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Abstract

This paper is dedicated to an axiomatic study of the Myerson value for cooperative games in which the set of feasible coalitions is a union-stable system. This type of partial cooperation structure generalizes well-known communication graph games and contains the widely studied union-closed and voting structures. In this framework, we provide a simple and intuitive characterization of the Myerson value using five appealing and independent axioms. We show that the Myerson value is the only allocation rule on the set of union-stable structures that satisfies component-efficiency, additivity, modularity, the extra-null player property, and equal treatment of veto players. We also show that this characterization is valid for the restricted class of union-closed structures.

Key words: TU-game, Union-stable structure, Harsanyi dividends, Harsanyi power solution, Myerson value.

1 Introduction

A cooperative game with transferable utilities (TU-game) is a game in which individuals (or agents, or players) can cooperate with each other in order to increase their benefits. A *coalition* is a group of players who are able to work together. In cooperative game theory, it is often assumed that any coalition can form; that is, any player can cooperate with any other player, making all coalitions feasible. However, many real-life situations highlight the plausible lack of cooperation between certain agents, which implies that the set of feasible coalitions cannot be the collection of all subsets of the set of players. One of the most appealing models of cooperative games with limited cooperation was introduced by [Myerson \(1977\)](#), who defined communication graph games. These games are modeled by an undirected graph in which each player is represented by a vertex and edges represent possible cooperation between players. A feasible coalition is therefore a coalition of players who are connected within the graph.

The foundational work of [Myerson \(1977\)](#) paved the way for subsequent research. Notably, [Gilles et al. \(1992\)](#) introduced cooperative games on permission structures, in which some players must obtain permission from others before joining any coalition. In addition, [Algaba et al. \(2000, 2001\)](#) studied cooperative games on union-stable structures, where the union of any two feasible coalitions is feasible whenever they share at least one common member. More recently, [van den](#)

Brink et al. (2011b) introduced cooperative games on union-closed systems, in which the union of any two feasible coalitions is feasible even if they are disjoint. Furthermore, Algaba et al. (2019) and Gassi (2025) examined cooperative games on voting structures, where any superset of a feasible coalition is also feasible. All these models are highly relevant for apprehend many economic applications, as emphasized by van den Brink (2017).

The study by Algaba et al. (2000) is particularly notable for introducing cooperative games on union-stable structures, which generalize the communication graph structures proposed by Myerson (1977). A union-stable structure is characterized by the property that if two feasible coalitions have a non-empty intersection, then their union is also feasible. Intuitively, the players in the intersection ensure the connection between the two coalitions, allowing all members to cooperate through these common players. This interpretation highlights how this framework extends the communication graph setting. Moreover, the class of union-stable structures encompasses other important cooperation structures, such as union-closed structures introduced by van den Brink et al. (2011b), where the union of any two feasible coalitions (even if disjoint) is feasible, and both the empty set and the grand coalition (the set of all players) are always feasible. Another relevant subclass is that of voting structures, recently studied by Algaba et al. (2019) and Gassi (2025). This class of cooperation structures is characterized by a monotonicity property: any coalition that contains a feasible coalition is also feasible. In this paper, we focus on the broad class of union-stable structures.

The problem of sharing the benefits arising from cooperation among agents remains a central issue in cooperative game theory. For the class of cooperative games on union-stable structures considered in this paper, a well-known family of allocation rules is the class of Harsanyi power solutions, studied by Algaba et al. (2015). These solutions distribute the worth of each coalition among its members proportionally to a *power measure*. A power measure is a mapping that assigns to each player a non-negative real number representing his/her cooperation power. The work of van den Brink et al. (2011a) introduces and provides an axiomatic characterization of Harsanyi power solutions for graph-restricted games, while Algaba et al. (2015) extends this characterization to cooperative games on union-stable structures. Subsequently, Algaba et al. (2019) focuses on the specific class of voting structures. From these latter works, the Myerson value (Myerson, 1977), defined as the Shapley value of the TU-game associated with a given union-stable structure, can be interpreted as the Harsanyi power solution corresponding to the equal power measure yielding the same cooperation power to all players who can cooperate with at least one other player. It is worth noting that Harsanyi power solutions generally require prior knowledge of the players' power measures, as payoffs are distributed accordingly. This is consistent with their axiomatic characterization, which relies on these given measures. In contrast, since the power measure underlying the Myerson value assigns equal power to all relevant players, this value can then be characterized independently of any exogenous specification of power. The work of Gassi (2025) provides an axiomatic characterization of the Myerson value on the subclass of voting structures using five appealing and independent axioms. In this paper, we adapt these axioms to obtain an equivalent characterization of the Myerson value for the broader class of union-stable structures. We show that the Myerson value is the unique allocation rule on union-stable structures satisfying Component-Efficiency, Additivity, Modularity, the Extra-Null

Player property, and Equal Treatment of Veto players. The first three axioms have already been used by [Algaba et al. \(2015\)](#) and [Algaba et al. \(2019\)](#) to characterize Harsanyi power solutions for union-stable and voting structures. The remaining two axioms capture how an allocation rule should account for both the production and cooperation abilities of players. The Extra-Null Player property requires that a player with neither cooperation ability nor productive contribution receives a zero payoff. The Equal Treatment of Veto players requires that a player who is decisive for cooperation receives the same payoff as a player who is decisive for production. Finally, we show that this characterization also holds for the subclass of union-closed structures.

The rest of the paper is organized as follows: Section 2 lays out the model with some preliminary definitions and results, Section 3 is devoted to defining the axioms and presenting the characterization on the class of union-stable structures. Section 4 establish the validity of the characterization on the subclass of union closed structures. Section 5 show the independence of the axioms on the class of union-stable structures and, Section 6 concludes and outlines the main direction for further research.

2 Preliminaries

Throughout the paper, the cardinality of a finite set S will be denoted by $|S|$ and the collection of the subsets of S will be denoted by 2^S . Moreover, for simplicity, any singleton set $\{i\}$ will be written as i .

2.1 Cooperative games

Let $N = \{1, \dots, n\}$ be a finite set of agents called *players*. Any subset S of N is called *coalition* and N itself is called the *grand coalition*. A cooperative game with transferable utilities (TU-game) is defined as a pair (N, v) , where N is the set of players and $v : 2^N \rightarrow \mathbb{R}$ is the coalition function satisfying $v(\emptyset) = 0$. For each coalition $S \subseteq N$, $v(S)$ represents the worth of S if its members cooperate. We denote by G^N the set of all TU-games on the player set N . Since the set of players is fixed throughout the paper, we will identify any TU-game (N, v) simply by v for ease of notations.

We say that two distinct players $i, j \in N$ are symmetric in the game v if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$. Player i is called null player in v if $v(S \cup i) = v(S)$ for all $S \subseteq N \setminus i$.

For each coalition $S \subseteq N$, the *unanimity game* u_S is defined by $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. It is well-known that G^N is a vector space and that the family of unanimity games $(u_S)_{\emptyset \neq S \subseteq N}$ forms a basis of G^N . More precisely, each TU-game v can be uniquely decomposed as

$$v = \sum_{S \subseteq N, S \neq \emptyset} \Delta_v(S) u_S \quad (1)$$

where the coefficients $\Delta_v(S)$ are called the *Harsanyi dividends* ([Harsanyi, 1959](#)) of the coalitions $S \subseteq N$ in the game v , and are computed using the following recursive formula:

$$\Delta_v(S) = v(S) - \sum_{T \subset S} \Delta_v(T), \quad \forall S \subseteq N, S \neq \emptyset. \quad (2)$$

A *solution* on G^N is any mapping φ that assigns to each game $v \in G^N$ and each player $i \in N$, a numerical value $\varphi_i(v) \in \mathbb{R}$ called the *payoff* of player i in the game v . The most well-known solution in the literature of cooperative games is the *Shapley value* Φ (Shapley, 1953), which shares the benefit of cooperation (the Harsanyi dividend) of each coalition equally among its members. Formally, the Shapley value is defined as follows:

$$\Phi_i(v) = \sum_{S \subseteq N, i \in S} \frac{1}{|S|} \Delta_v(S), \quad \forall i \in N. \quad (3)$$

2.2 Union stable structures

Let $N = \{1, \dots, n\}$ be a finite set of players and $\mathcal{F} \subseteq 2^N$ be a set system of feasible coalitions. The set system \mathcal{F} is called *union-stable* if for all $A, B \in \mathcal{F}$ with $A \cap B \neq \emptyset$, we have $A \cup B \in \mathcal{F}$. In other words, if any two feasible coalitions have at least one common player, then that player ensures the connection between the two coalitions in such a way that their union remains feasible. It is not hard to verify that for any two union-stable systems $\mathcal{F}_1, \mathcal{F}_2 \subseteq 2^N$, the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ is also union-stable. However, $\mathcal{F}_1 \cup \mathcal{F}_2$ may fail to be union-stable. To see this, consider the player set $N = \{1, 2, 3, 4\}$, $\mathcal{F}_1 = \{\{1, 2\}, \{2, 3, 4\}, N\}$, and $\mathcal{F}_2 = \{\{1, 3\}, \{2, 3, 4\}, N\}$. It can be checked that $\{1, 2\} \cap 1, 3 \neq \emptyset$, but $\{1, 2, 3\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$. Thus, $\mathcal{F}_1 \cup \mathcal{F}_2$ is not union-stable.

A *union-stable structure* is a triplet (N, v, \mathcal{F}) where (N, v) is a TU-game and \mathcal{F} is a union-stable system on the player set N . We denote by US^N the set of all union-stable structures. Union-stable structures were introduced in Algaba et al. (2000) to generalize communication graph games by Myerson (1977). Many real-world situations find a natural framework in these structures. For instance (see also Algaba et al., 2015), suppose that player 1 is a homeowner who wants to sell his/her house. Player 1 has signed a contract with a real estate agent, player 2, and therefore can only sell the house through player 2. There are two buyers, players 3 and 4. In this setting, the family of feasible coalitions that can generate a surplus consists of those that allow the seller to complete the transaction. Therefore, the coalitions that can trade are

$$\mathcal{F} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}.$$

Let \mathcal{F} be a union-stable system. The *basis* of \mathcal{F} is the subset $\mathcal{B}(\mathcal{F})$ defined by

$$\mathcal{B}(\mathcal{F}) = \mathcal{F} \setminus \mathcal{E}(\mathcal{F})$$

where

$$\mathcal{E}(\mathcal{F}) = \{G \in \mathcal{F} : G = A \cup B, A \neq G, B \neq G, A, B \in \mathcal{F}, A \cap B \neq \emptyset\}.$$

Indeed, defining the sequence of set systems: $\mathcal{B}^{(0)} = \mathcal{B}(\mathcal{F})$, and $\mathcal{B}^{(p)} = \{A \cup B : A, B \in \mathcal{B}^{(p-1)}, A \cap B \neq \emptyset\}$ (for all $p \geq 1$), it holds that there exists an integer k , such that $\mathcal{B}^{(k)} = \mathcal{F}$. The elements of the basis are called the *supports* of \mathcal{F} .

Let $\mathcal{F} \subseteq 2^N$ be a union-stable structure and $\mathcal{B}(\mathcal{F}) = \{B_1, B_2, \dots, B_m\}$ its basis of \mathcal{F} . For any $j \in \{1, 2, \dots, m\}$, let

$$\mathcal{F}_j = \{A \in \mathcal{F} | B_j \subseteq A\}; \quad (4)$$

that is, \mathcal{F}_j stands for the set of all feasible coalitions according to \mathcal{F} , containing the support B_j . Note that $\mathcal{F} = \cup_{j=1}^m \mathcal{F}_j$.

Proposition 1 *If $\mathcal{B}(\mathcal{F}) = \{B_1, B_2, \dots, B_m\}$ is the basis of \mathcal{F} , then for any nonempty $J \subseteq \{1, 2, \dots, m\}$, $\cup_{j \in J} \mathcal{F}_j$ is union-stable.*

Proof. Let $\emptyset \neq J \subseteq \{1, 2, \dots, m\}$, and $A, B \in \cup_{j \in J} \mathcal{F}_j$. There exists $j_1, j_2 \in J$ such that $A \in \mathcal{F}_{j_1}$ and $B \in \mathcal{F}_{j_2}$. This implies that $B_{j_1} \subseteq A$ and $B_{j_2} \subseteq B$. Therefore, $B_{j_1}, B_{j_2} \subseteq A \cup B$, and $A \cup B \in \cup_{j \in J} \mathcal{F}_j$. Hence, $\cup_{j \in J} \mathcal{F}_j$ is union-stable. \blacksquare

Let $\mathcal{F} \subseteq 2^N$ a union-stable system and let $S \subseteq N$. A set $T \subseteq S$ is called a \mathcal{F} -component of S if $T \in \mathcal{F}$ and there is no $T' \in \mathcal{F}$ such that $T' \subset T \subseteq S$. In other words, the \mathcal{F} -component of S are the maximal feasible coalitions (according to \mathcal{F}) that are contained in S . We denote by $C_{\mathcal{F}}(S)$ the set all \mathcal{F} -components of S . Note that for all $S \subseteq N$ with $C_{\mathcal{F}}(S) \neq \emptyset$, the \mathcal{F} -components of S are pairwise disjoint. The \mathcal{F} -restricted game associated to the union-stable structure (N, v, \mathcal{F}) is the TU-game $v^{\mathcal{F}} : 2^N \rightarrow \mathbb{R}$ defined on the player set N and given by $v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(S)$ if $C_{\mathcal{F}}(S) \neq \emptyset$, and $v^{\mathcal{F}}(S) = 0$ otherwise.

Proposition 2 *Let (N, v, \mathcal{F}_1) and (N, v, \mathcal{F}_2) be any two union-stable structures. If $\mathcal{F}_1 \cup \mathcal{F}_2$ is union-stable, then we have*

$$\forall S \subseteq N, v^{\mathcal{F}_1 \cup \mathcal{F}_2}(S) = v^{\mathcal{F}_1}(S) + v^{\mathcal{F}_2}(S) - v^{\mathcal{F}_1 \cap \mathcal{F}_2}(S).$$

Proof. Consider a coalition $S \subseteq N$. Four possibilities arise:

- If $C_{\mathcal{F}_1}(S) = C_{\mathcal{F}_2}(S) = \emptyset$, then $v^{\mathcal{F}_1 \cup \mathcal{F}_2}(S) = 0 = v^{\mathcal{F}_1}(S) = v^{\mathcal{F}_2}(S) = v^{\mathcal{F}_1 \cap \mathcal{F}_2}(S)$.
- If $C_{\mathcal{F}_1}(S) \neq \emptyset$ and $C_{\mathcal{F}_2}(S) = \emptyset$, then $C_{\mathcal{F}_1 \cup \mathcal{F}_2}(S) = C_{\mathcal{F}_1}(S)$ and $v^{\mathcal{F}_1 \cap \mathcal{F}_2}(S) = v^{\mathcal{F}_2}(S) = 0$. Therefore, $v^{\mathcal{F}_1 \cup \mathcal{F}_2}(S) = v^{\mathcal{F}_1}(S) = v^{\mathcal{F}_1}(S) + v^{\mathcal{F}_2}(S) - v^{\mathcal{F}_1 \cap \mathcal{F}_2}(S)$.
- If $C_{\mathcal{F}_2}(S) \neq \emptyset$ and $C_{\mathcal{F}_1}(S) = \emptyset$, then $C_{\mathcal{F}_1 \cup \mathcal{F}_2}(S) = C_{\mathcal{F}_2}(S)$ and $v^{\mathcal{F}_1 \cap \mathcal{F}_2}(S) = v^{\mathcal{F}_1}(S) = 0$. Therefore, $v^{\mathcal{F}_1 \cup \mathcal{F}_2}(S) = v^{\mathcal{F}_2}(S) = v^{\mathcal{F}_1}(S) + v^{\mathcal{F}_2}(S) - v^{\mathcal{F}_1 \cap \mathcal{F}_2}(S)$.
- If $C_{\mathcal{F}_2}(S) \neq \emptyset$ and $C_{\mathcal{F}_1}(S) \neq \emptyset$, then

$$\begin{aligned} v^{\mathcal{F}_1}(S) + v^{\mathcal{F}_2}(S) - v^{\mathcal{F}_1 \cap \mathcal{F}_2}(S) &= \sum_{T \in C_{\mathcal{F}_1}(S)} v(T) + \sum_{T \in C_{\mathcal{F}_2}(S)} v(T) - \sum_{T \in C_{\mathcal{F}_1 \cap \mathcal{F}_2}(S)} v(T) \\ &= \sum_{T \in C_{\mathcal{F}_1 \setminus \mathcal{F}_1 \cap \mathcal{F}_2}(S)} v(T) + \sum_{T \in C_{\mathcal{F}_2}(S)} v(T) \\ &= \sum_{T \in C_{\mathcal{F}_1 \cup \mathcal{F}_2}(S)} (S)v(T) \\ &= \sum_{T \in C_{\mathcal{F}_1 \cup \mathcal{F}_2}(S)} v(T). \end{aligned}$$

■

An *allocation rule* on US^N is any mapping φ assigning to each union-stable structure (N, v, \mathcal{F}) and each player $i \in N$ a numerical value $\varphi_i(N, v, \mathcal{F}) \in \mathbb{R}$, which represents the payoff of player i in the structure (N, v, \mathcal{F}) . A well-known family of allocation rules on US^N is the class of *Harsanyi power solutions*, which distribute the Harsanyi dividends of the coalitions in the \mathcal{F} -restricted game among their members proportionally to a power measure (see [Algaba et al., 2015](#)). A *power measure* is a mapping σ assigning to each player $i \in N$ a non-negative real number $\sigma_i(\mathcal{F})$, representing the cooperation power of player i in the collection \mathcal{F} . Given a power measure σ , the *Harsanyi power solution* associated with σ is the allocation rule φ^σ defined by:

$$\forall i \in N, \varphi_i(N, v, \mathcal{F}) = \sum_{\substack{S \subseteq N, i \in S \\ \sum_{j \in S} \sigma_j(\mathcal{F}_S) > 0}} \frac{\sigma_i(\mathcal{F}_S)}{\sum_{j \in S} \sigma_j(\mathcal{F}_S)} \Delta_{v, \mathcal{F}}(S) \quad (5)$$

where $\mathcal{F}_S = \{F \in \mathcal{F} : F \subseteq S\}$.

For any union-stable structure $(N, v, \mathcal{F}) \in US^N$, it is clear that for all coalitions $S \notin \mathcal{F}$, $\Delta_{v, \mathcal{F}}(S) = 0$.¹

The *Myerson value* μ on US^N is the allocation rule defined as the Shapley value of the \mathcal{F} -restricted game associated to each union-stable structure (N, v, \mathcal{F}) ; that is, for each $(N, v, \mathcal{F}) \in US^N$,

$$\mu_i(N, v, \mathcal{F}) = \sum_{S \subseteq N, i \in S} \frac{1}{|S|} \Delta_{v, \mathcal{F}}(S) = \sum_{\substack{S \subseteq N, S \in \mathcal{F} \\ i \in S}} \frac{1}{|S|} \Delta_{v, \mathcal{F}}(S), \quad \forall i \in N. \quad (6)$$

From (5) and (6), we can see that the Myerson value μ on US^N is the Harsanyi power solution associated to the equal power measure $\sigma = E$, defined by

$$E_i(\mathcal{F}) = \begin{cases} 1 & \text{if } \mathcal{C}_i(\mathcal{F}) \neq \emptyset \\ 0 & \text{if } \mathcal{C}_i(\mathcal{F}) = \emptyset \end{cases}$$

where for all $i \in N$, $\mathcal{C}_i(\mathcal{F}) = \{B \in \mathcal{B}(\mathcal{F}) : i \in B, |B| \geq 2\}$.

3 Characterization of the Myerson value on US^N

The work of [van den Brink et al. \(2011a\)](#) provides a characterization of the Harsanyi power solutions on the class of graph-restricted games and [Algaba et al. \(2015\)](#) provide an extension of this characterization on the class of union-stable structures. This latter work characterizes the Myerson value within the family of Harsanyi power solutions, based on the underlying power measure. In this paper, we provide a simple characterization of the Myerson value for union-stable structures, without considering the family of Harsanyi solutions and without using any power measure. We consider two common axioms used by [Algaba et al. \(2015\)](#) to characterize the family of Harsanyi power solutions, namely Component-Efficiency (CE) and Additivity (A).

Component-efficiency (CE): An allocation rule φ on US^N satisfies component-efficiency if for all $(N, v, \mathcal{F}) \in US^N$ and $M \in \mathcal{C}_{\mathcal{F}}(N)$, we have $\sum_{i \in M} \varphi_i(N, v, \mathcal{F}) = v(M)$.

¹The proof is given in [Algaba et al. \(2015\)](#).

Additivity (A): An allocation rule φ on US^N satisfies additivity if for all $(N, u, \mathcal{F}), (N, v, \mathcal{F}) \in US^N$, we have

$$\varphi(N, u + v, \mathcal{F}) = \varphi(N, u, \mathcal{F}) + \varphi(N, v, \mathcal{F}).$$

It is known from [Algaba et al. \(2001\)](#) that the Myerson value satisfies component-efficiency and additivity.

We consider three axioms that are extensions of the axioms introduced by [Gassi \(2025\)](#) for voting structures, namely Extra-Null Player (ENP), Modularity (M), and Equal Treatment of Veto (ETV).

Extra-Null Player (ENP): Given $(N, v, \mathcal{F}) \in US^N$, a player $i \in N$ is an extra-null player in the structure (N, v, \mathcal{F}) , if

- for all $S \subseteq N, S \in \mathcal{F} \Rightarrow S \setminus i \in \mathcal{F}$;
- for all $S \subseteq N, v(S) = v(S \setminus i)$.

In other words, an extra-null player is a null player in the game (N, v) , who does not help any coalition to turn feasible (so he has no cooperation power).

An allocation rule φ on US^N satisfies the extra-null player property if for all $(N, v, \mathcal{F}) \in US^N$, $\varphi_i(N, v, \mathcal{F}) = 0$ whenever i is an extra-null player.

Modularity (M): An allocation rule φ on US^N satisfies modularity if for all $(N, v, \mathcal{F}_1), (N, v, \mathcal{F}_2) \in US^N$ such that $\mathcal{F}_1 \cup \mathcal{F}_2$ is union-stable, it holds that

$$\varphi_i(N, v, \mathcal{F}_1 \cup \mathcal{F}_2) = \varphi_i(N, v, \mathcal{F}_1) + \varphi_i(N, v, \mathcal{F}_2) - \varphi_i(N, v, \mathcal{F}_1 \cap \mathcal{F}_2)^2.$$

The last property aims to ensure fairness between a player who is decisive for cooperation in \mathcal{F} , and a player who is decisive for the production in the game (N, v)

Equal Treatment of Veto (ETV): Let (N, v, \mathcal{F}) and a $i \in N$. We say that player i has:

- a *cooperation veto* if for all $S \subseteq N, S \in \mathcal{F} \Rightarrow i \in S$;
- a *production veto* if for all $S \subseteq N, v(S) \neq 0 \Rightarrow i \in S$.

Denote by $CV(N, v, \mathcal{F})$ and $PV(N, v, \mathcal{F})$ the set of players who have the cooperation veto and the production veto in (N, v, \mathcal{F}) , respectively.

An allocation rule φ on US^N satisfies the equal treatment of veto if for all $(N, v, \mathcal{F}) \in US^N$ and $i, j \in N$, if $i \in CV(N, v, \mathcal{F})$ and $j \in PV(N, v, \mathcal{F})$, then $\varphi_i(N, v, \mathcal{F}) = \varphi_j(N, v, \mathcal{F})$.

Theorem 1 *The Myerson value $\mu: US^N \rightarrow \mathbb{R}^n$ satisfies (ENP), (M), and (ETV).*

Proof.

1. Let us show that μ satisfies (ENP). Consider a structure $(N, v, \mathcal{F}) \in US^N$ and an extra-null player $i \in N$. We just have to show that i is a null player in the TU-game $v^{\mathcal{F}}$.

²The modularity axiom is similar to the transfer axiom for simple games by [Dubey \(1975\)](#), and another version can be found in [Laruelle and Valenciano \(2001\)](#).

Consider $S \subseteq N \setminus i$. First, assume that there exists some $T \subseteq S$ such that $T \cup i \in C_{\mathcal{F}}(S \cup i)$. Then by definition of $C_{\mathcal{F}}(S \cup i)$, it follows that $T \cup i \in \mathcal{F}$ and $K \subseteq S \setminus T$ for all $K \in C_{\mathcal{F}}(S \cup i)$ and $K \neq T$. Since i is an extra-null player in the structure (N, v, \mathcal{F}) , it follows that $T \in \mathcal{F}$ and $T \in C_{\mathcal{F}}(S)$. Therefore,

$$\begin{aligned} v^{\mathcal{F}}(S \cup i) &= v(T \cup i) + \sum_{K \subseteq S \setminus T: K \in C_{\mathcal{F}}(S \cup i)} v(K) \\ &= v(T) + \sum_{K \subseteq S \setminus T: K \in C_{\mathcal{F}}(S)} v(K) \text{ since } i \text{ is a null player in the game } v \\ &= v^{\mathcal{F}}(S) \end{aligned}$$

Now assume that there exists no $T \subseteq S$ such that $T \cup i \in C_{\mathcal{F}}(S \cup i)$. Then $C_{\mathcal{F}}(S \cup i) = C_{\mathcal{F}}(S)$. Therefore $v^{\mathcal{F}}(S \cup i) = v^{\mathcal{F}}(S)$ by definition of $v^{\mathcal{F}}$ and $C_{\mathcal{F}}(S)$. This proves that i is a null player in the game in $v^{\mathcal{F}}$ and consequently, we have $\mu_i(N, v, \mathcal{F}) = \Phi_i(N, v) = 0$ since the Shapley value has the null player property.

2. Let us show that μ satisfies (M). Consider $(N, v, \mathcal{F}_1), (N, v, \mathcal{F}_2) \in US^N$ such that $\mathcal{F}_1 \cup \mathcal{F}_2$ is union-stable. By Proposition 2, it holds that

$$\forall S \subseteq N, v^{\mathcal{F}_1 \cup \mathcal{F}_2}(S) = v^{\mathcal{F}_1}(S) + v^{\mathcal{F}_2}(S) - v^{\mathcal{F}_1 \cap \mathcal{F}_2}(S).$$

Now by definition of μ , we have

$$\mu_i(v, \mathcal{F}_1 \cup \mathcal{F}_2) = \Phi_i(v^{\mathcal{F}_1 \cup \mathcal{F}_2}) = \sum_{S \subseteq N, i \in N} \frac{1}{|S|} \Delta_{v^{\mathcal{F}_1 \cup \mathcal{F}_2}}(S). \quad (7)$$

Recall that the Harsanyi dividend $\Delta_v(S)$ of a coalition S in a game v is defined by the recursive formula

$$\Delta_v(S) = v(S) - \sum_{T \subsetneq S} \Delta_v(T),$$

which can also be written as

$$\Delta_v(S) = \sum_{T \subsetneq S} (-1)^{|S|-|T|} v(T) \text{ (see Harsanyi, 1959).}$$

We can see that $\Delta_v(S)$ is linear with respect to v . Hence, applying Proposition 2 in Equation (7), we have

$$\Delta_{v^{\mathcal{F}_1 \cup \mathcal{F}_2}}(S) = \Delta_{v^{\mathcal{F}_1}}(S) + \Delta_{v^{\mathcal{F}_2}}(S) - \Delta_{v^{\mathcal{F}_1 \cap \mathcal{F}_2}}(S).$$

Therefore, it follows that

$$\begin{aligned}
\mu_i(v, \mathcal{F}_1 \cup \mathcal{F}_2) &= \sum_{S \subseteq N, i \in S} \frac{1}{|S|} \Delta_{v^{\mathcal{F}_1}}(S) + \sum_{S \subseteq N, i \in S} \frac{1}{|S|} \Delta_{v^{\mathcal{F}_2}}(S) - \sum_{S \subseteq N, i \in S} \frac{1}{|S|} \Delta_{v^{\mathcal{F}_1 \cap \mathcal{F}_2}}(S) \\
&= \Phi_i(N, v^{\mathcal{F}_1}) + \Phi_i(N, v^{\mathcal{F}_2}) - \Phi_i(N, v^{\mathcal{F}_1 \cap \mathcal{F}_2}) \\
&= \mu_i(N, v, \mathcal{F}_1) + \mu_i(N, v, \mathcal{F}_2) - \mu_i(N, v, \mathcal{F}_1 \cap \mathcal{F}_2).
\end{aligned}$$

Hence, μ satisfies modularity.

3. Let us show that μ satisfies (ETV). Consider a structure $(N, v, \mathcal{F}) \in US^N$ and two players $i, j \in N$ such that $i \in CV(N, v, \mathcal{F})$ and $j \in PV(N, v, \mathcal{F})$. To prove that $\mu_i(N, v, \mathcal{F}) = \mu_j(N, v, \mathcal{F})$, it is sufficient to prove that i and j are symmetric players in the games $v^{\mathcal{F}}$. Consider $S \subseteq N \setminus \{i, j\}$. Note that $i \notin S \cup j$ and $j \notin S \cup i$. Therefore, we have $v^{\mathcal{F}}(S \cup j) = 0$ since $C_{\mathcal{F}}(S \cup j) = \emptyset$ because any feasible coalition should contain i . On the other hand, we have

$$v^{\mathcal{F}}(S \cup i) = \sum_{T \in C_{\mathcal{F}}(S \cup i)} v(T) = 0 \text{ since } j \notin T \text{ for all } T \in C_{\mathcal{F}}(S \cup i).$$

It follows that $v^{\mathcal{F}}(S \cup i) = v^{\mathcal{F}}(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$. Thus i and j are symmetric in the game $v^{\mathcal{F}}$. Since the Shapley value on G^N is symmetric, it follows that $\mu_i(N, v, \mathcal{F}) = \Phi_i(N, v^{\mathcal{F}}) = \Phi_j(N, v^{\mathcal{F}}) = \mu_j(N, v, \mathcal{F})$. ■

To prove the uniqueness of the Myerson value, we need to establish a crucial proposition on the behavior of allocation rules satisfying modularity.

Let $(N, v, \mathcal{F}) \in US^N$ with the basis $\mathcal{B}(\mathcal{F}) = \{B_1, B_2, \dots, B_m\}$. For any $\emptyset \neq K \subseteq J = \{1, 2, \dots, m\}$, denote $\mathcal{F}_K = \bigcap_{k \in K} \mathcal{F}_k$, where each family \mathcal{F}_k is defined as in (4). Recall that \mathcal{F}_K is union-stable for all $\emptyset \neq K \subseteq J$.

Proposition 3 *Let $\varphi : US^N \rightarrow \mathbb{R}^n$ be allocation rule that satisfies modularity. Then for all $(N, v, \mathcal{F}) \in US^N$, we have*

$$\varphi(N, v, \mathcal{F}) = \sum_{\emptyset \neq K \subseteq J} (-1)^{|K|-1} \varphi(N, v, \mathcal{F}_K). \quad (8)$$

Proof. Let us first recall that $\mathcal{F} = \bigcup_{j \in J} \mathcal{F}_j$. The proof is driven by induction the cardinality m of $|J|$.

- If $m = 1$ the result is straightforward.
- If $m = 2$, we have $J = \{1, 2\}$ and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Therefore, by modularity, we have $\varphi(N, v, \mathcal{F}) = \varphi(N, v, \mathcal{F}_1 \cup \mathcal{F}_2) = \varphi(N, v, \mathcal{F}_1) + \varphi(N, v, \mathcal{F}_2) - \varphi(N, v, \mathcal{F}_1 \cap \mathcal{F}_2)$, which is exactly the formula 8 for $m = 2$.

- Now assume that the result holds for an integer $m \geq 2$, and let us show that it remains true for $|J| = m + 1$. Let $J = \{1, 2, \dots, m, m + 1\}$, $J' = \{2, \dots, m, m + 1\}$, and $\mathcal{F}' = \cup_{j \in J'} \mathcal{F}_j$. Note that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}'$. By modularity we have

$$\begin{aligned}
\varphi(N, v, \mathcal{F}) &= \varphi(N, v, \mathcal{F}_1) + \varphi(N, v, \mathcal{F}') - \varphi(N, v, \mathcal{F}_1 \cap \mathcal{F}') \\
&= \varphi(N, v, \mathcal{F}_1) + \sum_{\emptyset \neq K \subseteq J \setminus \{1\}} (-1)^{|K|-1} \varphi(N, v, \mathcal{F}_K) - \sum_{\emptyset \neq K \subseteq J \setminus \{1\}} (-1)^{|K|-1} \varphi(N, v, \mathcal{F}_{K \cup \{1\}}) \\
&\quad \text{by applying assumption on } \mathcal{F}' \text{ and } \mathcal{F}_1 \cup \mathcal{F}' = \mathcal{F}_1 \cup \left(\bigcap_{j \in J'} \mathcal{F}_j \right) \\
&= \sum_{\emptyset \neq K \subseteq J \setminus \{1\}} (-1)^{|K|-1} \varphi(N, v, \mathcal{F}_K) + \sum_{1 \in K \subseteq J \setminus \{1\}} (-1)^{|K|-1} \varphi(N, v, \mathcal{F}_K) \\
&= \sum_{\emptyset \neq K \subseteq J} (-1)^{|K|-1} \varphi(N, v, \mathcal{F}_K)
\end{aligned}$$

This proves that the result holds for $|J| = m + 1$. Thus it holds for all integers m . ■

Hence, combining additivity and modularity leads to the following corollary.

Corollary 1 *Let $\varphi : US^N \rightarrow \mathbb{R}^n$ be an allocation rule. If φ satisfies Additivity (A) and Modularity (M), then for any union-stable structure $(N, v, \mathcal{F}) \in US^N$, $\varphi(N, v, \mathcal{F})$ is entirely defined by the family of union-stable structures $(N, \alpha_S u_S, \mathcal{F}_K)$, with $S \subseteq N$ and $\emptyset \neq K \subseteq J$.*

We are now ready to establish the uniqueness of the Myerson value on US^N .

Theorem 2 *The Myerson value $\mu : US^N \rightarrow \mathbb{R}^n$ is the unique allocation rule on US^N that satisfies (CE), (A), (M), (ENP), (ETV).*

Proof. It is clear from [Algaba et al. \(2001\)](#) and [Theorem 1](#) that the Myerson value satisfies (CE), (A), (M), (ENP), and (ETV). Let us now show the uniqueness.

Let $\varphi : US^N \rightarrow \mathbb{R}^n$ that satisfies (CE), (A), (M), (ENP), and (ETV). Let $(N, v, \mathcal{F}) \in US^N$ with basis $\mathcal{B}(\mathcal{F}) = \{B_1, B_2, \dots, B_m\}$. Recall that the TU-game v can be written as $v = \sum_{S \in 2^N} \alpha_S u_S$ where for all $S \subseteq N$, α_S is the Harsanyi dividend given in [\(2\)](#). Since φ satisfies (A) and (M), we just have to show that $\varphi(N, \alpha_S u_S, \mathcal{F}_K) = \mu(N, \alpha_S u_S, \mathcal{F}_K)$ for all $S \subseteq N$ and $\emptyset \neq K \subseteq J$.

First, remark that for all $\emptyset \neq K \subseteq J$,

$$\mathcal{F}_K = \bigcap_{k \in K} \mathcal{F}_k = \{A \in \mathcal{F} : \cup_{k \in K} B_k \subseteq A\}.$$

Moreover, it can be checked that $CV(N, \alpha_S u_S, \mathcal{F}_K) = \cup_{k \in K} B_k$ and $PV(N, \alpha_S u_S, \mathcal{F}_K) = S$. Since φ satisfies ETV, we have

$$\forall i, j \in S \cup (\cup_{k \in K} B_k), \varphi_i(N, \alpha_S u_S, \mathcal{F}_K) = \varphi_j(N, \alpha_S u_S, \mathcal{F}_K).$$

Furthermore, any player i who is not in $S \cup (\cup_{k \in K} B_k)$ is an extra-null player in the structure $S \cup (\cup_{k \in K} B_k)$ and since φ satisfies (ENP), it holds that

$$\varphi_i(N, \alpha_{SUS}, \mathcal{F}_K) = 0, \forall i \notin S \cup (\cup_{k \in K} B_k).$$

Now, two cases should be considered:

- If $\{T \in \mathcal{F}_K : S \cup (\cup_{k \in K} B_k) \subseteq T\} = \emptyset$, (which means that there is no $T \in \mathcal{F}_K$ containing S), then $(\alpha_{SUS})^{\mathcal{F}_K}(T) = 0$ for all $T \subseteq N$. Therefore, $(\alpha_{SUS})^{\mathcal{F}_K}$ is the null game and by component efficiency, we have $\varphi_i(N, \alpha_{SUS}, \mathcal{F}_K) = 0 = \Phi_i(N, (\alpha_{SUS})^{\mathcal{F}_K}) = \mu_i(N, \alpha_{SUS}, \mathcal{F}_K)$ for all $i \in N$.
- If $\{T \in \mathcal{F}_K : S \cup (\cup_{k \in K} B_k) \subseteq T\} \neq \emptyset$, let $M \in C_{\mathcal{F}_K}(N)$ such that $S \cup (\cup_{k \in K} B_k) \subseteq M$. M is the \mathcal{F}_K -component of N containing S (it contains $\cup_{k \in K} B_k$ by definition of \mathcal{F}_K). Since φ satisfies (CE), we have

$$\sum_{i \in M} \varphi_i(N, \alpha_{SUS}, \mathcal{F}_K) = \alpha_{SUS}(M) = \alpha_S$$

which implies that

$$\sum_{i \in S \cup (\cup_{k \in K} B_k)} \varphi_i(N, \alpha_{SUS}, \mathcal{F}_K) = \alpha_{SUS}(M) = \alpha_S.$$

Therefore, it follows that for all $i \in N \setminus S \cup (\cup_{k \in K} B_k)$, $\varphi_i(N, \alpha_{SUS}, \mathcal{F}_K) = 0 = \Phi_i(N, (\alpha_{SUS})^{\mathcal{F}_K}) = \mu_i(N, \alpha_{SUS}, \mathcal{F}_K)$, and for all $i \in S \cup (\cup_{k \in K} B_k)$, $\varphi_i(N, \alpha_{SUS}, \mathcal{F}_K) = \frac{\alpha_S}{|S \cup (\cup_{k \in K} B_k)|} = \Phi_i(N, (\alpha_{SUS})^{\mathcal{F}_K}) = \mu_i(N, \alpha_{SUS}, \mathcal{F}_K)$.

Finally, $\varphi(N, \alpha_{SUS}, \mathcal{F}_K) = \mu(N, \alpha_{SUS}, \mathcal{F}_K)$ and by Corollary 1, we deduce that $\varphi(N, v, \mathcal{F}) = \mu(N, v, \mathcal{F})$. \blacksquare

4 Restriction on union closed structures

In this section, we show that the characterization of the Myerson value for union-stable structures provided in the previous section remains valid for the class of union-closed structures introduced by [van den Brink et al. \(2011b\)](#).

Let $N = \{1, \dots, n\}$ be a finite set of players and $\mathcal{G} \subseteq 2^N$ be a collection of feasible coalitions. The collection \mathcal{G} is said to be union-closed if $\emptyset, N \in \mathcal{G}$ and for all $A, B \in \mathcal{G}$, we have $A \cup B \in \mathcal{G}$. By definition, a union-closed system differs from a union-stable structure in that it always contains the empty set and the grand coalition, and the union of any two feasible coalitions is feasible, even if they are disjoint.

Given a union-closed system \mathcal{G} , we can similarly define the basis of \mathcal{G} as

$$\mathcal{B}(\mathcal{G}) = \mathcal{G} \setminus \mathcal{E}(\mathcal{G})$$

where

$$\mathcal{E}(\mathcal{G}) = \{G \in \mathcal{G} : G = A \cup B, A, B \in \mathcal{G}, A \neq G, B \neq G\}.$$

Note that for any two union-closed systems \mathcal{G}_1 and \mathcal{G}_2 , $\mathcal{G}_1 \cap \mathcal{G}_2$ is union closed, but $\mathcal{G}_1 \cup \mathcal{G}_2$ it not necessary union-closed. However, if we denote by $\mathcal{B}(\mathcal{G}) = \{B_1, \dots, B_m\}$ the basis of \mathcal{G} , it can be checked that for all $J \subseteq \{1, \dots, m\}$, $\bigcup_{j \in J} \mathcal{G}_j$ is union-closed, where for all $j \in \{1, \dots, m\}$, \mathcal{G}_j is defined similarly as in (4).

We use the terminology *union-closed structure* to refer to the triplet (N, v, \mathcal{G}) where (N, v) is a TU-game and \mathcal{G} is a union closed system on N , and we denote by UC^N the set of union-closed structures on N . Given a union-closed structure (N, v, \mathcal{G}) , we can define the its associated \mathcal{G} -restricted game by

$$\forall S \subseteq N, v^{\mathcal{G}}(S) = v(T)$$

where T is the largest feasible coalition contained in S . Note that, unlike the union-stable structures, the largest feasible coalition according to \mathcal{G} included in any subset of players is unique.

Proposition 4 *Let $(N, v, \mathcal{G}_1), (N, v, \mathcal{G}_2) \in UC^N$ such that $\mathcal{G}_1 \cup \mathcal{G}_2$ is union closed. Then we have*

$$\forall S \subseteq N, v^{\mathcal{G}_1 \cup \mathcal{G}_2}(S) = v^{\mathcal{G}_1}(S) + v^{\mathcal{G}_2}(S) - v^{\mathcal{G}_1 \cap \mathcal{G}_2}(S).$$

The proof is very similar to that of Proposition 2 and is even simpler, since the largest feasible coalition (i.e., the \mathcal{G} -component) is unique. From Proposition 4, it is straightforward to verify that the Myerson value satisfies the modularity property (M) on UC^N . It also clearly satisfies additivity (A), because $(v+w)^{\mathcal{G}} = v^{\mathcal{G}} + w^{\mathcal{G}}$ for all TU-games v, w and for all union-closed systems \mathcal{G} . We now turn to the remaining three axioms.

Since the grand coalition is always feasible in union-closed systems, component-efficiency (CE) coincides with the standard efficiency property. An allocation rule $\gamma : UC^N \rightarrow \mathbb{R}^n$ satisfies efficiency (E) if $\sum_{i \in N} \gamma_i(N, v, \mathcal{G}) = v(N)$ for all $(N, v, \mathcal{G}) \in UC^N$. Moreover, the extra-null player (ENP) and equal treatment of veto (ETV) are defined in the same way as for union-stable structures.

Proposition 5 *The Myerson value $\mu : UC^N \rightarrow \mathbb{R}^n$ satisfies efficiency (E), the extra-null player property (ENP), and equal treatment of veto (ETV).*

Proof.

Efficiency (E): Let $(N, v, \mathcal{G}) \in UC^N$.

$$\begin{aligned} \sum_{i \in N} \mu_i(N, v, \mathcal{G}) &= \sum_{i \in N} \sum_{S \in \mathcal{G}, i \in S} \frac{1}{|S|} \Delta_{v^{\mathcal{G}}}(S) \text{ recall that } \Delta_{v^{\mathcal{G}}}(S) = 0 \forall S \notin \mathcal{G} \\ &= \sum_{S \in \mathcal{G}} \sum_{i \in S} \frac{1}{|S|} \Delta_{v^{\mathcal{G}}}(S) \\ &= \sum_{S \in \mathcal{G}} \Delta_{v^{\mathcal{G}}}(S) \\ &= v^{\mathcal{G}}(N) = v(N). \end{aligned}$$

Hence, μ satisfies (E) on UC^N .

Extra-Null Player property (ENP): Let $(N, v, \mathcal{G}) \in UC^N$ and a player $i \in N$ who is an extra-null player in (N, v, \mathcal{G}) . Let $S \subseteq N \setminus i$.

- If $\{T \subseteq S, T \in \mathcal{G}\} = \emptyset$, then $T \cup i \notin \mathcal{G}$ for all $T \subseteq S$, and $v^{\mathcal{G}}(S \cup i) = v^{\mathcal{G}}(S) = 0$.
- If $\{T \subseteq S, T \in \mathcal{G}\} \neq \emptyset$, let us consider K , the \mathcal{G} -component of S . Firstly, if $K \cup i \notin \mathcal{G}$, then $v^{\mathcal{G}}(S \cup i) = v(K) = v^{\mathcal{G}}(S)$. Secondly, if $K \cup i \in \mathcal{G}$, then $v^{\mathcal{G}}(S \cup i) = v(K \cup i) = v(K) = v^{\mathcal{G}}$. Therefore, i is a null player in the TU-game $v^{\mathcal{G}}$. It follows that $\mu_i(N, v, \mathcal{G}) = \Phi_i(N, v^{\mathcal{G}}) = 0$.

Hence, μ satisfies (ENP) on UC^N .

Equal Treatment of Veto (ETV): Let $(N, v, \mathcal{G}) \in UC^N$. Denote by $CV(N, v, \mathcal{G})$ and $PV(N, v, \mathcal{G})$ the set of players with the a cooperation and production veto respectively. Let $i \in CV(N, v, \mathcal{G})$ and $j \in PV(N, v, \mathcal{G})$ and let $S \subseteq N \setminus \{i, j\}$. On the one hand, we have

$$\begin{aligned} v^{\mathcal{G}}(S \cup i) &= v(T) \text{ where } T \text{ is the } \mathcal{G} \text{ - component of } S \cup i \\ &= 0 \text{ since } j \notin T. \end{aligned}$$

On the other hand, we have $v^{\mathcal{G}}(S \cup j) = 0$ since $i \notin S \cup j$. Therefore, i and j are symmetrical in the TU-game $v^{\mathcal{G}}$. It then follows that $\mu_i(N, v, \mathcal{G}) = \Phi_i(N, v^{\mathcal{G}}) = \Phi_j(N, v^{\mathcal{G}}) = \mu_j(N, v, \mathcal{G})$. Hence; μ satisfies (ETV) on UC^N . \blacksquare

Theorem 3 *The Myerson value $\mu : UC^N \rightarrow \mathbb{R}^n$ is the unique allocation rule that satisfies (E), (A), (M), (ENP), and (ETV).*

Proof. It is already proven that the Myerson value satisfies the five axioms on UC^N . The uniqueness is shown very similarly to that of Theorem 2. For all $\emptyset \neq K \subseteq \{1, \dots, m\}$, we define $\mathcal{G}_K = \cap_{k \in K} \mathcal{G}_k = \mathcal{G}_{\cup_{k \in K} B_k}$. For any structure $(N, \alpha_S u_S, \mathcal{G}_K)$, it still holds that $CV(N, \alpha_S u_S, \mathcal{G}_K) = \cup_{k \in K} B_k$ and $PV(N, \alpha_S u_S, \mathcal{G}_K) = S$. The only point to emphasize is that since $N \in \mathcal{G}_K$, efficiency immediately implies that $\sum_{i \in S \cup (\cup_{k \in K} B_k)} \gamma_i(N, \alpha_S u_S, \mathcal{G}_K) = \alpha_S$ and $\gamma_i(N, \alpha_S u_S, \mathcal{G}_K) = 0$ for all $i \notin S \cup (\cup_{k \in K} B_k)$ by (ENP). By (ETV) we have that $\gamma_i(N, \alpha_S u_S, \mathcal{G}_K) = \frac{\alpha_S}{|S \cup (\cup_{k \in K} B_k)|}$ for all $i \in S \cup (\cup_{k \in K} B_k)$. \blacksquare

5 Independence of axioms

In this section, we show that the five axioms used in our characterization are logically independent. To prove this, we present five simple allocation rules on US^N each of them satisfying four of these axioms but failing to satisfy the remaining one.³

- It is not hard to see that the null value $\tilde{\mathbf{0}}$ defined by $\tilde{\mathbf{0}}_i(N, v, \mathcal{F}) = 0$ for all $i \in N$ satisfies (A), (M), (ENP), and (ETV) but it does not satisfy (CE).

³It is worth mentioning that the independence of the original formulations of these axioms has been shown in Gassi (2025) for voting structures. However, it seems relevant to show the independence of the extended formulations used in this paper highlighting, some features that proper to union-stable structures in general.

- Consider the allocation rule ϕ on US^N defined by

$$\phi_i(N, v, \mathcal{F}) = \begin{cases} \mu_i(N, v, \mathcal{F}) & \text{if } v^{\mathcal{F}}(N) \neq 0 \\ 0 & \text{if } v^{\mathcal{F}}(N) = 0 \end{cases}$$

where μ is the Myerson value on US^N . It can be checked that ϕ satisfies (CE), (ENP), (M), and (ETV). However, it does not satisfy (A). To verify this, consider any two voting structures $(N, u, \mathcal{F}), (v, \mathcal{F}) \in US^N$ such that $u^{\mathcal{F}}(N) \neq 0$ and $v^{\mathcal{F}}(N) = 0$ (but $v^{\mathcal{F}}$ is not the null game). Since $(u + v)^{\mathcal{F}}(N) = u^{\mathcal{F}}(N) + v^{\mathcal{F}}(N) \neq 0$, it follows that for all players $i \in N$, $\phi_i(N, u + v, \mathcal{F}) = \mu_i(N, u + v, \mathcal{F}) = \mu_i(N, u, \mathcal{F}) + \mu_i(N, v, \mathcal{F})$. On the other hand, we have $\phi_i(N, u, \mathcal{F}) + \phi_i(N, v, \mathcal{F}) = \mu_i(N, u, \mathcal{F}) \neq \mu_i(N, u, \mathcal{F}) + \mu_i(N, v, \mathcal{F})$ in general (since $\mu_i(N, v, \mathcal{F})$ can be non-null).

- Consider an allocation rule φ on US^N defined as follows:

$$\varphi_i(N, v, \mathcal{F}) = \sum_{S \in \mathcal{F}, i \in S} \frac{\lambda_i}{\sum_{j \in S} \lambda_j} \Delta_{v^{\mathcal{F}}}(S)$$

where λ_j are arbitrary non-negative numbers that do not depend on the system \mathcal{F} . It can be checked that φ satisfies (CE), (A), (ENP)⁴ and (M). However, φ does not satisfy (ETV) since we can have $\varphi_i(N, v, \mathcal{F}) \neq \varphi_j(N, v, \mathcal{F})$ whenever $\lambda_i \neq \lambda_j$.

- Consider the allocation rule \mathcal{E} on US^N defined by:

$$\forall M \in C_{\mathcal{F}}(N), \forall i \in M, \mathcal{E}_i(N, v, \mathcal{F}) = \begin{cases} \frac{v(M)}{|M \setminus ENP(M)|} & \text{if } i \in M \setminus ENP(M) \\ 0 & \text{if } i \in ENP(M) \end{cases}$$

where $ENP(M)$ is the set of extra-null players belonging in M . It can easily be checked that \mathcal{E} satisfies (CE), (A), (ENP) and (ETV), but does not satisfy (M) since the component M depends on \mathcal{F} and a such component may differs from $\mathcal{F}_1 \cup \mathcal{F}_2$ to $\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{F}_1 \cap \mathcal{F}_2$, where \mathcal{F}_1 and \mathcal{F}_2 are two union-stable systems such that $\mathcal{F}_1 \cup \mathcal{F}_2$ is also union-stable. However, an essential point should be emphasized regarding (ETV). Indeed, it is worth mentioning that if there exists a player with a cooperation veto in the structure (N, v, \mathcal{F}) , then there is a unique \mathcal{F} -component of N and that component contains any player with a cooperation veto, as well as any player with a production veto.

- Finally, consider an allocation rule Ψ defined as follows: fix a player $i_0 \in N$. For any union-stable structure (N, v, \mathcal{F}) , if i_0 is not an extra-null player, then $\Psi_i(N, v, \mathcal{F}) = \mu_i(N, v, \mathcal{F})$ for all $i \in N$ (where μ is the Myerson value). If i_0 is an extra-null player, then $\Psi_{i_0}(N, v, \mathcal{F}) = v(M_{i_0})$, $\Psi_i(N, v, \mathcal{F}) = 0$ for all $i \in M_{i_0} \setminus i_0$, where M_{i_0} is the \mathcal{F} -component of N containing i_0 , and $\Psi_i(N, v, \mathcal{F}) = \mu_i(N, v, \mathcal{F})$ for all $i \notin M_{i_0}$. It can be checked that Ψ satisfies (CE), (A), (M) and (ETV), but fails to satisfy (ENP), since i_0 can gain $v(M_{i_0}) \neq 0$ while being an extra-null player. However, A clarification should bring on modularity. Indeed, note that if i_0 is an extra-null player in $(N, v, \mathcal{F}_1 \cup \mathcal{F}_2)$, then it is

⁴For (ENP), remark that $\Delta_{v^{\mathcal{F}}}(S) = 0$ for any coalition that contains a null player in $v^{\mathcal{F}}$.

an extra-null player both in (N, v, \mathcal{F}_1) and (N, v, \mathcal{F}_2) (therefore in $(N, v, \mathcal{F}_1 \cap \mathcal{F}_2)$), and modularity follows from Proposition 2.

6 Concluding remarks

The goal of this paper was to provide an axiomatic characterization of the Myerson value for the class of cooperative games on union-stable structures, without referring to the broader family of Harsanyi power solutions to which this value belongs. The axioms used in this paper are intuitive, independent, and effectively highlight both the features of these structures and how an efficient allocation rule should perform in relation to them.

Another well-known and widely studied allocation rule for this class of games is the position value, originally introduced by Meessen (1988), which also belongs to the family of Harsanyi power solutions. This value first assigns to each support (an element of the basis) a value representing the impact of that support to the overall production, computed via the Shapley value, and then distributes this value equally among the players in the support. Based on the underlying intuition of this allocation rule, we believe that an important pending challenge is to provide a similar axiomatic characterization of the position value without identifying it as an Harsanyi power solution; that is, without relying on any underlying power measure

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