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OVERTAKING CRITERIA AND SUPERMODULAR PAYOFF FUNCTIONS*

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ABSTRACT

This study revisits the overtaking criterion in economics, in a framework without the usual convex structures that are often used to prove existence and convergence of optimal paths. We show that the supermodular structure plays a decisive role in shaping the behavior of the economy. We further prove that, as the discount factor approaches 1, optimal paths under standard discounting criteria converge in the product topology to an optimal path under the overtaking criterion. This supports the view that the overtaking criterion can be interpreted as a limiting case of discounting criteria with discount factor equal to 1. Our arguments rely on turnpike properties, one for the overtaking problem and one for the discounted problems.

Keywords: Overtaking criteria; Multiple discounting; Turnpike theory; Time preference; Dynamic Programming

JEL Classification: C61, D15, D90, O41

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1. INTRODUCTION

In dynamic economic theory, a fundamental question is how to evaluate and compare infinite sequences of intertemporal utilities. The standard approach based on a positive discount factor strictly smaller than one is often criticized for treating the near and distant future asymmetrically. By contrast, setting the discount factor equal to one gives equal weight to all periods, but then raises the difficulty that the infinite sum of utilities may fail to converge. The seminal paper of [Gale \(1967\)](#) proposed a way to evaluate infinite utility streams without using the sum of the differences between the utility obtained in each period and the utility generated by the *golden rule*, namely the capital stock that yields the highest possible constant level of consumption.¹ Any such sum is either a well defined real value or $-\infty$. Under appropriate conditions, there exists an optimal program, which gives the highest value for this criterion and converges to the golden rule.

Gale's approach has had a major influence on research in dynamic economics. As noted by [Mitra \(1991\)](#), this approach requires only a few tools from convex analysis to establish the existence and characterize the properties of an optimal allocation of resources over time. However, the question remains open when the standard convexity and concavity assumptions are not satisfied. In this paper, we provide some natural examples in which these properties no longer hold. We then consider a more general framework and focus on three main results under the overtaking criterion: the existence of optimal paths and the convergence of the economy to the golden rule, the turnpike property, and the Bellman functional equation.

[Majumdar and Mitra \(1982\)](#) and [Mirman and Nyarko \(1991\)](#) prove the existence of optimal paths. They are even able to relax either the convexity of the production set or the stationarity of the criterion.² They also prove the monotonicity of the optimal policy function with respect to the capital stock. In the deterministic setting, this monotonicity guarantees the convergence of the economy to the golden rule.

The main idea of the turnpike property is that it is optimal for the economy to enter quickly into the desired region (namely a region close to a steady state, the golden rule, or an efficient balanced path), and to remain there thereafter. Among the important contributions to this literature, we may mention the pioneering book by [Dorfman et al. \(1958\)](#), the continuous time analysis of [Cass \(1966\)](#), and the seminal paper by [McKenzie \(1976\)](#), which provided an insightful treatment of turnpike notions and an

¹See [Basu and Mitra \(2007\)](#) for an axiomatic foundation of these criteria and [Majumdar \(2009\)](#) for a detailed discussion of Gale's contribution. The articles by [Diamond \(1965\)](#) and [Basu and Mitra \(2003\)](#) show the difficulties of finding a function that treats the near and distant future equally while preserving the Pareto property.

²The utility function may change over time.

approach that has been widely used in subsequent work.

Following the approach of McKenzie (1976), Dana and Le Van (1990) established the corresponding Bellman functional equation and showed that the value function is a solution. They also proved that the economy converges to the golden rule. Le Van and Morhaim (2006) showed that any limit of solutions to discounted problems, as the discount factor converges to 1, is a solution under the overtaking criterion.

These results are based either on the concavity of the utility function with respect to the consumption level, or on the concavity of the payoff function with respect to the current and subsequent capital stocks. This concavity allows one to use a price-system technique. Yet this condition may be violated in many situations, for example when the externality generated by the capital stock is taken into account in the utility function. In this work, we provide examples in which the usual concavity assumption no longer holds, while only the supermodular structure remains, and we show that our approach is useful in such configurations. We propose a new approach based on an order-theoretic structure, with an ascending choice correspondence and a supermodular single peaked payoff function. This framework is more general and is sufficient to obtain the results listed above.

Our work echoes the contributions of Topkis (1978) and Amir (1996), in which the monotonicity and convergence of the optimal path do not result from local properties such as convexity or concavity, but rather from a more global property, satisfied by a much larger class of utility functions, namely supermodularity. Under the discounting criteria, if the payoff function is supermodular, then, once an optimal path exists, it is monotone, and convergence to a steady state follows directly.³

To proceed, we first look at feasible capital sequences such that, from some period T onwards, all capital stocks are equal to the golden rule. The advantage of using these “truncated economies” is that, the obtained sum of the differences between the utility level and the utility at the golden rule is always determined, and is finite. Continuity and compactness properties allow us to ensure that we can always find an optimal path for each T . The relationship between the value function associated with $T + 1$ and the one with T has the form of a Bellman functional equation.

Two turnpike results, one for the overtaking criterion and the other for discounted criteria, play a central role in our arguments. Given any open neighborhood of the golden rule, the first result states that, from some period onward, every optimal path of the auxiliary problems enters this neighborhood. The second result asserts that, provided the discount factor is sufficiently close to 1, every optimal path

³We should note that, in the absence of the standard concavity of the indirect utility function, several different optimal paths may exist. As shown in Dechert and Nishimura (1983) and Le Van and Dana (2003), in the context of a Ramsey–Cass–Koopmans economy with a convex-concave production function, the set of capital levels from which more than one optimal path may exist has measure zero, and the value function is differentiable almost everywhere.

of the corresponding discounted problems enters this neighborhood from some period onward. These results are in the same spirit as McKenzie (1976): although in the truncated economies, the number of periods before the terminal golden rule may increase, it remains optimal to approach the golden rule quickly, rather than to delay outside its neighborhood. These turnpike properties are needed to establish the continuity between the auxiliary problems, the discounting problems and the problem with the overtaking criterion.

As the period from which all capital stocks are constrained to be equal to the golden rule tends to infinity, the finite-horizon optimal paths converge to an optimal path under the overtaking criterion. Moreover, the auxiliary value function converges to the value function associated with the overtaking criterion, which is a solution to the Bellman equation.

This work is organized as follows. Section 2 presents the fundamentals of the economy, the overtaking criterion, and an example where standard conditions no longer hold. Section 3 studies the relation between optimal paths under the auxiliary criteria and under the overtaking criterion. Section 4 examines the convergence of optimal paths under discounting criteria to an optimal path under the overtaking criterion, when the discount factor tends to 1. All proofs are given in the Appendix.

2. FUNDAMENTALS

Time is discrete. Let Γ be a correspondence from $[0, \infty)$ to $[0, \infty)$ describing the technology. For any capital stock $x_t \geq 0$ at time t , $\Gamma(x_t)$ denotes the set of feasible capital stocks at time $t + 1$.⁴ Given an initial condition $x_0 \geq 0$, let $\Pi(x_0)$ be the set of feasible capital sequences $\{x_t\}_{t=0}^\infty$ such that $x_{t+1} \in \Gamma(x_t)$ for every $t \geq 0$, with per-period payoff is $V(x_t, x_{t+1})$. By a slight abuse of notation, we will also call $(x_0, x_1, \dots, x_T)$ a *feasible sequence* (or a feasible path) if $x_{t+1} \in \Gamma(x_t)$ for all $t = 0, 1, \dots, T - 1$.

We present the *overtaking criterion*, in the spirit of Gale (1967). Let

$$v^* = \sup_{x \geq 0} V(x, x).$$

Starting from $x_0 \geq 0$, a sequence $\{x_t\}_{t=0}^\infty \in \Pi(x_0)$, the evaluation of the corresponding intertemporal utility is defined as

$$W(x_0, x_1, x_2, \dots) = \liminf_{T \rightarrow \infty} \sum_{t=0}^T [V(x_t, x_{t+1}) - v^*].$$

This criterion divides the set of feasible sequences into two subsets: *good programs*, whose value is not $-\infty$, and *bad programs*, whose value is $-\infty$. It is straightforward that bad programs are dominated by any good program. Moreover, if every good program has a finite value, then good programs can be compared with one another. Under suitable conditions, an optimal program even exists.

⁴For example, in a standard Ramsey-Cass-Koopman model with production function f , one can take $\Gamma(x) = [0, f(x)]$.

In the next part of this section, we provide conditions ensuring not only that v^* exists and is finite, but also that there exists a capital stock level x^* at which the corresponding utility $V(x^*, x^*)$ attains this value. They also allow us to compare good programs and to establish the existence of an optimal program.

Let $\text{Graph}(\Gamma)$ denote the set of pairs $(x, y) \in \mathbb{R}_+^2$ such that $y \in \Gamma(x)$. When (x, y) belongs to the interior of $\text{Graph}(\Gamma)$, we denote by $V_1(x, y)$ and $V_2(x, y)$ the partial derivatives of V with respect to its first and second arguments.

Assumption 1. i) Γ is nonempty, convex- and compact-valued, and continuous on $[0, \infty)$, with $\Gamma(0) = \{0\}$ and $\text{int}(\Gamma(x)) \neq \emptyset$ for $x > 0$.

ii) There exist $x_m \geq 0$ and $x_M \in (0, +\infty]$ such that:

- a) If $x_m > 0$, then for $0 < x < x_m$ we have $x > y$ for every $y \in \Gamma(x)$.
- b) If $x_M < +\infty$ and $x > x_M$, then $x > y$ for every $y \in \Gamma(x)$.
- c) If $x_m < x < x_M$, then $x \in \text{int}(\Gamma(x))$.

iii) The payoff function V is twice differentiable in the interior of $\text{Graph}(\Gamma)$, strictly increasing in its first argument and strictly decreasing in its second argument.

iv) For $(x, y) \in \text{int}(\text{Graph}(\Gamma))$,

$$V(x, y) > \inf_{\text{Graph}(\Gamma)} V(x', y').$$

v) The function $V(x, \min \Gamma(x))$ is nondecreasing with respect to x .

vi) The function $V(x, x)$ is single-peaked on $[x_m, x_M]$: there exists $x_m < x^* < x_M$ such that $V(x, x)$ is strictly increasing on $[x_m, x^*]$ and strictly decreasing on $[x^*, x_M]$.

The conditions in Assumption 1 are standard in the dynamic programming literature. Condition (ii.a) is introduced to cover the case where capital stock must be sufficiently large to be able to replicate itself. If $x_m = 0$, then we simply return to the case where productivity is sufficiently high near the origin. Condition (ii.b) covers the case where there exists existence of a maximum level of capital that can replicate itself. Condition (iii) represents the trade-off between payoff today and investment for tomorrow.

Condition (iv) ensures that if the economy does not invest its whole capacity for the future, then the payoff today is higher than the infimum level. Condition (v) says that, the payoff that we obtain by investing the minimum for tomorrow is nondecreasing with the capital stock. Observe that, under the free disposal assumption: $0 \in \Gamma(x)$ for all $x \geq 0$, this conditions holds if $\max \Gamma(x)$ is nondecreasing with x , a condition that is already included in the ascending property of Γ , as we will see in Assumption 2.

Condition (vi) ensures the existence of a *golden rule*, the level of capital that maximizes the payoff when capital is kept constant. This condition also assumes the single-peaked property, which always holds implicitly in results in the literature, where the utility function is supposed to be concave.

Let

$$x^* = \operatorname{argmax}_{x \in [x_m, x_M]} V(x, x) \text{ and then } v^* = V(x^*, x^*).$$

It is obvious that the feasible set $\Pi(x_0)$ contains a good program if and only if $x_0 > x_m$. If $x_0 \leq x_m$, then every feasible sequence is a bad program.

After the Assumption 1, which is standard and guarantees the existence properties that we need, we introduce the core structure of the economy, through the supermodularity of the payoff function. We begin with the standard strong set order, which in our one dimensional setting, takes the of the *ascending* property, which states that if $x < x'$ and $y \in \Gamma(x)$, $y' \in \Gamma(x')$, then $\min\{y, y'\} \in \Gamma(x)$ and $\max\{y, y'\} \in \Gamma(x')$. In the context of this work, this property can be summarized as follows: the functions $\min \Gamma(x)$ and $\max \Gamma(x)$ are both non decreasing with respect to x .

The strict supermodularity of the payoff function V is defined as follows. For all $x < x'$ and $y < y'$, we have

$$V(x', y') - V(x', y) > V(x, y') - V(x, y),$$

whenever the terms in that inequality are feasible. It is also well known under the name the *increasing differences* property.⁵ If function V is twice differentiable, the strict supermodularity can be summarized as $V_{12}(x, y) > 0$ for all $(x, y) \in \operatorname{int}(\operatorname{Graph}(\Gamma))$.

Assumption 2. i) *The technology correspondence Γ is ascending.*

ii) *The function V is strictly supermodular.*

Let v denote the value function under the overtaking criterion:

$$v(x_0) = \sup_{\Pi(x_0)} \liminf_{T \rightarrow \infty} \sum_{t=0}^T [V(x_t, x_{t+1}) - v^*].$$

Clearly, $v(x_0) = -\infty$ whenever $x_0 \leq x_m$. Lemma 3.1 will imply that $v(x_0) > -\infty$ for $x_0 > x_m$. Since for any (if exists) optimal path $(x_0, x_1, x_2, \dots)$ starting from x_0 , the path $(x_1, x_2, \dots)$ is an optimal path starting from x_1 , we may guess that v may be a solution to the Bellman equation:

$$v(x_0) = \max_{x_1 \in \Gamma(x_0)} [V(x_0, x_1) - v^* + v(x_1)].$$

⁵Observe that the roles played by x and y are symmetric: this inequality can also be written as $V(x', y') - V(x, y') > V(x', y) - V(x, y)$.

At this stage, we do not yet know whether $v(x_0)$ is always finite or may take the value $+\infty$. The analysis in Section 3 answers this question and establishes the main properties that allow us to solve the optimization problem under the overtaking criterion. Before turning to deeper analysis, we provide an example of an economy with externality of the capital stock in the consumption function, where the usual conditions in concavity and convexity no longer hold, and yet the economy satisfies the two assumptions presented above.

Example In this economy, we denote by x_t the stock of natural resources at the end of period $t - 1$, and let f be the regeneration function, which gives the available stock $f(x_t)$ at time t . Let c_t be the extraction in period t . At the end of that period, we have $x_{t+1} = f(x_t) - c_t$. The instantaneous utility depends not only on the consumption level, but also on the stock of natural resources. The utility function is therefore denoted by a function with two arguments, and the felicity at period t is $u(c_t, x_t)$. The technology correspondence is $\Gamma(x_t) = [0, f(x_t)]$. Given the stocks x_t and $x_{t+1} \in \Gamma(x_t)$ at periods t and $t + 1$, the utility of period t is $V(x_t, x_{t+1}) = u(f(x_t) - x_{t+1}, x_t)$.

Let the regeneration function of natural resources be $f(x) = x^{\frac{2}{3}}$. With this function, $x_m = 0$ and $x_M = 1$. The maximum level of available stock is normalized to be equal to 1. We consider the following two utility functions, defining on $c \geq 0$ and $x \geq 0$:

$$u(c, x) = (1 + x)(1 - e^{-c}),$$

and

$$u(c, x) = \begin{cases} -(2 - x)e^{-3c} & \text{if } 0 \leq x \leq 1, \\ -\frac{1}{x}e^{-3c} & \text{if } x > 1. \end{cases}$$

In both cases, $u_c(c, x)$ and $u_x(c, x)$ are positive. The utility level increases with both consumption and the stock of natural resources. However, in the first case, $u_{cx}(c, x) > 0$: natural resources and consumption are complements. An increase in the level of one increases the marginal utility of the other. In the second case, $u_{cx}(c, x) < 0$. The two goods are substitutes: increasing one decreases the marginal utility of the other.

In both cases u is nonconcave in (c, x) . Moreover, with $V(x, y) = u(f(x) - y, x)$, the function V is nonconcave, strictly supermodular, and $V(x, x)$ is single-peaked on $[x_m, x_M]$. The proof is given in detail in the Appendix.

3. VALUE FUNCTIONS AND OPTIMAL PATHS

This section proposes a new approach to study the economy under overtaking criterion. The main idea is to consider the “truncated versions” of the overtaking criterion, or in other words, the optimal

problem determined on the set of feasible sequences that reach the golden rule in a finite number of periods and remains there afterwards. The advantage of this approach is that, since the terms become zero from some period onward, the infinite sums are always well defined. The description of the main properties of these intermediary optimal paths gives us informations to overcome the difficulties caused by the lack of usual convexity and concavity structure.

As a preparation step, we prove that, beginning from $x_0 > x_m$, the economy can reach x^* in a finite number of periods.

LEMMA 3.1. *For any $x_0 > x_m$, there exist $T \geq 0$ and a finite feasible sequence $(x_0, x_1, x_2, \dots, x_T)$ such that $x^* \in \Gamma(x_T)$.*

For each $x_0 > x_m$, let $\Pi_T(x_0)$ be the set of feasible paths linking x_0 and x^* :

$$\Pi_T(x_0) = \{ (x_0, x_1, x_2, \dots, x_T, x_{T+1}) \text{ that is feasible and } x_{T+1} = x^* \}.$$

If $\Pi_T(x_0) = \emptyset$, we define $w_T(x_0) = -\infty$. Otherwise, let

$$w_T(x_0) = \max_{\Pi_T(x_0)} \sum_{t=0}^T [V(x_t, x_{t+1}) - v^*]. \quad (1)$$

Since $\Pi_T(x_0)$ is compact, that truncated version has a solution for any T sufficiently great. Observe that, the optimization problem (1) is equivalent to

$$\sup_{\Pi_T(x_0)} \sum_{t=0}^T V(x_t, x_{t+1}).$$

Therefore, this suggests us that Problem (1) can be regarded as an approximation of the discounting problem with the discount factor equals to 1. We will return to the question in Section 4.

From now on, we will denote by $\mathcal{O}_T(x_0)$ the set of optimal paths in $\Pi_T(x_0)$ corresponding to (1), the set of sequences $(x_0, x_1, \dots, x_T, x^*) \in \Pi_T(x_0)$ such that:

$$w_T(x_0) = \sum_{t=0}^T [V(x_t, x_{t+1}) - v^*].$$

It is obvious that if $\Pi_{T+1}(x_0) \neq \emptyset$, then for any feasible path $(x_0, x_1, x_2, \dots, x_T, x_{T+1}, x^*)$ in $\Pi_{T+1}(x_0)$, the sequence $(x_1, x_2, \dots, x_T, x_{T+1}, x^*)$ is a feasible path in $\Pi_T(x_1)$. Thus, $\Pi_T(x_1) \neq \emptyset$ and we can establish a relation between w_T and w_{T+1} that resembles the Bellman equation:

$$w_{T+1}(x_0) = \max_{x_1 \in \Gamma(x_0)} [V(x_0, x_1) - v^* + w_T(x_1)]. \quad (2)$$

Lemma 3.2 shows that any optimal path in $\mathcal{O}_T(x_0)$ is monotonic, and $w_T(x_0)$ is increasing with respect to T . The inequalities become strict if and only if $x_0 \neq x^*$. Based on this result, taking the limit of

w_T when T goes to infinity, we obtain a function w that is a candidate for the solution to the Bellman equation. Moreover, when T becomes great, intuitively, we guess that an optimal path of $\mathcal{O}_T(x_0)$ will be close, in some sense, to an optimal path of the overtaking criterion problem.

LEMMA 3.2. *Consider $x_0 > 0$ and $T \geq 0$ such that $\Pi_T(x_0) \neq \emptyset$. Let $(x_0, x_1, \dots, x_T, x^*)$ be an optimal path in $\mathcal{O}_T(x_0)$.*

- i) a) *If $x_0 < x^*$, then $x_0 < x_1 < \dots < x_T < x_{T+1} = x^*$.*
- b) *If $x_0 = x^*$, then $x_t = x^*$ for $0 \leq t \leq T + 1$, and $w_T(x^*) = 0$.*
- c) *If $x_0 > x^*$, then $x_0 > x_1 > \dots > x_T > x_{T+1} = x^*$.*
- ii) *The value of $w_T(x_0)$ is increasing with respect to T ,*

$$w_T(x_0) \leq w_{T+1}(x_0) \leq w_{T+2}(x_0) \leq \dots \rightarrow w(x_0) < +\infty,$$

and the inequalities become strict if $x_0 \neq x^$.*

- iii) *For any $T \geq 0$ such that $\Pi_T(x_0)$ is not emptyset, function $w_T(x)$ is strictly increasing in the interval (x_0, ∞) .*
- iv) *Function w is continuous at x^* .*

Let

$$w(x_0) = \lim_{T \rightarrow \infty} w_T(x_0),$$

for all $x_0 \geq 0$. It is clear that $w(x_0) > -\infty$ if and only if $x_0 > x_m$. To prove that letting T go to infinity, we will obtain the Bellman equation, we present our first main result, which is based on the monotonicity of the optimal paths of the truncated economies.

This result establishes a *turnpike* property: given any open neighborhood $\mathcal{V}$ of x^* , there exists T^* such that for all $T \geq T^*$, every optimal path of $\mathcal{O}_T(x_0)$ enters $\mathcal{V}$ from period T^{*th} onward. This property is in the spirit of the one described in McKenzie (1976), where regardless of the length of the planning horizon, it is optimal to move quickly towards the golden rule and to remain there afterwards.

As we show in the proof, this turnpike property is crucial. The idea is that, when T goes to infinity, only finitely many capital stocks lie outside the neighborhood $\mathcal{V}$. Hence, their convergence can be controlled in a compact set. Although the number of stocks inside the neighborhood increases to infinity, the fact that they are all sufficiently close to the golden rule allows us to control the infinite sum of differences between the corresponding payoff levels and v^* . We therefore can establish the upper semicontinuity of w and prove that it solves the Bellman functional equation.

PROPOSITION 3.3. Fix $x_0 > x_m$. For each $T \geq 0$, let $(x_0, x_1^T, x_2^T, \dots, x_T^T, x^*)$ be an optimal path in $\mathcal{O}_T(x_0)$.

- i) For any $d > 0$, there exists T^* large enough such that for all $T \geq T^*$, $x_t^T \in (x^* - d, x^* + d)$ for all $t \geq T^*$.
- ii) Function w is upper semicontinuous in $(x_m, +\infty)$, strictly increasing, and is a solution to Bellman functional equation: for all $x_0 > x_m$,

$$w(x_0) = \max_{x_1 \in \Gamma(x_0)} [V(x_0, x_1) - v^* + w(x_1)].$$

To establish that w is also the value function under the overtaking criterion, we need a preliminary result that generalizes a well-known result in the literature: under suitable conditions, any good program converges to the golden rule. In contrast with the similar result obtained under the concavity of the utility function, we are not certain that there is convergence to the golden rule for every good program. We therefore obtain a weaker result: every good program has the limit superior greater than or equal to x^* and limit inferior smaller than or equal to x^* . Consequently, if this sequence is monotonic, then it converges to x^* .

LEMMA 3.4. Consider $x_0 > x_m$ and a good program $\{x_t\}_{t=0}^\infty$ starting from x_0 . Then

$$\limsup_{t \rightarrow \infty} x_t \geq x^* \text{ and } \liminf_{t \rightarrow \infty} x_t \leq x^*.$$

If $\{x_t\}_{t=0}^\infty$ is monotonic, then $\lim_{t \rightarrow \infty} x_t = x^*$.

Lemma 3.4 provides the key idea for proving Proposition 3.5, which states that the function w is also the value function under the overtaking criterion. While it is easy to show that for all $x_0 \geq 0$, we have $v(x_0) \geq w(x_0)$, the proof of $w(x_0) \geq v(x_0)$ requires the result that any good program under the overtaking criterion either converges to the golden rule or vacillates around it the way that is described in Lemma 3.4.

The second part of Proposition 3.5 gives more detailed properties of the optimal path(s). The corresponding optimal policy correspondence is increasing. This implies the strict monotonicity of every optimal that does not start from the golden rule. Using the Bellman function equation, we can build an optimal path for the overtaking criterion.

PROPOSITION 3.5. i) The function w is also the value function under the overtaking criterion:

$$v(x_0) = w(x_0) = \lim_{T \rightarrow \infty} w_T(x_0).$$

- ii) Starting from x_0 such that $x_0 > x_m$ and $x_0 \neq x^*$, every optimal path $\{x_t\}_{t=0}^\infty$ under the overtaking criterion is strictly monotonic and converges to x^* . For all $t \geq 0$,

$$v(x_t) = V(x_t, x_{t+1}) - v^* + v(x_{t+1}),$$

and

$$v(x_0) = \sum_{t=0}^{\infty} [V(x_t, x_{t+1}) - v^*].$$

- iii) Assume $x_0 > x_m$ and the sequence $\{x_t\}_{t=0}^\infty$ satisfies:

$$v(x_t) = V(x_t, x_{t+1}) - v^* + v(x_{t+1}),$$

for all $t \geq 0$. Then this sequence is optimal under the overtaking criterion.

4. DISCOUNTING AND OVERTAKING CRITERIA

In this section, we recall the well-known main results in the literature on dynamic programming under discounting criteria, and we establish a continuity property linking these criteria, when the discount factor converges to 1, to the overtaking criterion. The latter can therefore be considered as a special case with a discount factor equal to 1.

For each discount factor $\delta \in (0, 1)$ and a given capital stock $x_0 \geq 0$, let the economy $\mathcal{E}(\delta)$ solve the following optimization problem:

$$v(\delta, x_0) = \sup \left[\sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right], \quad (3)$$

such that $x_{t+1} \in \Gamma(x_t) \forall t \geq 0, .$

In order to simplify the presentation, we assume from the outset that the following conditions hold for every $\delta \in (0, 1)$, or at least for every δ sufficiently close to 1. They are all known classical results under suitable conditions. Curious readers may refer to [Le Van and Morhaim \(2002\)](#) for details on conditions ensuring continuity and existence. The monotonicity of optimal correspondences is ensured by the supermodularity of V , as presented in [Amir \(1996, 2005\)](#).

- a) The sum $\sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1})$ is upper semi-continuous with respect to the product topology on the set $\Pi(x_0)$.
- b) The economy $\mathcal{E}(\delta)$ has a solution to (3).
- c) The value function $v(\delta, \cdot)$ is upper semi-continuous with respect to x_0 and is a solution to the Bellman equation:

$$v(\delta, x_0) = \max_{x_1 \in \Gamma(x_0)} [V(x_0, x_1) + \delta v(\delta, x_1)].$$

d) Let $\varphi(\delta, x_0)$ be the optimal policy correspondence:

$$\varphi(\delta, x_0) = \operatorname{argmax}_{x_1 \in \Gamma(x_0)} [V(x_0, x_1) + \delta v(\delta, x_1)].$$

If $x_0 < x'_0$, $\delta < \delta'$, and $x_1 \in \varphi(\delta, x_0)$, $x'_1 \in \varphi(\delta', x'_0)$, then $x_1 \leq x'_1$.

Under these assertions, we obtain a continuity property between the economies with different discount factors. First, we prove the following *turnpike* property, stating that for any open neighborhood of x^* and any initial condition $x_0 > x_m$, there exists a period T^* such that, every optimal path of the economies $\mathcal{E}(\delta)$ enters this neighborhood from T^* onward, provided that δ is sufficiently close to 1.

PROPOSITION 4.1. *Fix $x_0 > x_m$. For each $\delta \in (0, 1)$, let $\{x_t^\delta\}_{t=0}^\infty$ be an optimal path of the economy $\mathcal{E}(\delta)$ starting from x_0 . Then for any $d > 0$, there exists $\bar{\delta} \in (0, 1)$ and $T^* \geq 0$ such that, for all $\delta \in (\bar{\delta}, 1)$ and $t \geq T^*$, we have $x_t^\delta \in (x^* - d, x^* + d)$.*

Observe that the problem $\mathcal{E}(\delta)$ is equivalent to the following one:

$$\sup_{\Pi(x_0)} \left[\sum_{t=0}^{\infty} \delta^t (V(x_t, x_{t+1}) - v^*) \right].$$

As in the previous section, the fact that, when T goes to infinity, only finitely many capital stocks of problem $\mathcal{E}(\delta)$ lie outside the neighborhood allows us to control the convergence of the infinite sum as δ converges to 1. This enables us to establish that the limit is equal to the value of the problem under the overtaking criterion. More precisely, as δ converge to 1, any limit in the product topology of a subsequence of $\{x_t^\delta\}_{t=0}^\infty$ is an optimal path under the overtaking criterion.

PROPOSITION 4.2. *For each $\delta \in (0, 1)$, let $\{x_t^\delta\}_{t=0}^\infty$ be an optimal path of the economy $\mathcal{E}(\delta)$. Assume that for a sequence $\{\delta_n\}_{n=0}^\infty$ converging to 1 such that $\{x_t^{\delta_n}\}_{t=0}^\infty$ converges to $\{\bar{x}_t\}_{t=0}^\infty$ in the product topology, then the sequence $\{\bar{x}_t\}_{t=0}^\infty$ is an optimal path under overtaking criterion:*

$$v(x_0) = \sum_{t=0}^{\infty} [V(\bar{x}_t, \bar{x}_{t+1}) - v^*].$$

Corollary 4.3 states that if δ is close to 1, then the economy $\mathcal{E}(\delta)$ admits a non-decreasing optimal path. The meaning of this result is clear, if the economy is sufficiently patient, it may choose to increase, or at least not to decrease the investment.

To the best of our knowledge, surprisingly, this result is new. Indeed, without appropriate convexity conditions, or at least some condition ensuring high productivity at the origin, it is not clear whether the economy collapse to the origin or not, nor whether an increasing optimal path exists. We know now that, for δ close to 1, such a path does exist under weak assumptions on the structure of the economy. This result is reminiscent of *mild discounting* condition in [Dechert and Nishimura \(1983\)](#).

In their Ramsey-Cass-Koopmans model with a convex-concave production function, if δ is sufficiently large, then there exists an increasing optimal path that converges to a capital level that maximizing the net gain of investment.

COROLLARY 4.3. *Fix $x_0 \in (x_m, x^*)$. Then there exists $\bar{\delta} \in (0, 1)$ such that, for all $\bar{\delta} < \delta < 1$, there exists an optimal path of the economy $\mathcal{E}(\delta)$ starting from x_0 that is non-decreasing.*

5. CONCLUSION

In this work, we reconsider dynamic optimization under the overtaking criterion, imposing relatively mild assumptions: supermodularity and single-peakedness of the payoff function. As in the discounted configuration, these conditions are sufficient to ensure the existence of an optimal path and its convergence to the *golden rule*. Our argument proceeds via “truncated economies”, the dynamic problems being defined on sequences that reach the golden rule in finitely many periods. The crucial step is a *turnpike* property, which states that every optimal path of a truncated economy enters any neighbourhood of the golden rule, regardless of the horizon length. We also show that the limit of optimal paths under discounting criteria, as the discount factor converges to 1, is an optimal path under the overtaking criterion. Therefore, we can reconstruct the well-known results in the literature without relying on local conditions such as convexity or concavity.

APPENDIX

PROOF OF THE CLAIM IN THE EXAMPLE

First utility function. Some calculus give us:

$$\begin{aligned} V_1(x, y) &= 1 - e^{-f(x)+y} + (1+x)f'(x)e^{-f(x)+y}, \\ V_2(x, y) &= -(1+x)e^{-f(x)+y}, \\ V_{12}(x, y) &= [-1 + (1+x)f'(x)]e^{-f(x)+y}, \\ V_{11}(x, y) &= [2f'(x) + (1+x)f''(x) - (1+x)(f'(x))^2]e^{-f(x)+y}, \\ V_{22}(x, y) &= -(1+x)e^{-f(x)+y}. \end{aligned}$$

For all $0 \leq y \leq 1$, we have $V_{11}(1, y) = 0$ and $V_{12}(1, y) > 0$. Let $H(x, y)$ be the Hessian matrix of V . Clearly, $\det H(1, y) < 0$ for all $0 \leq y \leq 1$. Hence, V is not concave.

Now, we prove that V is strict supermodular. Let

$$\psi(x) = x^{-\frac{1}{3}} + x^{\frac{2}{3}}.$$

We have

$$\psi'(x) = -\frac{1}{3}x^{-\frac{4}{3}} + \frac{2}{3}x^{-\frac{1}{3}} = \frac{1}{3}x^{-\frac{4}{3}}(-1 + 2x).$$

Obviously, ψ is decreasing in $(0, \frac{1}{2})$, and increasing in $(\frac{1}{2}, 1)$. We obtain for all $0 \leq y \leq f(x)$ with $x \in (0, 1)$:

$$e^{f(x)-y}V_{12}(x, y) = \frac{1}{3}\psi(x) > \frac{1}{3}\psi\left(\frac{1}{2}\right) > 0.$$

Hence, V has positive derivatives.

Observe that

$$\log V(x, x) = \log(1 + x) + \log(1 - e^{-f(x)+x}),$$

is a concave function taking value 0 at $x = 0$ and $x = 1$. Therefore, this function is single peaked in $[0, 1]$, and the same property applies for $V(x, x)$.

Second utility function. Some calculus give us:

$$\begin{aligned} V_1(x, y) &= [1 + 3(2 - x)f'(x)]e^{-3f(x)+3y}, \\ V_2(x, y) &= -3(2 - x)e^{-3f(x)+3y}, \\ V_{12}(x, y) &= 3[1 + 3(2 - x)f'(x)]e^{-3f(x)+3y}, \\ V_{11}(x, y) &= [-6f'(x) + 3(2 - x)f''(x) - 9(2 - x)(f'(x))^2]e^{-3f(x)+3y}, \\ V_{22}(x, y) &= -9(2 - x)e^{-3f(x)+3y}. \end{aligned}$$

It is easy to verify that $\det H(1, y) = -3e^{-6+6y} < 0$. Hence, $V(x, y)$ is not concave. Since $f'(x)$ is decreasing, the function $1 + 3(2 - x)f'(x)$ is decreasing, and

$$V_{12}(x, y)e^{3f(x)-3y} = 3[1 + 3(2 - x)f'(x)] > 3[1 + 3(2 - 1)f'(1)] = 9,$$

for all $0 < x < 1$. The function V is strictly supermodular, having positive cross derivatives.

We have

$$V_1(x, x) + V_2(x, x) = \left[-5 + 3x + 2(2 - x)x^{-\frac{1}{3}}\right]e^{-3f(x)+3x}.$$

Let

$$\psi(x) = -5 + 3x + 2(2 - x)x^{-\frac{1}{3}}.$$

We have

$$\psi'(x) = 3 - \frac{4}{3}\left[x^{-\frac{1}{3}} + x^{-\frac{4}{3}}\right],$$

which is an increasing function, with $\psi'(0) = -\infty$, and $\psi'(1) > 0$. Hence, there exists unique $\tilde{x} \in (0, 1)$ such that $\psi'(\tilde{x}) = 0$. Function ψ is decreasing in $(0, \tilde{x})$ and increasing in $(\tilde{x}, 1)$. Since $\psi(1) = 0$, we have $\psi(\tilde{x}) < 0$. Thus, there exists unique $x^* \in (0, \tilde{x})$ such that $\psi(x^*) = 0$.

If $0 < x < x^*$, we have $\psi(x) > \psi(x^*) = 0$. If $x^* < x < \tilde{x}$, we have $\psi(x) < \psi(x^*) = 0$. And if $\tilde{x} < x < 1$, $\psi(x) < \psi(1) = 0$. In other words, $\psi(x) > 0$ in $(0, x^*)$ and $\psi(x) < 0$ in $(x^*, 1)$. This is also the property of $V_1(x, x) + V_2(x, x)$. The function $V(x, x)$ is single peaked in $(0, 1)$. $\square$

PROOF OF LEMMA 3.1

First consider the case $x_m < x_0 < x^*$. Let the sequence $\{x_t\}_{t=0}^\infty$ be defined as follows: $x_{t+1} = \max \Gamma(x_t)$, for all $t \geq 0$. Clearly, this sequence converges to x_m . Let T be the period such that $x_T < x^* \leq x_{T+1}$. Since $x_T, x_{T+1} \in \Gamma(x_T)$, x^* also belongs to $\Gamma(x_T)$.

The case $x_0 > x^*$, we use the same arguments with $x_{t+1} = \min \Gamma(x_t)$ for all $t \geq 0$, with the observation that this sequence converges to x_m . $\square$

PROOF OF LEMMA 3.2

(i) We will prove this by induction on T . First, consider $T = 0$. Clearly, $\Pi_0(x_0) \neq \emptyset$ if and only if $x^* \in \Gamma(x_0)$. In that case, $w_0(x_0) = V(x_0, x^*) - v^*$. Obviously, this lemma is verified for $T = 0$.

Now, assume that this lemma is true for $T \geq 0$. We will prove the same conclusion for $T + 1$. Fix $x_0 > x_m$ such that $\Pi_{T+1}(x_0) \neq \emptyset$. Fix an optimal path $(x_0, x_1, \dots, x_T, x_{T+1}, x^*)$ in $\mathcal{O}_{T+1}(x_0)$. Recall that $\Pi_T(x_1) \neq \emptyset$ and $(x_1, \dots, x_T, x_{T+1}, x^*)$ is an optimal path in $\mathcal{O}_T(x_1)$.

Consider the case $x_0 < x^*$. We will prove that $x_0 < x_1 < x^*$. Assume the contrary, then either $x_1 \leq x_0$, or $x_1 \geq x^*$.

Consider the first sub-case, $x_1 \leq x_0$. By the induction hypothesis, we have $x_1 < x_2 < \dots < x_{T+1} < x^*$. The single-peaked property of $V(x, x)$ implies

$$V_2(x_1, x_1) + V_1(x_1, x_1) > 0.$$

Since the cross derivative of V is positive, we have

$$V_2(x_0, x_1) + V_1(x_1, x_2) \geq V_2(x_1, x_1) + V_1(x_1, x_1) > 0.$$

Since $x_1 \in \text{int}(\Gamma(x_1))$, $x_2 > \min \Gamma(x_1)$. Obviously $x_2 \leq \max \Gamma(x_1)$.

By the continuity and the ascending properties of Γ , there exists ε positive and small enough such that:

$$\min \Gamma(x_1 + \varepsilon) < x_2 < \max \Gamma(x_1 + \varepsilon),$$

and therefore $x_2 \in \Gamma(x_1 + \varepsilon)$.

The sequence $(x_0, x_1 + \varepsilon, x_2, \dots, x_{T+1}, x^*)$ is feasible and links x_0 and x^* , satisfying the inequality:

$$V(x_0, x_1 + \varepsilon) + V(x_1 + \varepsilon, x_2) > V(x_0, x_1) + V(x_1, x_2).$$

This implies

$$\begin{aligned} w_{T+1}(x_0) &\geq V(x_0, x_1 + \varepsilon) - v^* + V(x_1 + \varepsilon, x_2) - v^* + \sum_{t=2}^T [V(x_t, x_{t+1}) - v^*] \\ &> \sum_{t=0}^{T+1} [V(x_t, x_{t+1}) - v^*], \end{aligned}$$

which contradicts the choice of $(x_0, x_1, \dots, x_T, x_{T+1})$.

Consider the sub-case $x_1 \geq x^*$. By the induction hypothesis, we have $x_1 \geq x_2 \geq \dots \geq x_{T+1} \geq x_{T+2} = x^*$. Every inequality becomes strict if $x_1 > x^*$ and becomes equality if $x_1 = x^*$.

Since $x_1 \in \text{int}(\Gamma(x_1))$, we have $x_2 < \max \Gamma(x_1)$. Obviously, $x_2 \geq \min \Gamma(x_1)$. Therefore, for $\varepsilon > 0$ small enough, we obtain $x_1 - \varepsilon \in \Gamma(x_0)$, and

$$\min \Gamma(x_1 - \varepsilon) \leq x_2 < \max \Gamma(x_1 - \varepsilon),$$

with the consequence that $x_2 \in \Gamma(x_1 - \varepsilon)$, with all ε positive and small enough. By the supermodularity and single-peaked properties of V , we have

$$V_2(x_0, x_1) + V_1(x_1, x_2) < V_2(x_1, x_1) + V_1(x_1, x_1) \leq 0.$$

Hence, for small $\varepsilon > 0$,

$$V(x_0, x_1 - \varepsilon) + V(x_1 - \varepsilon, x_2) > V(x_0, x_1) + V(x_1, x_2),$$

which implies

$$\begin{aligned} w_{T+1}(x_0) &\geq V(x_0, x_1 - \varepsilon) - v^* + V(x_1 - \varepsilon, x_2) - v^* + \sum_{t=2}^T [V(x_t, x_{t+1}) - v^*] \\ &> \sum_{t=0}^{T+1} [V(x_t, x_{t+1}) - v^*], \end{aligned}$$

a contradiction with the choice of $(x_0, x_1, \dots, x_T, x_{T+1})$.

Hence, $x_0 < x_1 < x^*$. Observe that

$$w_T(x_1) = \sum_{t=1}^{T+1} [V(x_t, x_{t+1}) - v^*].$$

By the induction hypothesis, we have $x_1 < x_2 < \dots < x_{T+1} < x_{T+2} = x^*$.

The case $x_0 > x^*$, using the similar arguments, with the observation that $V_2(x, x) + V_1(x, x) < 0$ if $x > x^*$, we obtain $x_0 > x_1 > \dots > x_{T+1} > x^*$.

Consider the case $x_0 = x^*$. Clearly, $\underbrace{(x^*, x^*, \dots, x^*)}_{T+2 \text{ times}}$ being feasible, $w_{T+1}(x^*) \geq 0$.

Assume that $x_1 < x^*$. By the first part of this lemma, we have $x_1 < x_2 < \dots < x_{T+1} < x^*$. Using the same argument in the first part of the proof, for $\varepsilon > 0$ small enough,

$$V(x^*, x_1 + \varepsilon) + V(x_1 + \varepsilon, x_2) > V(x^*, x_1) + V(x_1, x_2),$$

contradicts the choice of $(x^*, x_1, x_2, x_{T+1}, x^*)$ as an optimal path in $\mathcal{O}_{T+1}(x_0)$.

The case $x_1 > x^*$, a similar argument leads to a contradiction. Hence, $x_t = x^*$ for $1 \leq t \leq T+1$. As a consequence, $w_{T+1}(x^*) = 0$.

(ii) Assume that $\Pi_T(x_0) \neq \emptyset$ and $x_0 \neq x^*$.

Consider the case $x_0 < x^*$. Fix an optimal path $(x_0, x_1, \dots, x_T, x_{T+1})$ in $\mathcal{O}_T(x_0)$. By Lemma 3.2, $x_0 < x_1 < \dots < x_T < x_{T+1} = x^*$. Fix $\tilde{x}_{T+1} \in \Gamma(x_T)$ such that

$$w_1(x_T) = V(x_T, \tilde{x}_{T+1}) - v^* + V(\tilde{x}_{T+1}, x^*) - v^* = \max_{x \in \Gamma(x_T)} [V(x_T, x) - v^* + w_0(x)].$$

Applying Lemma 3.2, we have $x_T < \tilde{x}_{T+1} < x^*$, and $w_1(x_T) > V(x_T, x^*)$. Since $(x_0, x_1, \dots, x_T, \tilde{x}_{T+1}, x^*)$ belongs to $\Pi_{T+1}(x_0)$, we obtain

$$\begin{aligned} w_{T+1}(x_0) &\geq \sum_{t=0}^{T-1} [V(x_t, x_{t+1}) - v^*] + V(x_T, \tilde{x}_{T+1}) - v^* + V(\tilde{x}_{T+1}, x^*) - v^* \\ &> \sum_{t=0}^{T-1} [V(x_t, x_{t+1}) - v^*] + V(x_T, x^*) - v^* \\ &= w_T(x_0). \end{aligned}$$

The case $x_0 > x^*$, we use the same argument to prove that $w_{T+1}(x_0) > w_T(x_0)$.

We now prove that $\lim_{T \rightarrow \infty} w_T(x_0) < +\infty$. Clearly, since $w_T(x^*) = 0$ for all T , $w(x^*) = 0$.

Consider the case $x_0 \neq x^*$. Assume that $w(x_0) = +\infty$. Using the same arguments as in the proof of Lemma 3.1, we can show the existence of T_0 large enough such that x^* can reach x_0 in T_0 steps: there exists a finite feasible sequence $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{T_0}, x_{T_0+1})$ such that $\tilde{x}_0 = x^*$ and $x_{T_0+1} = x_0$.

Fix T large enough and a finite feasible sequence $(x_0, x_1, \dots, x_T, x^*)$ in $\Pi_T(x_0)$ such that

$$\sum_{t=0}^T [V(x_t, x_{t+1}) - v^*] + \sum_{t=0}^{T_0} [V(\tilde{x}_t, \tilde{x}_{t+1}) - v^*] > 0.$$

Since $(x^*, \tilde{x}_1, \dots, \tilde{x}_{T_0}, x_0, x_1, \dots, x_T, x^*)$ is a feasible path in $\Pi_{T_0+T}(x^*)$, $w_{T_0+T}(x^*) > 0$, a contradiction. Hence, $w(x_0) < +\infty$.

(iii) First, we prove that if $x_m < x_0 < x'_0 \leq x^*$, with both $\Pi_T(x_0)$ and $\Pi_T(x'_0)$ nonempty, then $w_T(x_0) < w_T(x'_0)$.

For each $x \in [x_0, x'_0]$, let $x < x_1(x) < x_2(x) < \dots < x_T(x) < x^*$ be an optimal path in $\mathcal{O}_T(x)$ starting from x . We will prove the following claim: there exists a sequence $x_0^0 < x_0^1 < x_0^2 < \dots < x_0^N < x_0^{N+1}$

such that $x_0^0 = x_0$, $x_0^{N+1} = x'_0$, and $x_1(x_0^n) \in \Gamma(x_0^{n+1})$, for all $0 \leq n \leq N$. This sequence serves as a bridge allowing us to compare $w(x_0)$ and $w(x'_0)$.

By part (i) of this lemma, for $x \in [x_0, x'_0]$, we have $\min \Gamma(x) < x \leq x_1(x) \leq \max \Gamma(x)$. Hence, for $y > x$ sufficiently close to x , we have $x_1(x) \in \Gamma(y)$. Let

$$d(x) = \sup\{d \text{ such that } x_1(x) \in \Gamma(y) \text{ for all } x < y < x + d\}.$$

Let $U(x) = (x, x + d(x))$ if $x_0 < x \leq x'_0$ and $U(x_0) = [x_0, x_0 + d(x_0))$.

Observe that for every $y \in (x_0, x'_0]$, there exists $x_0 < x < y$ such that $x_1(x) \in \Gamma(y)$. Indeed, we can simply choose any $x \in (\min \Gamma(y), y)$. Since $\max \Gamma(x) \leq \max \Gamma(y)$, we obtain the inclusion $[x, \max \Gamma(x)] \subset \Gamma(y)$, which guarantees that $x_1(x) \in \Gamma(y)$. Therefore,

$$[x_0, x'_0] \subset \bigcup_{x \in [x_0, x'_0]} U(x).$$

By the compactness of $[x_0, x'_0]$, there exist finitely many points $x_0^1 < x_0^2 < \dots < x_0^N$ such that

$$[x_0, x'_0] \subset U(x_0) \cup U(x_0^1) \cup \dots \cup U(x_0^N) \cup U(x'_0).$$

Let $x_0^0 = x_0$ and $x_0^{N+1} = x'_0$. Clearly, $U(x_0^n) \cap U(x_0^{n+1}) \neq \emptyset$, which implies that $x_1(x_0^n) \in \Gamma(x_0^{n+1})$ for all $0 \leq n \leq N$. The claim is proven.

The claim having been established, for all $0 \leq n \leq N$,

$$\begin{aligned} w_T(x_0^n) &= V(x_0^n, x_1(x_0^n)) - v^* + \sum_{t=1}^T [V(x_t(x_0^n), x_{t+1}(x_0^n)) - v^*] \\ &< V(x_0^{n+1}, x_1(x_0^n)) - v^* + \sum_{t=1}^T [V(x_t(x_0^n), x_{t+1}(x_0^n)) - v^*] \\ &\leq w_T(x_0^{n+1}). \end{aligned}$$

Therefore, $w_T(x_0) < w_T(x_0^1) < w_T(x_0^2) < \dots < w_T(x_0^N) < w_T(x'_0)$.

The case $x^* \leq x_0 < x'_0$ follows by similar arguments. The case $x_0 < x^* < x'_0$ is then immediate, once the first two cases have been proven.

(iv) We prove that the function w is continuous at x^* . For any $\varepsilon > 0$, choose $d > 0$ such that for every $x \in (x^* - d, x^* + d)$, we have $x \in \Gamma(x^*)$, $x^* \in \Gamma(x)$, both $V(x^*, x), V(x, x^*)$ are feasible, and $V(x^*, x), V(x, x^*) \in (v^* - \varepsilon/2, v^* + \varepsilon/2)$.

Let $x_0 \in (x^* - d, x^* + d)$. We will prove that $-\varepsilon \leq w(x_0) \leq \varepsilon$. Suppose, to the contrary, this is not the case. Since $w(x^*) = 0$, this implies that $x_0 \neq x^*$, and hence $w_0(x_0) < w(x_0)$, by part (ii) of Lemma

3.2.

If $w(x_0) < -\varepsilon$, then

$$w(x_0) < -\frac{\varepsilon}{2} < V(x_0, x^*) - v^* = w_0(x_0) < w(x_0),$$

a contradiction.

Now, consider the case $w(x_0) > \varepsilon$. This implies the existence of T sufficiently large such that $w_T(x_0) > \varepsilon$. Let $(x_0, x_1, x_2, \dots, x_T, x^*)$ be an optimal path in $\mathcal{O}_T(x_0)$. Observe that $(x^*, x_0, x_1, x_2, \dots, x_T, x^*)$ is a feasible path in $\Pi_{T+1}(x^*)$. Then we have

$$\begin{aligned} w(x^*) &\geq w_{T+1}(x^*) \\ &\geq V(x^*, x_0) - v^* + \sum_{t=0}^T [V(x_t, x_{t+1}) - v^*] \\ &= V(x^*, x_0) - v^* + w_T(x_0) \\ &> \frac{\varepsilon}{2}, \end{aligned}$$

which implies $w(x^*) > 0$, a contradiction. $\square$

PROOF OF PROPOSITION 3.3

(i) Fix $d > 0$. For each $T \geq 0$, let

$$t^*(T) = \min \left\{ t \text{ such that } x_t^T \in (x^* - d, x^* + d) \right\}.$$

Consider the case $x_0 < x^*$. Then $\{x_t^T\}_{t=0}^{t^*(T)-1}$ is an increasing path with respect to t and bounded above by $x^* - d$. By part (i) of Lemma 3.2, $x_t^T < x_{t+1}^T < x^*$ for all t , and since $w_{T-t^*(T)}$ is increasing,

$$\begin{aligned} w_T(x_0) &= \sum_{t=0}^{t^*(T)-1} [V(x_t^T, x_{t+1}^T) - v^*] + w_{T-t^*(T)}(x_{t^*(T)}^T) \\ &< \sum_{t=0}^{t^*(T)-1} [V(x_{t+1}^T, x_{t+1}^T) - v^*] + w_{T-t^*(T)}(x^*) \\ &< \sum_{t=0}^{t^*(T)-1} [V(x^* - d, x^* - d) - v^*] + w(x^*) \\ &\leq t^*(T) \times [V(x^* - d, x^* - d) - v^*] + w(x^*). \end{aligned}$$

Hence, if $t^*(T)$ is not bounded, when T go to infinity, while the left hand side converges to $w(x_0)$, the right hand side goes to $-\infty$, a contradiction. We conclude that there exists $T^* \geq 0$ such that $x_t^T > x^* - d$ for every $t \geq T^*$.

Consider the case $x_0 > x^*$. Clearly, it is enough to consider the case that $x_0 > x^* + d$ and the optimal path $\{x_t^T\}_{t=0}^{t^*(T)-1}$ is decreasing and bounded below by $x^* + d$.

Fix $\tilde{d} > d$ sufficiently close to d such that $V(x^* + \tilde{d}, x^* + d) < v^*$. Let

$$\mathcal{T}(T) = \left\{ t \leq t^*(T) \text{ such that } x_t^T > x^* + \tilde{d} \text{ and } V(x_t^T, x_{t+1}^T) \geq V(x^* + d, x^* + d) \right\}.$$

We will prove that the number of elements of $\mathcal{T}(T)$ is bounded by some T_0 independently of T . To do this, we construct a sequence $\{\hat{x}_t\}_{t=0}^\infty$ such that $V(\hat{x}_t, \hat{x}_{t+1}) = V(x^* + d, x^* + d)$ for all $t \geq 0$, and show that this sequence enters the interval $(x^* + d, x^* + \tilde{d})$ in a finite number of steps T_0 .

Let $\hat{x}_0 = x_0$. Observe that $V(x_0, x_0) < V(x^* + d, x^* + d)$ and

$$V(x_0, \min \Gamma(x_0)) \geq V(x^* + d, \min \Gamma(x^* + d)) \geq V(x^* + d, x^* + d).$$

Hence, there exists $\hat{x}_1 \in \Gamma(x_0)$ such that $V(x_0, \hat{x}_1) = V(x^* + d, x^* + d)$. Clearly, $x_0 > \hat{x}_1 > x^* + d$. We continue the process with $\hat{x}_1$ in place of x_0 , and so on, to obtain the sequence $\{\hat{x}_t\}_{t=0}^\infty$ such that $V(\hat{x}_t, \hat{x}_{t+1}) = V(x^* + d, x^* + d)$ for all t . This sequence is decreasing and converges to $x^* + d$.

Fix T_0 large enough such that $\hat{x}_t < x^* + \tilde{d}$ for all $t \geq T_0$. Take any K periods in $\mathcal{T}(T)$: $t_0 < t_1 < \dots < t_k < \dots < t_K$. Since $x_{t_0}^T \leq x_0$ and $V(x_0, \hat{x}_1) = V(x^* + d, x^* + d) \leq V(x_{t_0}^T, x_{t_0+1}^T)$, we obtain $x_{t_1}^T \leq x_{t_0+1}^T \leq \hat{x}_1$. By induction, for all $1 \leq k \leq K$, we have $x_{t_k}^T \leq \hat{x}_k$. It follows that $K \leq T_0$, since $\hat{x}_{T_0} < x^* + \tilde{d}$. In other words, the number of elements in $\mathcal{T}(T)$ does not exceed T_0 .

For all $t \leq t^*(T)$, if $t \in \mathcal{T}(T)$, then $x_t^T > x^* + \tilde{d}$. If $t \notin \mathcal{T}(T)$, then either $V(x_t^T, x_{t+1}^T) < V(x^* + d, x^* + d)$, or $x_t^T < x^* + \tilde{d}$. In either case, we have $V(x_t^T, x_{t+1}^T) < V(x^* + \tilde{d}, x^* + d)$ whenever $t \notin \mathcal{T}(T)$.

By part (iii) of Lemma 3.2, the function $w_{T-t^*(T)}$ is strictly increasing. For all T large enough,

$$\begin{aligned} w_T(x_0) &= \sum_{t=0}^{t^*(T)-1} [V(x_t^T, x_{t+1}^T) - v^*] + w_{T-t^*(T)}(x_{t^*(T)}^T) \\ &\leq \sum_{t \in \mathcal{T}(T) \text{ and } t \leq t^*(T)} [V(x_t^T, x_{t+1}^T) - v^*] + \sum_{t \notin \mathcal{T}(T) \text{ and } t \leq t^*(T)} [V(x_t^T, x_{t+1}^T) - v^*] + w_{T-t^*(T)}(x_0) \\ &\leq T_0 \times [V(x_0, \min \Gamma(x_0)) - v^*] + (t^*(T) - T_0) \times [V(x^* + \tilde{d}, x^* + d) - v^*] + w(x_0). \end{aligned}$$

Let T go to infinity. If $t^*(T)$ goes to infinity, the right hand side converges to $-\infty$, while the left hand side converges to $w(x_0)$, which is finite by Lemma 3.2, a contradiction.

The case $x_0 = x^*$ is obvious.

(ii) First, we prove the following claim. For any $T \geq 0$, let correspondence φ_{T+1} be defined by

$$\varphi_{T+1}(x_0) = \operatorname{argmax}_{x_1 \in \Gamma(x_0)} [V(x_0, x_1) - v^* + w_T(x_1)].$$

Then, if $x_0 < \hat{x}_0$, $x_1 \in \varphi_{T+1}(x_0)$, and $\hat{x}_1 \in \varphi_{T+1}(\hat{x}_0)$, we have $x_1 \leq \hat{x}_1$. Moreover, $\min \varphi_{T+1}(x_0) \leq \min \varphi_{T+1}(\hat{x}_0)$, and $\max \varphi_{T+1}(x_0) \leq \max \varphi_{T+1}(\hat{x}_0)$.

Indeed, assume the contrary, that $\hat{x}_1 < x_1$. The ascending property of Γ ensures that $x_1 \in \Gamma(\hat{x}_0)$ and $\hat{x}_1 \in \Gamma(x_0)$. By the choice of x_1 and $\hat{x}_1$:

$$\begin{aligned} V(x_0, x_1) - v^* + w_T(x_1) &\geq V(x_0, \hat{x}_1) - v^* + w_T(\hat{x}_1), \\ V(\hat{x}_0, \hat{x}_1) - v^* + w_T(\hat{x}_1) &\geq V(\hat{x}_0, x_1) - v^* + w_T(x_1). \end{aligned}$$

Adding the two inequalities we obtain

$$V(x_0, x_1) + V(\hat{x}_0, \hat{x}_1) \geq V(x_0, \hat{x}_1) + V(\hat{x}_0, x_1),$$

which contradicts the supermodularity property of V . Thus $x_1 \leq \hat{x}_1$. It is clear that this implies $\min \varphi_{T+1}(x_0) \leq \min \varphi_{T+1}(\hat{x}_0)$, and $\max \varphi_{T+1}(x_0) \leq \max \varphi_{T+1}(\hat{x}_0)$.

We now return to the proof of this proposition. We need only to prove the Bellman equation holds for $x_0 > x_m$.

Fix $\varepsilon > 0$. Choose $d > 0$ such that, if $x^* - d < x < x^* + d$, then $-\varepsilon < w(x) < \varepsilon$.

Fix $\underline{x}_0 < x_0 < \bar{x}_0$ such that $\underline{x}_0 > x_m$. Consider a sequence $\{x_0^n\}_{n=0}^\infty$ converging to x_0 . Without loss of generality, we can assume that, for every n , $\underline{x}_0 < x_0^n < \bar{x}_0$.

We prove that w is upper semicontinuous at x_0 . By part (i), there exists T^* large enough such that, for all $T \geq T^*$, every optimal path in $\mathcal{O}_T(\underline{x}_0)$ and $\mathcal{O}_T(\bar{x}_0)$ enters the interval $(x^* - d, x^* + d)$ from the T^{*th} period onward.

For each n and T , let sequence $(x_0^{n,T}, x_1^{n,T}, \dots, x_{T^*+T}^{n,T}, x^*)$ be the optimal path in $\mathcal{O}_{T^*+T}(x_0^n)$ that is defined as: $x_{t+1}^{n,T} = \min \varphi_{T^*+T-t}(x_t^{n,T})$, for all $0 \leq t \leq T^* + T$. Let $(\underline{x}_0, \underline{x}_1, \dots, \underline{x}_{T^*+T}, x^*)$ and $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{T^*+T}, x^*)$ be two optimal paths in $\mathcal{O}_{T^*+T}(\underline{x}_0)$ and $\mathcal{O}_{T^*+T}(\bar{x}_0)$ such that $\underline{x}_{t+1} = \min \varphi_{T^*+T-t}(\underline{x}_t)$ and $\bar{x}_{t+1} = \min \varphi_{T^*+T-t}(\bar{x}_t)$, with $0 \leq t \leq T^* + T$.

Since $\underline{x}_0 < x_0 < \bar{x}_0$, we have $\underline{x}_1 \leq x_1^{n,T} \leq \bar{x}_1$. This implies $\underline{x}_2 \leq x_2^{n,T} \leq \bar{x}_2$, and so on. By induction, we obtain $\underline{x}_t \leq x_t^{n,T} \leq \bar{x}_t$, for all $0 \leq t \leq T^* + T$. Since $\underline{x}_{T^*}$ and $\bar{x}_{T^*}$ belong to $(x^* - d, x^* + d)$, it follows that $x_{T^*}^{n,T}$ belongs to that interval. We have:

$$w_{T^*+T}(x_0^n) = \sum_{t=0}^{T^*-1} [V(x_t^{n,T}, x_{t+1}^{n,T}) - v^*] + w_T(x_{T^*}^{n,T}).$$

Observe that $w_T(x_{T^*}^{n,T}) < w(x_{T^*}^{n,T}) < \varepsilon$. Let $\{x_0^{n_k}\}_{k=0}^\infty$ be a subsequence such that

$$\lim_{k \rightarrow \infty} w(x_0^{n_k}) = \limsup_{n \rightarrow \infty} w(x_0^n).$$

Fix n_k . Consider a sequence $\{T_\ell^k\}_{\ell=0}^\infty$ going to infinity such that, for $t \leq T^*$, we have

$$\lim_{\ell \rightarrow \infty} x_t^{n_k, T_\ell^k} = x_t^{n_k}.$$

For T_ℓ^k large enough, we have:

$$\sum_{t=0}^{T^*-1} \left[V \left(x_t^{n_k, T_\ell^k}, x_{t+1}^{n_k, T_\ell^k} \right) - v^* \right] < \sum_{t=0}^{T^*-1} [V(x_t^{n_k}, x_{t+1}^{n_k}) - v^*] + \varepsilon.$$

Hence, with $w_{T_\ell^k}(x_{T^*}^{n_k, T_\ell^k}) < \varepsilon$,

$$\begin{aligned} w_{T^*+T_\ell^k}(x_0^{n_k}) &= \sum_{t=0}^{T^*-1} \left[V \left(x_t^{n_k, T_\ell^k}, x_{t+1}^{n_k, T_\ell^k} \right) - v^* \right] + w_{T_\ell^k} \left(x_{T^*}^{n_k, T_\ell^k} \right) \\ &< \sum_{t=0}^{T^*-1} [V(x_t^{n_k}, x_{t+1}^{n_k}) - v^*] + 2\varepsilon. \end{aligned}$$

Letting T_ℓ^k go to infinity we obtain

$$w(x_0^{n_k}) \leq \sum_{t=0}^{T^*-1} [V(x_t^{n_k}, x_{t+1}^{n_k}) - v^*] + 2\varepsilon. \quad (4)$$

Take a subsequence $\{x_0^{n_{k_j}}\}_{j=0}^\infty$ such that for all $t \leq T^*$, $x_t^{n_{k_j}}$ converges to some value x_t : $\lim_{j \rightarrow \infty} x_t^{n_{k_j}} = x_t$. With n_{k_j} large enough, we have

$$\sum_{t=0}^{T^*-1} [V(x_t^{n_{k_j}}, x_{t+1}^{n_{k_j}}) - v^*] < \sum_{t=0}^{T^*-1} [V(x_t, x_{t+1}) - v^*] + \varepsilon. \quad (5)$$

Since x_{T^*} belongs to $(x^* - d, x^* + d)$, for every $T \geq 0$,

$$\begin{aligned} w(x_0) &\geq \sum_{t=0}^{T^*-1} [V(x_t, x_{t+1}) - v^*] + w_{T^*+T}(x_{T^*}) \\ &\geq \sum_{t=0}^{T^*-1} [V(x_t, x_{t+1}) - v^*] - \varepsilon. \end{aligned}$$

Combining this with (4) and (5), then

$$\limsup_{j \rightarrow \infty} w(x_0^{n_{k_j}}) \leq w(x_0) + 3\varepsilon.$$

By the choice of sequence $\{x_0^{n_k}\}_{k=0}^\infty$, we have

$$\limsup_{n \rightarrow \infty} w(x_0^n) \leq w(x_0) + 3\varepsilon.$$

Since ε is chosen arbitrarily, we obtain

$$\limsup_{n \rightarrow \infty} w(x_0^n) \leq w(x_0).$$

Therefore, the function w is upper semi-continuous. We now prove that w is a solution to the Bellman equation. First, observe that the upper semi-continuity of w and the compactness guarantee that $\operatorname{argmax}_{\Gamma(x_0)} [V(x_0, x_1) - v^* + w(x_1)]$ is nonempty, for all $x_0 > x_m$.

For all T such that $\Pi_T(x_0) \neq \emptyset$, choose $x_1^T \in \Gamma(x_0)$ such that

$$w_{T+1}(x_0) = V(x_0, x_1) - v^* + w_T(x_1^T).$$

Fix any $\varepsilon > 0$. Take a subsequence $\{x_1^{T_k}\}_{k=0}^\infty$ that converges to $x_1 \in \Gamma(x_0)$. By the upper-continuity of w , for every T_k large enough, $w(x_1^{T_k}) \leq w(x_1) + \varepsilon$. Therefore,

$$w_{T_k+1}(x_0) = V(x_0, x_1) - v^* + w_{T_k}(x_1^{T_k}) \leq V(x_0, x_1) - v^* + w(x_1^{T_k}) \leq V(x_0, x_1) - v^* + w(x_1) + \varepsilon.$$

Letting T_k go to infinity,

$$w(x_0) \leq V(x_0, x_1) - v^* + w(x_1) + \varepsilon.$$

Since ε is chosen arbitrarily,

$$w(x_0) \leq \max_{x_1 \in \Gamma(x_0)} [V(x_0, x_1) - v^* + w(x_1)].$$

For any $x_1 \in \Gamma(x_0)$,

$$w(x_0) \geq w_{T+1}(x_0) \geq V(x_0, x_1) - v^* + w_T(x_1).$$

Letting T go to infinity, we have

$$w(x_0) \geq V(x_0, x_1) - v^* + w(x_1).$$

Since x_1 is chosen arbitrarily,

$$w(x_0) \geq \max_{x_1 \in \Gamma(x_0)} [V(x_0, x_1) - v^* + w(x_1)].$$

We have thus proven that w is a solution to the Bellman functional equation. Using the same arguments as in the proof of part (iii) of Proposition 3.2, we can prove that w is strictly increasing. $\square$

PROOF OF LEMMA 3.4

Consider a sequence $\{x_t\}_{t=0}^\infty$ starting from x_0 such that

$$\limsup_{t \rightarrow \infty} x_t < x^*.$$

We will prove that this is not a good program. Clearly, we only need to consider the case $x_t > x_m$ for every t .

Let $\bar{x} = \limsup_{t \rightarrow \infty} x_t < x^*$. Fix $d > 0$ small enough such that $0 < \bar{x} - d < \bar{x} + d < x^*$, and $V(\bar{x} + d, \bar{x} - d) < V(x^*, x^*)$.

By the definition of $\bar{x}$, there exists a sequence of periods $T_0 < T_1 < T_2 < \dots \rightarrow +\infty$ such that $x_t < \bar{x} + d$ for $t \geq T_0$ and $\bar{x} - d < x_{T_n} < \bar{x} + d$ for every $n \geq 0$. We construct a new sequence $\{\tilde{x}_t\}_{t=0}^\infty$ as follows.

Let $\tilde{x}_t = x_t$ for all $0 \leq t \leq T_0$, and $\tilde{x}_{T_n} = x_{T_n}$ for every n . For each n such that $T_n + 1 < T_{n+1}$, let

$$(\tilde{x}_{T_n}, \tilde{x}_{T_n+1}, \dots, \tilde{x}_{T_{n+1}-1}, \tilde{x}_{T_{n+1}}) = \operatorname{argmax}_{t=T_n}^{T_{n+1}-1} V(y_t, y_{t+1}),$$

such that $y_{T_n} = x_{T_n}, y_{T_{n+1}} = x_{T_{n+1}}$,

for $T_n \leq t \leq T_{n+1} - 1, y_{t+1} \in \Gamma(y_t)$ and $y_{t+1} \leq \bar{x} + d$.

By the continuity of Γ , the problem is defined on a compact set, which is non-empty since $(x_{T_n}, x_{T_n}, \dots, x_{T_n}, x_{T_{n+1}})$ is feasible. Hence, there exists an optimal solution $\tilde{x}_{T_n}, \tilde{x}_{T_n+1}, \dots, \tilde{x}_{T_{n+1}-1}, \tilde{x}_{T_{n+1}}$. Since $x_t < \bar{x} + d$ for $t \geq T_0$, it is obvious that

$$\sum_{t=T_n}^{T_{n+1}-1} V(\tilde{x}_t, \tilde{x}_{t+1}) \geq \sum_{t=T_n}^{T_{n+1}-1} V(x_t, x_{t+1}).$$

Taking the sum for every $n \geq 1$,

$$\begin{aligned} \sum_{t=T_0}^{T_n-1} [V(\tilde{x}_t, \tilde{x}_{t+1}) - v^*] &\geq \sum_{k=0}^{n-1} \sum_{t=T_k}^{T_{k+1}-1} [V(\tilde{x}_t, \tilde{x}_{t+1}) - v^*] \\ &\geq \sum_{k=0}^{n-1} \sum_{t=T_k}^{T_{k+1}-1} [V(x_t, x_{t+1}) - v^*] \\ &= \sum_{t=T_0}^{T_n-1} [V(x_t, x_{t+1}) - v^*]. \end{aligned}$$

Hence, sequence $\{x_t\}_{t=0}^\infty$ is dominated by $\{\tilde{x}_t\}_{t=0}^\infty$.

We will prove that, for all $t \geq T_0$, we have $\tilde{x}_t \in (\bar{x} - d, \bar{x} + d)$. More precisely, for all $n \geq 0$ such that $T_n + 1 < T_{n+1}$,

$$\min_{T_n \leq t \leq T_{n+1}} \tilde{x}_t \geq \min\{x_{T_n}, x_{T_{n+1}}\}.$$

Indeed, assume the contrary, $\min_{T_n \leq t \leq T_{n+1}} \tilde{x}_t < \min\{x_{T_n}, x_{T_{n+1}}\}$. Fix t^* the largest index of time t between T_n and T_{n+1} such that $\tilde{x}_{t^*} = \min_{T_n \leq t \leq T_{n+1}} \tilde{x}_t$. By the choice of T_n and t^* , we have $T_n + 1 \leq t^* \leq T_{n+1} - 1$, and

$$\tilde{x}_{t^*-1} \geq \tilde{x}_{t^*} \text{ and } \tilde{x}_{t^*+1} > x_{t^*}.$$

Using the same arguments as in the proof of Lemma 3.2, we can find $\varepsilon > 0$ sufficiently small such that $\tilde{x}_{t^*} + \varepsilon < \bar{x} + d$, and

$$V(\tilde{x}_{t^*-1}, \tilde{x}_{t^*} + \varepsilon) + V(\tilde{x}_{t^*} + \varepsilon, \tilde{x}_{t^*+1}) > V(\tilde{x}_{t^*-1}, \tilde{x}_{t^*}) + V(\tilde{x}_{t^*}, \tilde{x}_{t^*+1}),$$

a contradiction with the choice of $\tilde{x}_{T_n}, \tilde{x}_{T_n+1}, \dots, x_{T_{n+1}-1}, \tilde{x}_{T_{n+1}}$. Hence, for every $n \geq 0$, we have

$$\min_{T_n \leq t \leq T_{n+1}} \tilde{x}_t \geq \min\{x_{T_n}, x_{T_{n+1}}\},$$

which implies $\bar{x} - d \leq \tilde{x}_t \leq \bar{x} + d$ for all $t \geq T_0$. Therefore $V(\tilde{x}_t, \tilde{x}_{t+1}) < V(\bar{x} + d, \bar{x} - d) < V(x^*, x^*)$, and

$$\lim_{n \rightarrow \infty} \sum_{t=T_0}^{T_n-1} [V(\tilde{x}_t, \tilde{x}_{t+1}) - v^*] = -\infty.$$

Hence,

$$\begin{aligned} \liminf_{T \rightarrow \infty} \sum_{t=0}^T [V(x_t, x_{t+1}) - v^*] &\leq \liminf_{n \rightarrow \infty} \sum_{t=0}^{T_n} [V(x_t, x_{t+1}) - v^*] \\ &= \sum_{t=0}^{T_0-1} [V(x_t, x_{t+1}) - v^*] + \liminf_{n \rightarrow \infty} \sum_{t=T_0}^{T_n-1} [V(\tilde{x}_t, \tilde{x}_{t+1}) - v^*] \\ &= -\infty. \end{aligned}$$

Now, assume that

$$\underline{x} = \liminf_{t \rightarrow \infty} x_t > x^*.$$

We will prove that this is not a good program. Observe that if $x_M < +\infty$, $\underline{x} \leq x_M$.

Choose $d > 0$ such that $\underline{x} - d > x^*$ and $V(\underline{x} + d, \underline{x} - d) < V(x^*, x^*)$. Consider a sequence $T_0 < T_1 < \dots < T_n < \dots$ such that $x_t > \underline{x} - d$ for $t \geq T_0$ and $\underline{x} - d < x_{T_n} < \underline{x} + d$ for every n . We define the sequence $\{\hat{x}_t\}$ in the same way as in the first part: $\hat{x}_t = x_t$ for $0 \leq t \leq T_0$, $\hat{x}_{T_n} = x_{T_n}$ for every n , and

$$(\hat{x}_{T_n}, \hat{x}_{T_n+1}, \dots, \hat{x}_{T_{n+1}-1}, \hat{x}_{T_{n+1}}) = \operatorname{argmax}_{t=T_n}^{T_{n+1}-1} V(y_t, y_{t+1}),$$

$$\text{such that } y_{T_n} = x_{T_n}, y_{T_{n+1}} = x_{T_{n+1}},$$

$$\text{for } T_n \leq t \leq T_{n+1} - 1, y_{t+1} \in \Gamma(y_t) \text{ and } y_{t+1} \geq \underline{x} - d.$$

Using the same argument as in the first part, we can prove the existence of a solution to the problem such that $\hat{x}_t \leq \max\{x_{T_n}, x_{T_{n+1}}\}$ for all $T_n \leq t \leq T_{n+1}$. This implies

$$\liminf_{T \rightarrow \infty} \sum_{t=0}^T [V(x_t, x_{t+1}) - v^*] \leq \lim_{n \rightarrow \infty} \sum_{t=0}^{T_n} [V(\hat{x}_t, \hat{x}_{t+1}) - v^*] = -\infty.$$

□

PROOF OF PROPOSITION 3.5

(i) We only need to consider the case $x_0 > x_m$. We first prove that $v(x_0) \geq w(x_0)$. Indeed, consider any optimal path $(x_0, x_1, \dots, x_T, x^*)$ in $\mathcal{O}_T(x_0)$. Since $(x_0, x_1, \dots, x_T, x^*, x^*, x^*, \dots)$ is a feasible path in $\Pi(x_0)$, we get

$$v(x_0) \geq \sum_{t=0}^T [V(x_t, x_{t+1}) - v^*] = w_T(x_0).$$

Then $v(x_0) \geq \sup_{T \geq 0} w_T(x_0) = w(x_0)$.

Now, we prove that $v(x_0) \leq w(x_0)$. By the continuity of w at x^* , we can fix any $\varepsilon > 0$ and $d > 0$ such that if $x^* - d < x < x^* + d$, then $-\varepsilon < w(x) < \varepsilon$. Moreover, $x^* \in \Gamma(x^* - d)$.

Fix any feasible path $\{x_t\}_{t=0}^\infty$ starting from x_0 such that

$$\liminf_{T \rightarrow \infty} \sum_{t=0}^T [V(x_t, x_{t+1}) - v^*] > -\infty.$$

By Lemma 3.4, there exists a sequence of periods $T_0 < T_1 < T_2 < \dots < T_n < \dots$ such that $x_{T_n} > x^* - d$ for every n .

Fix $n \geq 0$ and $T > 0$. Consider a sequence $(x_{T_n}, \hat{x}_{T_n} + 1, \hat{x}_{T_n+2}, \dots, \hat{x}_{T_n+T}, x^*)$ in $\mathcal{O}_T(x_{T_n})$. Since $(x_0, x_1, \dots, x_{T_n}, \hat{x}_{T_n+1}, \hat{x}_{T_n+2}, \dots, \hat{x}_{T_n+T}, x^*)$ belongs to $\Pi_{T_n+T}(x_0)$, we obtain:

$$\sum_{t=0}^{T_n} [V(x_t, x_{t+1}) - v^*] + \sum_{t=0}^T [V(\hat{x}_{T_n+t}, \hat{x}_{T_n+t+1}) - v^*] \leq w_{T_n+T}(x_0).$$

In other words,

$$\sum_{t=0}^{T_n} [V(x_t, x_{t+1}) - v^*] + w_T(x_{T_n}) \leq w_{T_n+T}(x_0).$$

Let T go to infinity, we obtain:

$$\sum_{t=0}^{T_n} [V(x_t, x_{t+1}) - v^*] + w(x_{T_n}) \leq w(x_0).$$

This implies

$$\sum_{t=0}^{T_n} [V(x_t, x_{t+1}) - v^*] \leq w(x_0) - w(x_{T_n}) < w(x_0) + \varepsilon.$$

Hence,

$$\liminf_{n \rightarrow \infty} \sum_{t=0}^{T_n} [V(x_t, x_{t+1}) - v^*] \leq w(x_0) + \varepsilon.$$

Since the sequence $\{x_t\}_{t=0}^\infty$ and ε are both chosen arbitrarily, we conclude that $v(x_0) \leq w(x_0)$.

(ii) Assume that the path $\{x_t\}_{t=0}^\infty$ is optimal under the overtaking criterion.

Consider the case $x_0 < x^*$. We first prove that

$$v(x_0) = V(x_0, x_1) - v^* + v(x_1).$$

Indeed, this follows from

$$\begin{aligned} V(x_0, x_1) - v^* + v(x_1) &\leq v(x_0) \\ &= \liminf_{T \rightarrow \infty} \sum_{t=0}^T [V(x_t, x_{t+1}) - v^*] \\ &= V(x_0, x_1) - v^* + \liminf_{T \rightarrow \infty} \sum_{t=1}^T [V(x_t, x_{t+1}) - v^*] \\ &\leq V(x_0, x_1) - v^* + v(x_1). \end{aligned}$$

By induction, for all $t \geq 0$, we have

$$v(x_t) = V(x_t, x_{t+1}) - v^* + v(x_{t+1}).$$

We show that, for all t , if $x_t \neq x^*$, then $x_t \neq x_{t+1}$. Indeed, assume the contrary. Then

$$v(x_t) = V(x_t, x_t) - v^* + v(x_t),$$

which implies $V(x_t, x_t) = v^*$, a contradiction.

Assume $x_1 < x_0$. Mimicking the argument used at the beginning for the proof of part (ii) of Proposition 3.3, from $x_0 > x_1$, we obtain $x_1 \geq x_2$. Since $x_1 \neq x^*$, we have $x_1 \neq x_2$, it follows that $x_0 > x_1 > x_2$. By induction, the sequence $\{x_t\}_{t=0}^\infty$ is strictly decreasing and converges to some $x < x^*$, contradicting Lemma 3.4.

Hence, $x_0 < x_1$. We prove that $x_1 < x^*$. If $x^* < x_1$, again mimicking the proof of Proposition 3.3, we obtain $x^* < x_1 \leq x_2$, thus $x_0 < x_1 < x_2$. By induction, we have $x^* < x_t < x_{t+1}$ for all t . Thus the sequence $\{x_t\}_{t=0}^\infty$ is increasing and then converges to some $x > x^*$, contradicting Lemma 3.4.

If $x_1 = x^*$, then by Lemma 3.2,

$$v(x_0) = V(x_0, x^*) - v^* < w_1(x_0) < w(x_0) = v(x_0),$$

a contradiction.

Thus, $x_0 < x_1 < x^*$. By induction, we can prove that the sequence $\{x_t\}_{t=0}^\infty$ is strictly increasing and is bounded above by x^* . Combining this result with Lemma 3.4, this path converges to x^* .

The case $x_0 > x^*$ follows by similar arguments.

(iii) Consider the case $x_0 < x^*$.

Since for every t , $v(x_t) = V(x_t, x_{t+1}) - v^* + v(x_{t+1})$, we have

$$x_{t+1} \in \operatorname{argmax}_{y \in \Gamma(x_t)} [V(x_t, y) - v^* + v(y)].$$

Mimicking the proof of part (iii), the sequence $\{x_t\}_{t=0}^\infty$ is strictly monotone and converges to x^* . Hence, $\lim_{T \rightarrow \infty} v(x_T) = 0$

Letting T go to infinity,

$$v(x_0) = \sum_{t=0}^{T-1} [V(x_t, x_{t+1}) - v^*] + v(x_T) = \sum_{t=0}^{\infty} [V(x_t, x_{t+1}) - v^*].$$

The sequence $\{x_t\}_{t=0}^\infty$ is optimal under the overtaking criterion.

The cases $x_0 > x^*$ and $x_0 = x^*$ follow by the same arguments.

□

PROOF OF PROPOSITION 4.1

First, assume that the sequence $\{x_t^\delta\}_{t=0}^\infty$ is defined as follows: $x_{t+1} = \min \varphi(\delta, x_t)$, for all $t \geq 0$. By Theorem 2 in Amir (1996) and an induction argument, x_t^δ is nondecreasing with respect to δ , for all t . We can therefore define $\bar{x}_t = \lim_{\delta \rightarrow 1} x_t^\delta$. The continuity of Γ guarantees that the sequence $\{\bar{x}_t\}_{t=0}^\infty$ is feasible.

Consider the case $x_0 < x^*$. We first prove that for all δ close enough to 1, the optimal sequence $\{x_t^\delta\}_{t=0}^\infty$ is nondecreasing with respect to t and converges to a steady state $x(\delta)$ that is strictly positive, when t goes to infinity.

Assume the contrary, then there exists a set of discount factors $\mathcal{D} = \{\delta_n\}_{n=0}^\infty$ converging to 1 such that, for all $\delta \in \mathcal{D}$, the corresponding optimal sequence $\{x_t^\delta\}_{t=0}^\infty$ is nonincreasing.

As a consequence, the sequence $\{\bar{x}_t\}_{t=0}^\infty$ is nonincreasing, bounded above by x_0 , and therefore is not a good program, by Lemma 3.4.

For each $\delta \in \mathcal{D}$, let $\mathcal{T}(\delta)$ be the set of periods t such that $V(x_t^\delta, x_{t+1}^\delta) \geq v^*$. We will prove that the number of this set is bounded above by some T_0 , independently of the choice of δ .

If $V(x_0, \min \Gamma(x_0)) < v^*$, then $V(x_t^\delta, x_{t+1}^\delta) \leq V(x_0, \min \Gamma(x_0)) \leq v^*$ for all $t \geq 0$. Hence, $\mathcal{T}(\delta)$ is empty.

Now assume that $V(x_0, \min \Gamma(x_0)) \geq v^*$. Let $\hat{x}_1 < x_0$ be defined such that $V(x_0, \hat{x}_1) = v^*$. If $V(\hat{x}_1, \min \Gamma(\hat{x}_1)) < v^*$, we stop. Otherwise, we define $\hat{x}_2 < \hat{x}_1$ such that $V(\hat{x}_1, \hat{x}_2) = v^*$, and so on. Clearly, this process can not continue indefinitely, otherwise we would obtain a decreasing sequence $\{\hat{x}_t\}_{t=0}^\infty$ such that $V(\hat{x}_t, \hat{x}_{t+1}) = v^*$ with all t , converging to some $x < x^*$, a contradiction.

Hence, there exists $T_0 > 0$ and $\hat{x}_0, x_0, \hat{x}_1, \hat{x}_2, \dots, \hat{x}_{T_0}$ such that $\hat{x}_0 = x_0$, $V(\hat{x}_t, \hat{x}_{t+1}) = v^*$ for all $0 \leq t \leq T_0 - 1$, and $V(\hat{x}_{T_0}, \min \Gamma(\hat{x}_{T_0})) < v^*$.

We return to $\mathcal{T}(\delta)$ and prove that the number of its elements is less than T_0 . Consider any set of K elements in $\mathcal{T}(\delta)$: $t_0 < t_1 < \dots < t_K$. For all $0 \leq k \leq K$, $V(x_{t_k}^\delta, x_{t_{k+1}}^\delta) \geq v^*$.

For $k = 0$, we have $V(x_{t_0}^\delta, x_{t_0+1}^\delta) \geq v^* = V(x_0, \hat{x}_1)$. Since $x_{t_0} \leq x_0$, this implies $x_{t_0+1}^\delta \leq \hat{x}_1$. Therefore, $x_{t_1}^\delta \leq x_{t_0+1}^\delta \leq \hat{x}_1$. By induction, we obtain $x_{t_k}^\delta \leq \hat{x}_k$ for all $0 \leq k \leq K$. Thus $K \leq T_0$, otherwise we would have $x_{T_0}^\delta \leq \hat{x}_{T_0}$ and

$$v^* \leq V(x_{T_0}^\delta, x_{T_0+1}^\delta) \leq V(x_{T_0}^\delta, \min \Gamma(x_{T_0}^\delta)) \leq V(\hat{x}_{T_0}, \min \Gamma(\hat{x}_{T_0})) < v^*,$$

a contradiction. Hence, the set $\mathcal{T}(\delta)$ contains at most T_0 elements, independently of the choice of δ .

Now, fix any optimal path $\{x_t^*\}_{t=0}^\infty$ starting from x_0 under overtaking criterion. By Proposition 3.5,

this sequence is strictly increasing and converges to x^* . Since

$$v(x_t^*) = V(x_t^*, x_{t+1}^*) - v^* + v(x_{t+1}^*) < v(x_{t+1}^*),$$

we have $V(x_t^*, x_{t+1}^*) < v^*$ for all t . Hence, for all $t \geq 0$,

$$\delta^t [V(x_t^*, x_{t+1}^*) - v^*] \geq V(x_t^*, x_{t+1}^*) - v^*. \quad (6)$$

Consider any $T \geq 0$. Recall that T_0 may be equal to zero in the case $\mathcal{T}(\delta)$ is empty, or positive otherwise. For $t \notin \mathcal{T}(\delta)$, $V(x_t^\delta, x_{t+1}^\delta) < v^*$. Combining with (6), we have:

$$\begin{aligned} & \sum_{t=0}^T \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] + T_0 [V(x_0, \min \Gamma(x_0)) - v^*] \\ & \geq \sum_{t=0}^T \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] + \delta^{T+1} T_0 [V(x_0, \min \Gamma(x_0)) - v^*] \\ & \geq \sum_{t=0}^T \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] + \delta^{T+1} \sum_{t \in \mathcal{T}(\delta) \text{ and } t \geq T+1} [V(x_t^\delta, x_{t+1}^\delta) - v^*] \\ & \geq \sum_{t=0}^T \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] + \sum_{t \in \mathcal{T}(\delta) \text{ and } t \geq T+1} \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] \\ & + \sum_{t \notin \mathcal{T}(\delta) \text{ and } t \geq T+1} \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] \\ & = \sum_{t=0}^{\infty} \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] \\ & \geq \sum_{t=0}^{\infty} \delta^t [V(x_t^*, x_{t+1}^*) - v^*] \\ & \geq \sum_{t=0}^{\infty} [V(x_t^*, x_{t+1}^*) - v^*] \\ & = v(x_0). \end{aligned}$$

Letting $\delta \in \mathcal{D}$ converge to 1, we obtain:

$$\sum_{t=0}^T [V(\bar{x}_t, \bar{x}_{t+1}) - v^*] + T_0 [V(x_0, \min \Gamma(x_0)) - v^*] \geq v(x_0).$$

Letting T go to infinity, we get

$$\liminf_{T \rightarrow \infty} \sum_{t=0}^T [V(\bar{x}_t, \bar{x}_{t+1}) - v^*] \geq v(x_0) - T_0 [V(x_0, \min \Gamma(x_0)) - v^*] > -\infty.$$

Hence, $\{\bar{x}_t\}_{t=0}^{\infty}$ is a good program being bounded above by a value that is strictly smaller than x^* , contradicting Lemma 3.4.

This guarantees the existence of $\tilde{\delta}$ such that, for all $\delta \geq \tilde{\delta}$, the optimal path $\{x_t^\delta\}_{t=0}^{\infty}$ of the economy $\mathcal{E}(\delta)$ is nondecreasing and converges to some value x^δ . It is clear that $\lim_{\delta \rightarrow 1} x^\delta = x^*$.

Fix $\bar{\delta}$ such that for all $\delta \in (\bar{\delta}, 1)$ we have $x^\delta \in (x^* - d, x^*)$. Next, choose T^* large enough such that $x_t^\delta > x^* - d$ for all $t \geq T^*$. The monotonicity of x_t^δ with respect to δ implies that $x_t^\delta > x^* - d$ for all $\delta \geq \bar{\delta}$ and $t \geq T^*$.

We now consider now the case $x_0 > x^*$. Again, let

$$\bar{x}_t = \lim_{t \rightarrow \infty} x_t^\delta.$$

Clearly, since $x_0 > x^*$, for all δ , the sequence $\{x_t^\delta\}_{t=0}^\infty$ is decreasing and converges to some $x^\delta < x^*$. Therefore, the sequence $\{\bar{x}_t\}_{t=0}^\infty$ is nonincreasing.

Choose d small enough such that $(x^* - d, x^* + d) \subset \Gamma(x^*)$. Let

$$\Delta = \sup_{x, x', y \in (x^* - d, x^* + d)} |V(x, y) - V(x', y)|.$$

Consider an optimal path under the overtaking criterion $\{x_t^*\}_{t=0}^\infty$, and choose T^* sufficiently large such that $x_t^* \in (x^*, x^* + d)$ for all $t \geq T^*$. Using the same arguments as in the first part of the proof, since $\{x_t^*\}_{t=0}^\infty$ is a strictly decreasing sequence,

$$v(x_t^*) = V(x_t^*, x_{t+1}^*) - v^* + v(x_{t+1}^*) > v(x_{t+1}^*),$$

which implies $V(x_t^*, x_{t+1}^*) > v^*$ for all t .

For each δ , there exists $t^*(\delta)$ such that $x_{t^*(\delta)-1}^\delta > x^* + d > x_{t^*(\delta)}^\delta$. We will prove that $t^*(\delta)$ is bounded as δ converges to 1. First, let $\mathcal{T}(\delta)$ be the set of periods $t \leq t^*(\delta)$ such that $V(x_t^\delta, x_{t+1}^\delta) \geq v^*$. We will prove that the number of elements in $\mathcal{T}(\delta)$ is bounded by some T_0 independently of the choice of δ .

If $V(x_0, \min \Gamma(x_0)) < v^*$, then obviously $\mathcal{T}(\delta)$ is empty, and we can choose $T_0 = 0$. Otherwise, let $\underline{x}_1$ be such that $V(x_0, \underline{x}_1) = v^*$. If $\underline{x}_1 < x^* + d$, then choose $T_0 = 1$. If $\underline{x}_1 \geq x^* + d$, we continue the process to define $\underline{x}_2$ using $\underline{x}_1$ in place of x_0 , and so on.

Since $V(x_0, x_0) < v^*$, we have $x_0 > \underline{x}_1 > x^*$. By induction, we have $\underline{x}_t > \underline{x}_{t+1} > x^*$, for all t . Therefore, the process has to stop after finitely many steps, otherwise, we obtain an infinite decreasing sequence that converges to some $x \geq x^* + d$ and

$$V(x, x) = \lim_{t \rightarrow \infty} V(\underline{x}_t, \underline{x}_{t+1}) \geq v^*,$$

a contradiction. Let T_0 be the number of steps in this process: $\underline{x}_{T_0-1} \geq x^* + \tilde{d} > \underline{x}_{T_0}$.

Consider the case in which $\mathcal{T}(\delta)$ is nonempty. Take any elements $t_0 < t_1 < t_2 < \dots < t_{K-1} < t_K$ in $\mathcal{T}(\delta)$. Since $x_{t_0}^\delta \leq x_0$, we obtain $x_{t_1}^\delta \leq x_{t_0+1}^\delta \leq \underline{x}_1$. By induction, we obtain $x_{t_k}^\delta \leq \underline{x}_{t_k}$ for all $0 \leq k \leq K$. Thus, K cannot be greater than T_0 , otherwise, we would have $x_{t_{T_0}}^\delta \leq \underline{x}_{T_0} < x^* + d$, a contradiction.

Fix δ_0 sufficiently close to 1 such that, for all $\delta \in (\delta_0, 1)$ we have $x^\delta \in (x^* - d, x^* + d)$. Since for each fixed t , x_t^δ is nondecreasing with respect to δ , $t^*(\delta)$ is also a nondecreasing function with respect to δ : $t^*(\delta) \leq t^*(\delta')$ for all $\delta < \delta'$.

Assume that

$$\lim_{\delta \rightarrow 1} t^*(\delta) = +\infty.$$

Consider δ sufficiently close to 1 such that $t^*(\delta) > T_0$. The following sequence is feasible:

$$(x_0, x_1^*, x_2^*, \dots, x_{t^*(\delta)}^*, x_{t^*(\delta)+1}^\delta, x_{t^*(\delta)+2}^\delta, x_{t^*(\delta)+3}^\delta, \dots).$$

We have

$$\sum_{t=0}^{\infty} \delta^t V(x_t^\delta, x_{t+1}^\delta) \geq \sum_{t=0}^{t^*(\delta)} \delta^t V(x_t^*, x_{t+1}^*) + V(x_{t^*(\delta)}^*, x_{t^*(\delta)+1}^\delta) - V(x_{t^*(\delta)}^*, x_{t^*(\delta)+1}^*) + \sum_{t=t^*(\delta)+1}^{\infty} \delta^t V(x_t^\delta, x_{t+1}^\delta).$$

This implies

$$\sum_{t=0}^{t^*(\delta)} \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] \geq \sum_{t=0}^{t^*(\delta)} \delta^t [V(x_t^*, x_{t+1}^*) - v^*] - \Delta.$$

Fix $T \geq 0$. Choose δ sufficiently close to 1 such that $t^*(\delta) > T$. Recall that, for $t \notin \mathcal{T}(\delta)$ and $t \leq t^*(\delta)$, we have $V(x_t^\delta, x_{t+1}^\delta) \leq v^*$. Thus

$$\begin{aligned} & \sum_{t=0}^T \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] + \sum_{t=T+1}^{t^*(\delta)} \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] \\ & \leq \sum_{t=0}^T \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] + \sum_{t \in \mathcal{T}(\delta)} \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] \\ & \leq \sum_{t=0}^T \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] + T_0 \times [V(x_0, \min \Gamma(x_0)) - v^*]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{t=0}^T \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] + T_0 \times [V(x_0, \min \Gamma(x_0)) - v^*] \\ & \geq \sum_{t=0}^T \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] + \sum_{t=T+1}^{t^*(\delta)} \delta^t [V(x_t^\delta, x_{t+1}^\delta) - v^*] \\ & \geq \sum_{t=0}^T \delta^t [V(x_t^*, x_{t+1}^*) - v^*] + \sum_{t=T+1}^{t^*(\delta)} \delta^t [V(x_t^*, x_{t+1}^*) - v^*] - \Delta \\ & \geq \sum_{t=0}^T \delta^t [V(x_t^*, x_{t+1}^*) - v^*] - \Delta. \end{aligned}$$

Letting δ converge to 1 we obtain

$$\sum_{t=0}^T [V(\bar{x}_t, \bar{x}_{t+1}) - v^*] \geq \sum_{t=0}^T [V(x_t^*, x_{t+1}^*) - v^*] - T_0 \times [V(x_0, \min \Gamma(x_0)) - v^*] - \Delta.$$

This implies

$$\lim_{T \rightarrow \infty} \sum_{t=0}^T [V(\bar{x}_t, \bar{x}_{t+1}) - v^*] > -\infty.$$

Hence, the sequence $\{\bar{x}_t\}_{t=0}^\infty$ is a good program. Since it is nonincreasing, Lemma 3.4 implies its convergence to x^* . There exists T sufficiently large such that $\bar{x}_T < x^* + d$. Hence, $x_T^\delta < x^* + d$ for all δ sufficiently close to 1, which implies $t^*(\delta) \leq T$, contradicting the hypothesis that $\lim_{\delta \rightarrow 1} t^*(\delta) = +\infty$.

Therefore, $T^* = \lim_{\delta \rightarrow 1} t^*(\delta) < +\infty$. Choose δ sufficiently close to 1 such that $x_t^\delta \in (x^* - d, x^* + d)$ for all $t \geq T^*$. We have x_t^δ belongs to that interval, with all $t \geq T^*$.

It is obvious that, if $x_0 = x^*$, then $t^*(\delta) = 0$ for all $\delta \in (0, 1)$.

Consider the case where the sequence $\{x_t^\delta\}_{t=0}^\infty$ is defined as follows: $x_{t+1}^\delta = \max \varphi(\delta, x_t^\delta)$, for all $t \geq 0$. By Amir (1996), for all t , x_t^δ is nondecreasing with respect to δ . We apply the same arguments as in the proof of this proposition to establish the turnpike property for these sequences.

Now, for each $\delta \in (0, 1)$, let $\{x_t^\delta\}_{t=0}^\infty$ be an optimal path of the economy $\mathcal{E}(\delta)$. Define two optimal sequences $\{x_t^{\delta,m}\}_{t=0}^\infty$ and $\{x_t^{\delta,M}\}_{t=0}^\infty$ as follows: $x_{t+1}^{\delta,m} = \min \varphi(\delta, x_t^{\delta,m})$ and $x_{t+1}^{\delta,M} = \max \varphi(\delta, x_t^{\delta,M})$, for all $t \geq 0$. By induction, we obtain $x_t^{\delta,m} \leq x_t^\delta \leq x_t^{\delta,M}$ for all $t \geq 0$.

For any convex open neighborhood $\mathcal{V}$ of x^* , there exist $\underline{\delta}$ and T^* such that, for all $t \geq T^*$, and $\delta > \underline{\delta}$, both $x_t^{\delta,m}$ and $x_t^{\delta,M}$ belong to $\mathcal{V}$. Hence, the same property is true for x_t^δ .

The *turnpike* property has been proved. □

PROOF OF PROPOSITION 4.2

Fix $\varepsilon > 0$. Let $\mathcal{V} = (x^* - d, x^* + d)$ with $d > 0$ small enough such that for all $x, x', y \in \mathcal{V}$, we have $-\varepsilon < V(x, y) - V(x', y) < \varepsilon$. By Proposition 4.1, choose T large enough and $\bar{\delta}$ close enough to 1 so that $x_t^\delta \in \mathcal{V}$ for all $t \geq T$, and $\delta \geq \bar{\delta}$, and such that $x_{T+1}^* \in \mathcal{V}$.

Choose n large enough such that $\delta_n \geq \bar{\delta}$. Since the sequence $(x_0, x_1^*, \dots, x_{T+1}^*, x_{T+2}^{\delta_n}, x_{T+3}^{\delta_n}, \dots)$ is feasible,

$$\begin{aligned} & \sum_{t=0}^T \delta_n^t [V(x_t^{\delta_n}, x_{t+1}^{\delta_n}) - v^*] + \delta_n^{T+1} [V(x_{T+1}^{\delta_n}, x_{T+2}^{\delta_n}) - v^*] + \sum_{t=T+2}^\infty \delta_n^t [V(x_t^{\delta_n}, x_{t+1}^{\delta_n}) - v^*] \\ & \geq \sum_{t=0}^T \delta_n^t [V(x_t^*, x_{t+1}^*) - v^*] + \delta_n^{T+1} [V(x_{T+1}^*, x_{T+2}^{\delta_n}) - v^*] + \sum_{t=T+2}^\infty \delta_n^t [V(x_t^{\delta_n}, x_{t+1}^{\delta_n}) - v^*]. \end{aligned}$$

This implies

$$\begin{aligned} \sum_{t=0}^T \delta_n^t [V(x_t^{\delta_n}, x_{t+1}^{\delta_n}) - v^*] & \geq \sum_{t=0}^T \delta_n^t [V(x_t^*, x_{t+1}^*) - v^*] + \delta_n^{T+1} [V(x_{T+1}^*, x_{T+2}^{\delta_n}) - V(x_{T+1}^{\delta_n}, x_{T+2}^{\delta_n})] \\ & \geq \sum_{t=0}^T \delta_n^t [V(x_t^*, x_{t+1}^*) - v^*] - \varepsilon. \end{aligned}$$

Letting n go to infinity, so that δ_n converges to 1, we get an inequality that holds for all T sufficiently

big:

$$\sum_{t=0}^T [V(\bar{x}_t, \bar{x}_{t+1}) - v^*] \geq \sum_{t=0}^T [V(x_t^*, x_{t+1}^*) - v^*] - \varepsilon.$$

Let T go to infinity,

$$\liminf_{T \rightarrow \infty} \sum_{t=0}^T [V(\bar{x}_t, \bar{x}_{t+1}) - v^*] \geq \sum_{t=0}^{\infty} [V(x_t^*, x_{t+1}^*) - v^*] - \varepsilon.$$

Since ε is chosen arbitrarily, this guarantees that $\{\bar{x}_t\}_{t=0}^{\infty}$ is an optimal path under the overtaking criterion.

□

PROOF OF COROLLARY 4.3

For each $\delta \in (0, 1)$, let $\{x_t^\delta\}_{t=0}^{\infty}$ be the optimal path of $\mathcal{E}(\delta)$ being defined as: $x_{t+1}^\delta = \min \varphi(\delta, x_t)$. By Amir (1996), this sequence is monotone in t . Moreover, for any fixed t , x_t^δ is non-decreasing in δ .

Let $\bar{x}_t = \lim_{\delta \rightarrow 1} x_t^\delta$. By Proposition 4.2, the sequence $\{\bar{x}_t\}_{t=0}^{\infty}$ is an optimal path under the overtaking criterion. Since $x_0 < x^*$, this sequence is strictly increasing and converges to x^* . In particular, $x_0 < \bar{x}_1$. Fix any δ sufficiently close to 1 such that $x_0 < \bar{x}_1^\delta$, and therefore the sequence $\{x_t^\delta\}_{t=0}^{\infty}$ is non decreasing.

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