_i is the quantity of labor time required for one unit of good i and w is the wage per unit of labor time to produce any good, then this lemma shows that the price of one unit of good i is greater than the cost of labor for one unit of that good.*

6.1 Price-wage rate of exploitation

Let p_{n+1}, w and b denote the price of the wage-good, the hour wage and the subsistence level respectively. Then, the value of b is bp_{n+1}. The laborer works bp_{n+1}/w hours in order to obtain b. Define $\tilde{\theta} \leq 1$ by $\tilde{\theta}T = bp_{n+1}/w$. We say there exists exploitation if $\tilde{\theta} < 1$. Thus, the price-wage rate of exploitation is given by

$$\tilde{e} = \frac{T - b\frac{p_{n+1}}{w}}{b\frac{p_{n+1}}{w}}$$

Denoting $\omega \equiv 1/T$, we find

$$1 + \tilde{e} = \frac{1}{b\omega\frac{p_{n+1}}{w}}$$

According to Lemma 6, we have $p_{n+1}/w \geq \lambda_{n+1}$. Since

$$1 + e = \frac{1}{b\omega\lambda_{n+1}}$$

[4] Marx [5] states: "On the basis of capitalist production, this necessary labour [$b\lambda_{n+1}$] can form a part only of the working-day [T], the working-day itself can never be reduced to this minimum." (*Capital*, vol.1, ch. 10).

we obtain $\tilde{e} \leq e$.

Workers are fully paid if $wT = bp_{n+1}$ and not fully paid if $bp_{n+1} < wT$. The difference between e and $\tilde{e}$ is that e is defined in terms of values, while $\tilde{e}$ is defined in terms of nominal wages and prices. Indeed, if, to obtain b, the required time for the worker is $b\lambda_{n+1}$ which is equal θT, with $\theta \leq 1$. We say that there is exploitation if $\theta < 1$. The rate of exploitation is

$$e = \frac{1-\theta}{\theta} = \frac{1 - b\omega\lambda_{n+1}}{b\omega\lambda_{n+1}}$$

Therefore, when the worker is fully paid, we do not have price-wage exploitation ($\tilde{e} = 0$), but we can have exploitation ($e > 0$).

6.2 Marx's main result

Definition 8 (uniform profit rate) *A rate of profit $\pi > 0$ is uniform if there exists a system of prices $\hat{p}$ and a wage w such that*

$$1 + \pi = \frac{p_i}{p_1 a_{1i} + \ldots + p_n a_{ni} + w l_i}$$

for any $i = 1, \ldots, n+1$.

Now, we can state Marx's main result.

Proposition 9 (larger exploitation rate) *If $wT > bp_{n+1}$, the rate of profit π is uniform and the economy is efficient, then $e > \pi$.*

Condition $wT > bp_{n+1}$ is equivalent to $w > b\omega p_{n+1}$ meaning that the worker can buy $b\omega$ units of wage-good with her hourly wages. The worker is said to be fully paid if $wT = bp_{n+1}$.

Proposition 10 (full pay) *If $wT \geq bp_{n+1}$, the rate of profit π is uniform and the economy is efficient, then*

$$e = \pi \Rightarrow (wT = bp_{n+1} \text{ and } e = \pi = 0)$$

7 Transformation of values into production prices

Let $\hat{p}$ denote the system of prices of capital goods and wage good. We assume that wages are fixed at the subsistence level (workers can purchase only their daily means of subsistence). Hence, $w = b\omega p_{n+1}$ (see Morishima [9], p. 62). This important political assumption completes the model.

To define profits, we first define the composition of capital in any sector. For $j = 1, \ldots, n+1$, we have

Constant capital		
in terms of values	$C_j = \lambda_1 a_{1j} + \ldots + \lambda_n a_{nj}$	
in terms of prices	$C_j^p = p_1 a_{1j} + \ldots + p_n a_{nj}$	
		(10)
Variable capital		
in terms of values	$V_j = b\omega\lambda_{n+1} l_j$	
in terms of prices	$V_j^p = b\omega p_{n+1} l_j$	

Values and prices determine the surplus and the profit, respectively, in any sector.

The surplus and the profit in sector $j = 1, \ldots, n+1$ are given by

Surplus	$S_j = \lambda_j - \lambda_1 a_{1j} - \ldots - \lambda_n a_{nj} - b\omega\lambda_{n+1} l_j = \lambda_j - (C_j + V_j)$
Profit	$\Pi_j = p_j - p_1 a_{1j} - \ldots - p_n a_{nj} - b\omega p_{n+1} l_j = p_j - \left(C_j^p + V_j^p\right)$

The rate of profit in sector j is given by

$$\pi_j \equiv \frac{\Pi_j}{C_j^p + V_j^p}$$

Definition 11 (production prices) *Consider a positive price system $\hat{p} > 0$. If the rate of profit is uniform across sectors, $\hat{p}$ is said to be a system of production prices:*[5]

$$\pi = \frac{\Pi_j}{C_j^p + V_j^p}$$

where π is the common rate. A system of production prices $\hat{q}$ is said to be normalized if $q_1 + \ldots + q_{n+1} = 1$.

Lemma 12 (uniqueness) *There exist a unique uniform rate of profit and a unique normalized system of production prices.*

Definition 13 (transformation) *A transformation of values into prices is possible if there exist a price system $\hat{p}$ and a positive number α such that $\lambda_j = \alpha p_j$ for $j = 1, \ldots, n+1$.*

Lemma 14 (productiveness) *The augmented square matrix*[6]

$$M \equiv \begin{bmatrix} a_{11} & \cdots & a_{1,n+1} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,n+1} \\ b\omega l_1 & \cdots & b\omega l_{n+1} \end{bmatrix} \tag{11}$$

is productive.

[5] Koskinas and Chatzarakis [3] define different production prices based on profit redistribution. After Morishima's transformation, these prices coincide with our production prices. However, Koskinas and Chatzarakis [3] do not address the FRP issue.

[6] See also Roemer [14].

7.1 Endogenous necessary and sufficient conditions to transform values into prices

The following assumption will hold in the rest of the paper.

Assumption 5 (full pay) $w = b\omega p_{n+1}$.

Lemma 15 (constant profit-surplus ratios) *Consider the unique system of production prices $\hat{p}$ (according to Lemma 12) and compute the corresponding profits Π_j with $j = 1, \ldots, n+1$. Then, the ratios profit-surplus are constant across the sectors ($\Pi_1/S_1 = \ldots = \Pi_{n+1}/S_{n+1}$) if and only if there exists $\alpha > 0$ such that $\lambda_j = \alpha p_j$ for $j = 1, \ldots, n+1$.*

Lemma 16 (capital composition and profit-surplus ratios) *(1) If $\hat{p}$ is a system of production prices, then*

$$\frac{S_1}{\Pi_1} = \ldots = \frac{S_{n+1}}{\Pi_{n+1}} \Rightarrow \frac{C_1}{V_1} = \ldots = \frac{C_{n+1}}{V_{n+1}}$$

(2) Conversely, assume

$$\frac{C_1}{V_1} = \ldots = \frac{C_{n+1}}{V_{n+1}} \tag{12}$$

Then, there exists a uniform rate of profits π and a system of corresponding production prices (defined up to a positive scalar) such that

$$\frac{S_1}{\Pi_1} = \ldots = \frac{S_{n+1}}{\Pi_{n+1}} \tag{13}$$

Proposition 17 (Transformation Problem characterization)

$$\frac{C_1}{V_1} = \ldots = \frac{C_{n+1}}{V_{n+1}} \tag{14}$$

if and only if there exist $\alpha > 0$ and a normalized system of prices $\hat{q}$ such that $\lambda_j = \alpha q_j$ for $j = 1, \ldots, n+1$.

Proposition 17 provides endogenous necessary and sufficient conditions for the transformation of values into production prices. These conditions involve prices and values. Our goal is rather to determine exogenous necessary and sufficient conditions for the Transformation Problem, involving only the means of production and labor times.

7.2 Exogenous necessary and sufficient for the transformation of values into prices

The following condition is fundamental to obtain the profit equation (1).

Condition $\mathcal{C}$

$$\frac{l\,(I-A)^{-1}\,A_1}{l_1} = \ldots = \frac{l\,(I-A)^{-1}\,A_{n+1}}{l_{n+1}} \tag{15}$$

Condition $\mathcal{C}$ allows us to state an important corollary of Proposition 17.

Corollary 18 (unique transformation) *There exists a unique transformation of values into production prices of the form* $\lambda_j = \alpha p_j$ *for* $j = 1, \ldots, n+1$ *and* $\alpha > 0$ *if and only if Condition* $\mathcal{C}$ *holds. In this case, the rate of profit is given by (1).*

8 FRP: from an old conjecture to a new law

Under the Marxian assumptions of uniform rate of profit and uniform composition of capital across sectors, we obtain an important economic result with significant political consequences: the failure of the FRP law.

To prove the failure, a counterexample is enough. For mathematical tractability, in the next proposition, we assume constant technical coefficients.

Assumption 6 (constant coefficients)

$$a_{ij} = a \in \left(0, \frac{1 - b\omega l}{n}\right)$$

with $i, j = 1, \ldots, n+1$.

Proposition 19 (exploitation and profit rates) *Under Assumption 6, the capital composition is given by*

$$\frac{C}{V} = \frac{na}{b\omega l} \tag{16}$$

while the rates of exploitation and profit by

$$e = \frac{1-\sigma}{b\omega l} > 0 \text{ and } \pi = \frac{1}{\sigma} - 1 > 0 \tag{17}$$

where $\sigma \equiv na + b\omega l < 1$ *is the real total unit cost of production. In particular, the uniform rate of profit decreases with* σ.

8.1 Limitations of the Marxian FRP conjecture

There are two interpretations of (1) depending on the nature of the exploitation rate e. If e is exogenous, say constant, then the law is trivial. As Robinson [12] points out, it is a tautology of little interest. The case of an endogenous exploitation rate is more interesting. Evidence show that e varies over time. Marx himself [5] states that the rate of exploitation changes due to the lengthening of the working day, technological progress, wage fluctuations, and class struggle (*Capital*, vol. 1).

Let us complement Proposition 19 with two corollaries to clarify the meaning of Marxian FRP conjecture and its political implication.

Corollary 20 (example of FPR) *Under Assumption 6, if we fix the term* $b\omega l$ *and we increase* a*, then the capital composition* C/V *increases, while the exploitation rate* e *and the profit rate* π *decrease.*

Thus, fixing the term $b\omega l$ and increasing a preserves the FRP law. However, the law fails in the reverse case, when a is fixed and $b\omega l$ increases.

Corollary 21 (counterexample) *Under Assumption 6, if we fix the technology a and decrease the term ωbl, then the capital composition C/V increases, while the exploitation rate e and the profit rate π increase.*

Remark 22 *σ is the spectral radius, that is the largest left eigenvalue of the $(n+1)\times(n+1)$ transformation matrix*

$$\tilde{M} \equiv \begin{bmatrix} a & \cdots & a \\ \vdots & & \vdots \\ a & \cdots & a \\ \omega bl & \cdots & \omega bl \end{bmatrix}$$

Indeed, consider the $(n+1)$-row vector $\tilde{p} \equiv 1/(n+1)$. It is easy to check that $\tilde{p}$ is the normalized left eigenvector associated with the largest left eigenvalue $\sigma = 1/(1+\pi)$ of the matrix: $\tilde{p}\tilde{M} = \sigma\tilde{p}$.

Corollaries 20 and 21 demonstrate that the Marxian conjecture of FRP fails in general: a simultaneous increase in the capital composition, the exploitation rate and the profit rate is possible. The conjecture becomes a law under additional restrictions. Our corollaries are consistent with Okishio [10] and Roemer [14].

In the following, we suggest to replace the FRP conjecture with a new law (Proposition 23), which refers to a general economy with many capital goods and stipulates that the rate of profit is inversely proportional to the total unit cost of production.

8.2 Revisiting the FRP law

How to replace the Marxian FRP conjecture with a new FRP law valid in any case? To do this, let us return to the model presented in Section 3 and prove what we consider the main result of our article: the rate of profit decreases if and only if the total value of inputs increases.

We use the transformation matrix (11) of values into prices in Section 7 under the condition (14). The vector $\hat{q}$ is the normalized left eigenvector associated with the largest left eigenvalue $1/(1+\pi)$ of the matrix M in (11), where π denotes the uniform rate of profit.

Denote by $\mathcal{K} \equiv q_1 a_1 + \ldots + q_n a_n$ the total value of the capital inputs, and by $\mathcal{L} \equiv q_{n+1}\omega bL$ the total value of labor, where $\hat{q}$ is a normalized system of production prices and $a_i \equiv \sum_{j=1}^{n+1} a_{ij}$ and $L \equiv \sum_{j=1}^{n+1} l_j$.

Proposition 23 (new FPR law) *If workers are fully paid ($w = b\omega q_{n+1}$) and capital structure is the same across the sectors (equalities (14)), then composition of capital equals the ratio between the total value of capital inputs $\mathcal{K}$ and the total value of labor $\mathcal{L}$:*

$$\frac{C}{V} = \frac{\mathcal{K}}{\mathcal{L}}$$

and the uniform rate of profit π decreases as the total value $\mathcal{K}+\mathcal{L}$ increases:

$$\pi = \frac{e}{1+\mathcal{K}/\mathcal{L}} = \frac{1}{\mathcal{K}+\mathcal{L}} - 1 \tag{18}$$

The left-hand equation in (18) also represents a new version of the fundamental equation (1) previously proven under Condition $\mathcal{C}$ (see Corollary 18).

9 Conclusion

We have reconsidered the FRP law and shown that it is violated through a counterexample. Our reasoning unfolds in several stages.

We have used unique undiscounted single production prices to compute profits. But, unlike Roemer [14], we have applied Condition $\mathcal{C}$ to show that these prices can be obtained by a transformation of values.

To obtain a positive exploitation rate e, several constraints must be met: the coefficients of technical inputs cannot be very high, the daily working hours T must be significant, the subsistence level b must be low, and (effective) labor time l must be short. Small labour time means that workers become more productive.

Any economy using Leontief technology to produce goods, in order to be viable (no negative profits), must exhibit a positive rate of exploitation. In such economies, profits are always accompanied by exploitation. We have shown that exploitation disappears if and only if the profit rate equals the exploitation rate.

To discuss the transformation of values into prices and the FRP law, we have assumed that the worker is fully paid, meaning that the price-wage rate of exploitation is zero. However, the profit rate is positive, meaning that profits can coexist with the absence of exploitation in terms of prices and wage. More precisely, under a Marxian definition of the rate of exploitation in terms of value, a positive profit necessarily implies a positive rate of exploitation. Conversely, under the concept of the price-wage rate of exploitation defined in Subsection 6.1, a positive profit can correspond to a zero rate of exploitation.

Due to the endogeneity of the rate of exploitation, the relationship between profit and exploitation remains ambiguous. To remove any ambiguity, in the spirit of Morishima [9], we have provided necessary and sufficient exogenous conditions to transform values into production prices, and proven that, in general, the Marxian FRP conjecture can not be considered a law: we have presented a counterexample where the composition of capital and the rate of profit increase simultaneously. More precisely, if technology remains unchanged, but working time is increased, or the subsistence level of workers is decreased, or the labor time required to produce one unit of a good is reduced, then the composition of capital increases. This increase in the composition of capital is accompanied by a rise in both the rate of profit and the rate of exploitation. Our results are consistent with Okishio [10] and Roemer [14].

Finally, we revisited the FRP conjecture from a different perspective: considering the total value of inputs instead of the composition of capital leads to a new FRP law: the rate of profit decreases with this total value.

10 Appendix

Proof of Proposition 2

Let $A = (a_{ij})$ be a $n \times n$ matrix with $a_{ij} > 0$ for any $i, j = 1, \ldots, n$. According to the definition, A is productive if there exists a row vector $p \in \mathbb{R}^n_{++}$ such that $p > pA$, that is if the row vector $p - pA$ has positive components.

Equivalence (1) $\Leftrightarrow$ (2) is the definition of productive matrix.

We prove that (1) $\Leftrightarrow$ (3) following Michel (1984, sec. 9.2).

Necessity: (1) $\Rightarrow$ (3).

Suppose that A is productive and $(I - A)^{-1}$ does not exist. Let $p \in \mathbb{R}^n_{++}$ satisfy $p > pA$. Define $x \equiv p - pA$. Then, $x_i > 0$ for any i. Let $Z \equiv \{z : z(I - A) = 0\}$ be the cokernel of $I - A$. $I - A$ is invertible if and only if $Z = \{0\}$. Hence, there exists $z \in Z$ with $z_i > 0$ for some i. Define $c \equiv \sup_i (z_i/p_i) = z_j/p_j > 0$ for some j. Then,

$$0 < cx_j = cp_j - \sum_{i=1}^{n} a_{ij} cp_i = z_j - \sum_{i=1}^{n} a_{ij} cp_i \leq z_j - \sum_{i=1}^{n} a_{ij} z_i = 0$$

because $z \in Z$: a contradiction. Hence $I - A$ is invertible.

Now, we prove that the coefficients of $(I - A)^{-1}$ are non-negative. Suppose the contrary. This matrix has at least a negative coefficient and, then, there exists a row vector $y \geq 0$ such that the row vector $z = y(I - A)^{-1}$ has at least a negative component. Define $c \equiv \sup_i \{-z_i/p_i\} = -z_j/p_j > 0$ for some j. We obtain

$$0 < cx_j = cp_j - \sum_{i=1}^{n} a_{ij} cp_i \leq -\left(z_j - \sum_{i=1}^{n} a_{ij} z_i\right) = -y_j \leq 0$$

since $y = z(I - A)$, a contradiction. Hence, the coefficients of $(I - A)^{-1}$ are non-negative.

Sufficiency: (3) $\Rightarrow$ (1).

We want to prove that, if $(I - A)^{-1}$ exists and has positive coefficients, then A is productive. Indeed, consider a positive row vector $q \in \mathbb{R}^n_{++}$. Let $p = q(I - A)^{-1}$. Then $p > 0$. We have $p(I - A) = q > 0$. Therefore, A is productive.

Finally, we want to demonstrate that (1) $\Leftrightarrow$ (4).

Necessity: (1) $\Rightarrow$ (4).

Noticing that $x = p(I - A)$, we have: (1) $\Rightarrow$ (3) $\Rightarrow p = x(I - A)^{-1} \geq 0 \Rightarrow$ (4).

Sufficiency: (4) $\Rightarrow$ (1).

It is enough to prove that (4) $\Rightarrow$ (2) because (2) $\Rightarrow$ (1). For any $x \geq 0$, there exists a non-negative solution to the equation $p = pA + x$. Fix $x > 0$ and let $p \geq 0$ be the corresponding solution. We have $p > pA$.

Consider a $\varepsilon > 0$ small enough such that $x' \equiv x + \varepsilon (1, \ldots, 1) (I - A) > 0$. Let $p' \equiv p + \varepsilon (1, \ldots, 1) > 0$. Then,

$$\begin{aligned} p' &= p + \varepsilon (1, \ldots, 1) = pA + x + \varepsilon (1, \ldots, 1) \\ &= [p + \varepsilon (1, \ldots, 1)] A + x + \varepsilon (1, \ldots, 1) (I - A) \\ &= p'A + x' > p'A \end{aligned}$$

Thus, there exists a row vector $p' > 0$ such that $p' > p'A$. ■

Proof of Proposition 5

Necessity.

If the economy is efficient then there exists $p > 0$ which satisfies $p - pA > 0$: A is productive.

Sufficiency.

If the matrix A is productive then the economy is efficient. Indeed, let p be the positive row vector such that $p - pA$ has positive components. Take $w > 0$ small enough such that (9) holds: the economy is efficient. ■

Proof of Lemma 6

If the economy is efficient with a system of prices p and wage w , we have $p \geq pA + wl$ or, equivalently,

$$\frac{p}{w} (I - A) \geq l$$

where l, p and A are given by (2), (4) and (5).

Since the matrix A is productive, $(I - A)^{-1}$ is positive. Then,

$$\frac{p}{w} \geq l (I - A)^{-1}$$

Recalling that $\lambda = l (I - A)^{-1}$, where λ is given by (3), we find $p_i/w \geq \lambda_i$ for any $i = 1, \ldots, n$.

The sector $n+1$ makes no negative profit: $p_{n+1} \geq p_1 a_{1,n+1} + \ldots + p_n a_{n,n+1} + wl_{n+1}$. Hence,

$$\begin{aligned} \frac{p_{n+1}}{w} &\geq \frac{p_1}{w} a_{1,n+1} + \ldots + \frac{p_n}{w} a_{n,n+1} + l_{n+1} \\ &\geq \lambda_1 a_{1,n+1} + \ldots + \lambda_n a_{n,n+1} + l_{n+1} = \lambda_{n+1} \end{aligned}$$

■

Proof of Proposition 9

We know that $\lambda_{n+1} = \lambda_1 a_{1,n+1} + \lambda_2 a_{2,n+1} + \ldots + \lambda_n a_{n,n+1} + l_{n+1}$. Therefore, using Lemma 6, we find

$$\begin{aligned} 1 + \pi &= \frac{p_{n+1}}{p_1 a_{1,n+1} + \ldots + p_n a_{n,n+1} + wl_{n+1}} \\ &\leq \frac{p_{n+1}/w}{\lambda_1 a_{1,n+1} + \ldots + \lambda_n a_{n,n+1} + l_{n+1}} = \frac{p_{n+1}}{w\lambda_{n+1}} < \frac{1}{\omega \lambda_{n+1} b} = 1 + e \end{aligned}$$

that is $e > \pi$. ■

Proof of Proposition 10

On the one hand, $1+e = 1/(b\omega\lambda_{n+1})$ and, on the other hand, according to Lemma 6,

$$\begin{aligned} 1+\pi &= \frac{p_{n+1}}{p_1 a_{1,n+1} + \ldots + p_n a_{n,n+1} + w l_{n+1}} \\ &\leq \frac{p_{n+1}/w}{\lambda_1 a_{1,n+1} + \ldots + \lambda_n a_{n,n+1} + l_{n+1}} = \frac{p_{n+1}}{w\lambda_{n+1}} \end{aligned}$$

Then,

$$e = \pi \Rightarrow \frac{1}{\omega\lambda_{n+1} b} \leq \frac{p_{n+1}}{w\lambda_{n+1}} \Leftrightarrow Tw \leq b p_{n+1}$$

Since $wT \geq b p_{n+1}$, we obtain $wT = b p_{n+1}$ and

$$\frac{1}{b\omega\lambda_{n+1}} = \frac{p_{n+1}}{w\lambda_{n+1}}$$

We claim that $p_i/w = \lambda_i$ for $i = 1, \ldots n$. Suppose the contrary: $p_j/w > \lambda_j$ for some j. Then,

$$\begin{aligned} 1+\pi &= \frac{p_{n+1}}{p_1 a_{1,n+1} + \ldots + p_n a_{n,n+1} + w l_{n+1}} \\ &< \frac{p_{n+1}/w}{\lambda_1 a_{1,n+1} + \ldots + \lambda_n a_{n,n+1} + l_{n+1}} = \frac{p_{n+1}}{w\lambda_{n+1}} = \frac{1}{b\omega\lambda_{n+1}} = 1+e \end{aligned}$$

leading to $\pi < e$: a contradiction.

Since

$$\left[\begin{array}{ccc} \frac{p_1}{w} & \cdots & \frac{p_n}{w} \end{array} \right] = (1+\pi)\left(\left[\begin{array}{ccc} \frac{p_1}{w} & \cdots & \frac{p_n}{w} \end{array} \right] A + l\right)$$

and $p_i/w = \lambda_i$ for $i = 1, \ldots, n$, we obtain $\lambda = (1+\pi)(\lambda A + l)$, where l, λ and A are given by (2), (3) and (5). But the equation of values determination gives $\lambda = \lambda A + l$. Therefore, $\pi = 0$. ■

Proof of Lemma 12

Let $\hat{p} > 0$ be a system of production prices associated with a uniform rate of profit π. Then,

$$p_j = (1+\pi)(p_1 a_{1j} + \ldots + p_n a_{nj} + b\omega p_{n+1} l_j)$$

for $j = 1, \ldots, n+1$.

The assumption of uniform rate of profit writes in matrix terms: $\hat{p}M = \hat{p}/(1+\pi)$. Then, $1/(1+\pi)$ is a left eigenvalue and $\hat{p}$ is a non-negative left eigenvector. One of the corollaries of the Perron-Frobenius Theorem is that there are no other non-negative left eigenvectors except positive multiples of $\hat{p}$, that is all other eigenvectors must have at least one negative or non-real component (see Michel [6]). Thus, $1/(1+\pi)$ is also the largest left eigenvalue associated to $\hat{p} > 0$.

Define $q_j \equiv p_j/(p_1 + \ldots + p_{n+1})$. Clearly, $\hat{q} > 0$ and $\hat{q}$ is a left eigenvector: $\hat{q}M = \hat{q}/(1+\pi)$. This eigenvector is unique because there is only one normalized and non-negative eigenvector representing the left eigenspace generated by $\hat{p}$. ■

Proof of Lemma 14

We know that

$$\begin{aligned}\lambda &= \lambda\hat{A} + l = \lambda\hat{A} + b\omega\lambda_{n+1}l + eb\omega\lambda_{n+1}l > \lambda\hat{A} + b\omega\lambda_{n+1}l \\ \lambda_{n+1} &= \lambda A_{n+1} + l_{n+1} = \lambda A_{n+1} + b\omega\lambda_{n+1}l_{n+1} + eb\omega\lambda_{n+1} > \lambda A_{n+1} + b\omega\lambda_{n+1}l_{n+1}\end{aligned}$$

where l, λ and $\hat{A}$ are given by (2), (3) and (5). More compactly, we have $\hat{\lambda} > \hat{\lambda}M$, where $\hat{\lambda}$ is given by (3), that is M is productive. ■

Proof of Lemma 15

We follow Morishima [9].

Necessity.

Let $\Pi_j/S_j = 1/\alpha > 0$. Since $S_j = \lambda_j - (C_j + V_j)$ and $\Pi_j = p_j - \left(C_j^p + V_j^p\right)$ for $j = 1, \ldots, n+1$, we obtain

$$\left[\begin{array}{ccc}\lambda_1 - \alpha p_1 & \cdots & \lambda_{n+1} - \alpha p_{n+1}\end{array}\right] = \left[\begin{array}{ccc}\lambda_1 - \alpha p_1 & \cdots & \lambda_{n+1} - \alpha p_{n+1}\end{array}\right] M$$

where M is given by (11) or, equivalently,

$$\left[\begin{array}{ccc}\lambda_1 - \alpha p_1 & \cdots & \lambda_{n+1} - \alpha p_{n+1}\end{array}\right](I - M) = 0$$

According to Lemma 14 and Proposition (2), M is productive and $I - M$ is invertible. Then, $\lambda_j = \alpha p_j$ for $j = 1, \ldots, n+1$.

Sufficiency.

Let $\lambda_j = \alpha p_j$ for $j = 1, \ldots, n+1$ for some $\alpha > 0$. Considering expressions $S_j = \lambda_j - (C_j + V_j)$ and $\Pi_j = p_j - \left(C_j^p + V_j^p\right)$ for $j = 1, \ldots, n+1$, we get immediately $S_j = \alpha\Pi_j$ for $j = 1, \ldots, n+1$. ■

Proof of Lemma 16

We follow Morishima [9].

(1) Let $S_1/\Pi_1 = \ldots = S_{n+1}/\Pi_{n+1}$. From Lemma 15, there exists $\alpha > 0$ such that $\lambda_j = \alpha p_j$ for $j = 1, \ldots, n+1$. Let π denote the associated uniform rate of profit. We have

$$\pi = \frac{\Pi_j}{C_j^p + V_j^p} = \frac{S_j}{C_j + V_j} = \frac{eV_j}{C_j + V_j} = \frac{e}{C_j/V_j + 1}$$

for $j = 1, \ldots, n+1$. This implies $C_j/V_j = e/\pi - 1$ for $j = 1, \ldots, n+1$.

(2) Conversely, assume that (12) holds: the ratios $eV_j/(C_j + V_j)$ are the same for any j. Denote this common value by π:

$$\pi = \frac{eV_j}{C_j + V_j}$$

We have $\lambda_i = (1 + \pi)(C_j + V_j)$. Fix $\alpha > 0$ and define the system of prices $\hat{p}$ by $p_j = \lambda_j/\alpha$ with $j = 1, \ldots, n+1$. Since $S_i = \lambda_i - (C_j + V_j) = \pi(C_j + V_j) = \alpha\pi\left(C_j^p + V_j^p\right) = \alpha\Pi_j$, we obtain $S_j/\Pi_j = \alpha$ for $j = 1, \ldots, n+1$.

To end the proof we have to show that the system of prices $\hat{p}$ is defined up to a positive scalar. Since $\lambda_j = (1 + \pi)(C_j + V_j)$, we find

$$p_j = (1 + \pi)(p_1 a_{1j} + \ldots + p_n a_{nj} + b\omega p_{n+1} l_j)$$

for $j = 1, \ldots, n+1$, or, more compactly, $\hat{p} = (1+\pi)\,\hat{p}M$. Since M is positive, $1/(1+\pi)$ is the largest left eigenvalue of M and $\hat{p}$ is the associated positive eigenvector (defined up to a positive scalar). ■

Proof of Proposition 17

Necessity.

Assume that (14) holds. Let e denote the rate of exploitation. Define π as

$$\pi \equiv \frac{eV_j}{C_j + V_j}$$

for $j = 1, \ldots, n+1$. Then, $\lambda_j = (1+\pi)(C_j + V_j)$ for $j = 1, \ldots, n+1$.

Define $q_j = \lambda_j/\alpha$ for $j = 1, \ldots, n+1$ and chose $\alpha > 0$ such that $q_1 + \ldots + q_{n+1} = 1$. In other words, $\hat{q}$ is a normalized system of prices. We obtain

$$q_j = (1+\pi)(q_1 a_{1j} + \ldots + q_n a_{nj} + q_{n+1} b\omega l_j) = (1+\pi)\left(C_j^p + V_j^p\right)$$

for $j = 1, \ldots, n+1$. The system of prices $\hat{q}$ is a system of normalized production prices.

Sufficiency.

Assume $\lambda_j = \alpha q_j$ for $j = 1, \ldots, n+1$ with $\alpha > 0$ and $\hat{q}$ a normalized system of production prices. Recall that $\lambda_j = C_j + V_j + S_j = C_j + (1+e)\,V_j$ for $j = 1, \ldots, n+1$. Let π be the uniform rate of profit associated with the system of prices $\hat{q}$. We have $p_j = (1+\pi)(p_1 a_{1j} + \ldots + p_n a_{nj} + b\omega p_{n+1} l_j)$ for $j = 1, \ldots, n+1$, that is

$$\frac{\lambda_j}{\alpha} = \frac{1+\pi}{\alpha}(\lambda_1 a_{1j} + \lambda_2 a_{2j} + \ldots + \lambda_n a_{nj} + b\omega\lambda_{n+1} l_j) = \frac{1+\pi}{\alpha}(C_j + V_j)$$

Then, $C_j + (1+e)\,V_j = (1+\pi)(C_j + V_j)$ and $C_j/V_j = e/\pi - 1$ for $j = 1, \ldots, n+1$. ■

Proof of Corollary 18

According to (10), we have

$$\frac{C_j}{V_j} = \frac{\lambda_1 a_{1j} + \ldots + \lambda_n a_{nj}}{b\omega\lambda_{n+1} l_j}$$

for $j = 1, \ldots, n+1$.

Consider the equations of value determination: $\lambda = l\,(I-A)^{-1}$ where l, λ and A are given by (2), (3) and (5). We have

$$\frac{C_j}{V_j} = \frac{l\,(I-A)^{-1} A_j}{\omega b \lambda_{n+1} l_j} \tag{19}$$

for $j = 1, \ldots, n+1$, where A_j is given by (6). Thus, $C_1/V_1 = \ldots = C_{n+1}/V_{n+1}$ if and only if (15) holds, that is Condition $\mathcal{C}$.

Moreover, under condition $\mathcal{C}$, we have $C_j/V_j = C/V$ for $j = 1, \ldots, n+1$, where $C \equiv C_1 + \ldots + C_{n+1}$ and $V \equiv V_1 + \ldots + V_{n+1}$ represent the total of constant and variable capital, respectively. Finally, the rate of profit can be written as (1). ■

Proof of Proposition 19

Focus on equations $C_j/V_j = e/\pi - 1$ with $j = 1, \ldots, n+1$. Since $C_j/V_j = C/V$, we have (1).

We know also that $(1+e)(b\omega\lambda_{n+1}) = 1$. Under Assumption 6, $A = aJ$, where

$$J \equiv \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

is the $n \times n$ square matrix with all elements equal to 1, and $na < 1$. In this case, the matrix A is productive.

We observe that

$$A = aJ = aJ\frac{1-na}{1-na} = \frac{aJ - (aJ)^2}{1-na} = \frac{A - A^2}{1-na} = \frac{A(I-A)}{1-na}$$

since $J^2 = nJ$. Thus,

$$I = I - A + \frac{A}{1-na}(I-A) = \left(I + \frac{A}{1-na}\right)(I-A)$$

and, finally,

$$(I-A)^{-1} = I + \frac{aJ}{1-na} \tag{20}$$

Denote by $\mathbf{1}$ the row with all elements equal to 1, and its transposed $\mathbf{1}^T$. The expressions of C_j/V_j become

$$\frac{C_j}{V_j} = (1+e)\frac{al(I-A)^{-1}\mathbf{1}^T}{l_j}$$

Condition $\mathcal{C}$ simplifies: $l_1 = \ldots = l_{n+1} \equiv l$ (labor times are the same across sectors) and $C_j/V_j = C/V$ for $j = 1, \ldots, n+1$.

We obtain

$$\frac{C}{V} = (1+e)\left[a\mathbf{1}(I-A)^{-1}\mathbf{1}^T\right] \tag{21}$$

and, using (20),

$$a\mathbf{1}(I-A)^{-1}\mathbf{1}^T = a\left(\mathbf{1}I\mathbf{1}^T + \frac{\mathbf{1}A\mathbf{1}^T}{1-na}\right) = a\left(n + \frac{an^2}{1-na}\right) = \frac{an}{1-an} \tag{22}$$

Therefore,

$$\lambda\mathbf{1} = l\mathbf{1}(I-A)^{-1} = l\left(\mathbf{1}I + \frac{a}{1-na}\mathbf{1}J\right) = l\left(\mathbf{1} + \frac{an}{1-na}\mathbf{1}\right)$$

that is $\lambda = l/(1-na)$ and

$$1 + e = \frac{1}{b\omega\lambda} = \frac{1-na}{b\omega l} \tag{23}$$

Under Assumption 6, the real total unit production cost is the same in any sector: $\sigma \equiv na + \omega bl < 1$, and we obtain explicit but simple expressions for the capital composition and the rates of exploitation and profit, based on the fundamental parameters.

More precisely, from (21), (22) and (23), we find the capital composition (16).

From (23), we obtain also the exploitation rate in (17).

Finally, we get

$$1 + \pi = 1 + \frac{e}{1 + C/V} = 1 + \frac{\frac{1-na}{b\omega l} - 1}{1 + \frac{na}{b\omega l}} = \frac{1}{\sigma} \tag{24}$$

that is the profit rate in (17).

We observe that Assumption 6 implies $\sigma < 1$, that is $e, \pi > 0$. ■

Proof of Corollary 20

Simply consider expressions (16) and (17). ■

Proof of Corollary 21

Simply consider expressions (16) and (17). ■

Proof of Proposition 23

Under condition (14), there exist $\alpha > 0$ and a normalized vector $\hat{q} > 0$ such that $\sum_{i=1}^{n+1} q_i = 1$ and $\lambda_i = \alpha q_i$ for $i = 1, \ldots, n+1$. According to (14),

$$\frac{C}{V} = \frac{\sum_{i=1}^{n+1} C_i}{\sum_{i=1}^{n+1} V_i} = \frac{\sum_{i=1}^{n+1} \frac{C_i}{V_i} V_i}{\sum_{i=1}^{n+1} V_i} = \frac{C_j}{V_j} \frac{\sum_{i=1}^{n+1} V_i}{\sum_{i=1}^{n+1} V_i} = \frac{C_j}{V_j}$$

Then,

$$\begin{aligned} \frac{C_i}{V_i} &= \frac{C}{V} = \frac{\sum_{j=1}^{n+1} C_j}{\sum_{j=1}^{n+1} V_j} = \frac{\lambda_1 \sum_{j=1}^{n+1} a_{1j} + \ldots + \lambda_n \sum_{j=1}^{n+1} a_{nj}}{b\omega\lambda_{n+1} \sum_{j=1}^{n+1} l_j} \\ &= \frac{\lambda_1 a_1 + \ldots + \lambda_n a_n}{\lambda_{n+1}\omega b L} = \frac{q_1 a_1 + \ldots + q_n a_n}{b\omega q_{n+1} L} = \frac{\mathcal{K}}{\mathcal{L}} \end{aligned}$$

Replacing $C/V = \mathcal{K}/\mathcal{L}$ in (1), which holds under Condition $\mathcal{C}$ (see Corollary 18), we obtain the left-hand equation in (18).

Recalling that $q_j = (1 + \pi)(q_1 a_{1j} + \ldots + q_n a_{nj} + b\omega q_{n+1} l_j)$ for $j = 1, \ldots, n+1$, and summing over j, we get

$$1 = \sum_{j=1}^{n+1} q_j = (1 + \pi)(q_1 a_1 + \ldots + q_n a_n + b\omega q_{n+1} L)$$

that is the right-hand equation in (18). ■

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