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ABSTRACT

We consider a general Ramsey model with endogenous discounting, dependent on current consumption or future capital, study the monotonicity properties of the optimal path and propose a new narrative of the existence of a poverty trap, an alternative to the literature on convex-concave production functions.

We first demonstrate the continuity and differentiability of the value function, the monotonicity of the policy correspondence and the strict monotonicity of the optimal path.

Secondly, to overcome the impossibility of computing the critical capital level that limits the poverty trap, we prove, through two examples where discounting depends either on consumption or on future capital, the existence of a poverty trap for a nonzero-measure set of parameter values.

Keywords: endogenous discounting; dynamic programming; Ramsey-Cass-Koopmans model.

JEL codes: C61, D11, O4.

1. INTRODUCTION

1.1 MOTIVATION AND RESULTS

Discounting is essential to shaping individual decisions, driving capital accumulation and general equilibrium dynamics. In order to put current and future generations on an equal footing, [Ramsey \(1928\)](#) assumes that there is no discounting. [Phelps \(1961\)](#), [Cass \(1965\)](#) and [Koopmans \(1965\)](#), building on [Fisher \(1930\)](#)'s approach, introduce an exogenous discount rate into the Ramsey model and obtain a Modified Golden Rule.

[Fisher \(1930\)](#) is also relevant to endogenous discounting, conjecturing that wealth increases the individual's level of patience and, consequently, reversing the causal nexus of [Ramsey \(1928\)](#)'s conjecture,

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which refers to an economy with exogenous discounting, but heterogeneous agents, and stipulates that the most patient agent becomes the richest in the long run. [Uzawa \(1968\)](#) is the first to model discounting based on past consumption experience, followed by [Epstein \(1983\)](#) with an axiomatic approach. Since then, a substantial body of literature has developed on the dynamic properties and time inconsistency of economies where discounting depends on other variables, particularly consumption or wealth.¹

We study optimal growth models in which the discount factor is not necessarily constant, but can depend on either consumption or the capital stock. In the first case, future utility is discounted by a function of current consumption, while, in the second, the discount function depends on future capital.²

Formally, let us focus on a capital path $(x_t)_{t=0}^{\infty}$ and the corresponding consumption path $(c_t)_{t=0}^{\infty}$, and consider two recursive forms:³

$$U(c_0, c_1, c_2, \dots) = u(c_0) + \delta(c_0)U(c_1, c_2, c_3, \dots)$$

and

$$W(x_0, x_1, \dots) = u(f(x_0) - x_1) + \delta(x_1)W(x_1, x_2, \dots).$$

In order to draw robust conclusions about the dynamic properties of the optimal path, we address general questions such as monotonicity or the existence of a poverty trap.

Before embarking on a more detailed dynamic analysis, we focus on two important aspects of endogenous discounting: the boundedness and sign of the utility function. We will see that lower boundedness is necessary to obtain the Pareto property of the intertemporal valuation. Without this property, a consumption sequence that Pareto dominates another may be assigned a lower valuation. The importance of this property is also emphasized in [Noor and Takeoka \(2021, 2022\)](#). [Epstein \(1983\)](#) also assumes a utility function bounded by zero. In the case where the discount factor depends on consumption, we greatly simplify the analysis by showing that it is possible, without loss of generality, to normalize the utility to zero.

Having clarified the role of Pareto property in preferences, we address the main question of this article: in the presence of endogenous discounting, can an underdeveloped economy converge to a higher level of capital, or will it remain trapped in poverty despite high initial productivity? The mechanism of

¹See [Vasilev \(2022\)](#), among others, for an overview of recent contributions.

²Under appropriate conditions guaranteeing convergence, the first and second formulations correspond respectively to those of [Epstein \(1983\)](#) and [Erol et al. \(2011\)](#).

³See [Bich, Druegon and Morhaim \(2017\)](#) for an axiomatic approach of recursive preferences and [Datta, Reffett and Wozny \(2017\)](#) for an analysis of various structural properties of recursive equilibria.

a pessimistic scenario is quite simple: when agents are poor, their discounting factors are also low, leading to underinvestment and fueling a vicious cycle that keeps the economy in poverty.

In the existing literature, this vicious cycle occurs when a low capital stock is accompanied by low productivity, which is indeed the case in the presence of fixed costs or the Allee effect. The economy then reaches a critical level of capital stock, called the Dechert-Nishimura-Skiba point, with the particularity that below this point, the economy collapses, and above it, it can converge toward a positive state.⁴

Endogenous discounting provides further support for the existence of a poverty trap. However, this alternative approach raises difficulties not encountered in studies of convex-concave production functions, where the existence of a critical level and a poverty trap relies solely on the interaction between a constant discount factor, productivity at the inflection point⁵ and maximum average output.⁶

The analysis becomes more complex when the discount factor is no longer constant. In this case, determining the general conditions for the existence of a poverty trap is far from easy. To overcome the theoretical obstacles, we specify the fundamentals in two examples, and show that the poverty trap is bounded by a positive threshold and exists for a nonzero-measure set of parameter values.

In many respects, these examples remain general enough to highlight robust properties. Indeed, the production function is Cobb-Douglas, meaning that productivity at the origin is infinite, while the utility function exhibits constant elasticity of intertemporal substitution. Furthermore, the discount function is hyperbolic, and the existence of a poverty trap depends solely on its curvature, indexed by a single parameter.⁷

More precisely, we prove the monotonicity of the optimal policy correspondence, which guarantees that if an optimal path does not start from a steady state, it is strictly monotonic. The existence of a poverty trap is linked to the existence of a critical capital level such that, if the initial condition is lower (higher), the optimal path converges to the origin (to a positive steady state). The monotonicity property of the optimal policy correspondence ensures that, if a critical level exists, it is unique.

⁴Interested readers may refer to the classic contributions of [Clark \(1971\)](#), [Skiba \(1978\)](#), [Majumdar and Mitra \(1982\)](#), and [Dechert and Nishimura \(1983\)](#), or consult a recent survey by [Akao et al. \(2025\)](#), as well as an extension of [Dechert and Nishimura \(1983\)](#) by [Ha-Huy \(2025\)](#). [Hung et al. \(2009\)](#) also demonstrate the existence of a threshold and interpret it as a poverty trap, but in the different context of a production function aggregating two distinct technologies.

⁵The critical level at which the initially convex production function becomes concave.

⁶See [Dechert and Nishimura \(1983\)](#) and [Le Van and Dana \(2003\)](#) for low, high, and intermediate discounting conditions. The poverty trap appears under high or intermediate discounting conditions.

⁷The hyperbolic function we use to represent the discounting should not be confused with the different notion of hyperbolic discounting, which gives rise to time inconsistency (see, among others, [Laibson \(1997\)](#) and [Krusell and Smith \(2003\)](#)).

1.2 RELATED LITERATURE

The novelty of our article lies in new results on monotonicity under endogenous discounting within a more general framework (regardless of the sign of the discount factor slope in the capital-dependent case) and in the characterization of the poverty trap. It is worthwhile to compare our model with the main contributions of recent literature, both in discrete and continuous time, in order to assess its added value.

Let us first focus on discrete-time research on endogenous discounting.

[Kamihigashi \(2002\)](#) considers a recursive utility function as the sum of the present utility and the discounted utility of all future flows. He examines the role of a nonlinear discount factor dependent on the level of future utility and demonstrates that indeterminacy can arise independently of the magnitude of productive externalities and increasing returns.

[Stern \(2006\)](#) proposes a model significantly different from ours. In his article, the economic agent devotes a portion of her output, thereby reducing her consumption or investment, in order to increase the discount factor. This “sacrifice” constitutes an additional choice variable. It could allow her to strengthen the parent-child bond, for example, by spending time with her children or caring for them. Another interpretation is that this fraction represents the cost of a better understanding of the future, acquired through means such as education, religion, or self-discipline. The author offers a thorough analysis of the economic dynamics, demonstrating that the optimal trajectory is monotonic and converges to a steady state, thanks to supermodularity properties, as in [Dechert and Nishimura \(1983\)](#) and in our own study. Numerical examples complement the theoretical analysis.

To our knowledge, the works closest to ours in discrete time are [Erol et al. \(2011\)](#) and [Noor and Takeoka \(2022\)](#). The former defines intertemporal utility as the sum of current-period utility and the present value of future intertemporal utility. The discount factor depends on the capital stock in the following period and increases with the stock. They demonstrate that any optimal capital trajectory is monotonic and converges to a steady state (equation (9) in [Erol et al. \(2011\)](#)). This yields a condition guaranteeing economic collapse. The paper also presents a numerical simulation illustrating the existence of a poverty trap.

Our article differs from theirs in many respects. In the first part, we also consider a discounting that, in the spirit of [Uzawa \(1968\)](#), depends on consumption as a proxy of wealth or consumption habits. In the second part, we also focus on future capital, but we remain more general, since our theoretical results remain valid for both increasing and decreasing discounting. Moreover, our dynamic analysis seems more intuitive to us because it relies on steady-state conditions that establish whether the optimal trajectory is increasing or decreasing (Proposition 5.5, Corollary 5.1 and Proposition 5.6).

Finally, [Erol et al. \(2011\)](#) illustrate the monotonicity of the optimal path and the existence of a poverty trap only numerically, for specific parameter values, while we demonstrate it analytically in the case of standard production, utility and discount functions, for a nonzero-measure set of parameter values.

[Noor and Takeoka \(2022\)](#) view the agent as a succession of “selves,” with the current self concerned with the well-being of future selves. Intertemporal valuation then takes the form of a sum of discounted utilities, with the utility of each period discounted as a nondecreasing function of consumption during that period. This introduces a fundamental difference from our model, where, from the perspective of the current period, the utility of each future period is discounted based on the entire consumption path between the two periods. [Noor and Takeoka \(2021, 2022\)](#) also raise the issue of the boundedness and sign of the utility function, arguing that this is a “substantial restriction” and that the solution to their model is sensitive to the sign of this function. In our view, constant discounting is not simply a special case of endogenous discounting; it differs fundamentally when working with unbounded utility functions, such as the logarithmic function.

[Balbus et al. \(2022\)](#) and [Borissov et al. \(2024\)](#) focus more on time consistency than on endogenous discounting. They study dynamic choice models with general discounting rules. It is well known that such models generate dynamically inconsistent preferences. However, they demonstrate the existence of a time-consistent stationary Markovian equilibrium for this class of models.

The article by [Borissov et al. \(2025\)](#) shares similarities with ours.

However, its authors consider an intertemporal utility defined as a recursive convex combination of present and future utilities, with the value calculated as a weighted average of the period utilities. In this respect, they are similar to [Chambers and Echenique \(2018\)](#), but unlike them, they take endogenous discounting into account, as we do. More precisely, their discount depends positively on current consumption or capital and negatively on future capital. In the first case, we assume, as they do, that the discount function is nondecreasing with respect to consumption. In the second case, we impose no restrictions on the sign of $\delta'(x_{t+1})$: forward-looking discounting can be increasing or decreasing. In the example, we simply consider a strictly increasing function. It is important to note that we find a monotonicity property and, therefore, we also exclude cycles in the less favorable case $\delta'(x_{t+1}) > 0$ (note that there is no room for cycles in [Borissov et al. \(2025\)](#) when $\delta'(x_{t+1}) < 0$).

The literature on continuous-time endogenous discounting, even if methodologically different, deserves to be mentioned.

In [Drugeon \(1996, 1998\)](#), impatience increases with consumption: the hypothesis is the inverse of that of [Fisher \(1930\)](#) and of our article. This opens the possibility of oscillating optimal trajectories. The intuition is similar to that of [Vasilev \(2022\)](#), where the agent is more patient when poor and therefore

saves more, but becomes less patient when rich, thus generating cycles around a steady state which, as in our article, may not be unique.

Das (2003) is another continuous-time model close to Drugeon (1996, 1998), where, unlike Uzawa (1968), the instantaneous time preference decreases with current consumption (the higher the consumption activity, the more patient the agent).

In this context, it highlights the possibility of multiple equilibria and poverty traps. We assume, as in Das (2003), that the agent's patience increases with consumption, but within a discrete-time framework. Unlike Das (2003), we can prove the continuity and differentiability of the value function, as well as the monotonicity of the policy correspondence, which implies the strict monotonicity of the optimal path. Furthermore, Das (2003) only demonstrates that the number of steady states is odd without concluding as to their number.⁸

Meng (2006)'s model is a continuous-time general equilibrium economy where local indeterminacy can arise if the individual rate of time preference (impatience) increases with average consumption (consumption standard) and decreases with average income (living standard). In other words, as the economy becomes wealthier, the agent becomes more patient, and as society consumes more, the agent becomes more impatient. Like Kamihigashi (2002), Meng (2006) takes into account endogenous discounting via the preference channel and the possibility of indeterminacy, while considering externalities related to preferences rather than technology. Unlike our approach, the discount rate does not depend on individual choices, but on the average behavior of society, which is considered an externality. Furthermore, the author focuses on a market economy and the possibility of local indeterminacy due to market imperfections (externalities), rather than on a poverty trap intrinsic to individual rationality in the absence of market imperfections.

In the continuous-time models by Schumacher (2009, 2011), the felicity at time t is discounted by $e^{-\theta(t)}$, but, here, the change in the impatience rate $\dot{\theta}(t)$ depends negatively on the stock of capital. This approach is rather unusual; however, Schumacher provides necessary and sufficient conditions for the existence of multiple steady states and, like us, a poverty trap in a framework with endogenous discounting rather than convex-concave technology. Furthermore, he highlights, for the first time (before Noor and Takeoka (2021, 2022)), the role of the sign of the utility function and demonstrates that, if this sign is negative (positive), the economy converges to a lower (higher) steady state than that of the Modified Golden Rule.⁹ Our Proposition 5.5, Corollary 5.1, and Proposition 5.6 echo these results in a discrete-time framework.

⁸See Proposition 1 in Das (2003).

⁹See section 3 in Schumacher (2011).

1.3 STRUCTURE

This article is structured as follows. We introduce the notion of recursive utility and define discounting specifications that depend on consumption and capital. We then study the case where discounting is a function of current consumption, and supplement the theoretical results with a simple example to demonstrate the existence of a poverty trap more explicitly and robustly. Finally, we consider the case where discounting depends on future capital and, similarly to the first case, we prove the existence of a poverty trap more explicitly. All the proofs are gathered in the appendix.

2. RECURSIVE UTILITIES

We define intertemporal utility recursively and consider two discounting specifications.

(1) First, we assume that intertemporal utility depends solely on the consumption path:

$$U(c_0, c_1, c_2, \dots) = u(c_0) + \delta(c_0)U(c_1, c_2, c_3, \dots).$$

The intertemporal utility of the consumption path $(c_0, c_1, \dots)$ is defined as the sum of the current utility $u(c_0)$ and the discounted utility derived from future consumption, determined by the sequence $(c_1, c_2, \dots)$.

(2) Alternatively, we assume that, rather than depending on current capital, the discount factor depends on future capital. The intertemporal utility generated by $(x_t)_{t=0}^\infty$ is given by:

$$W(x_0, x_1, \dots) = u(f(x_0) - x_1) + \delta(x_1)W(x_1, x_2, \dots).$$

To highlight the cumulative effect of the discounting, we can transform these recursive forms into series, under appropriate boundedness properties ensuring the convergence of infinite sums.

(1) Discounting based on consumption:

$$U(c_0, c_1, \dots) = u(c_0) + \sum_{t=1}^{\infty} \left(\prod_{s=0}^{t-1} \delta(c_s) \right) u(c_t).$$

(2) Discounting based on future capital with $c_t = f(x_t) - x_{t+1}$:

$$W(x_0, x_1, \dots) = u(c_0) + \sum_{t=1}^{\infty} \left(\prod_{s=0}^{t-1} \delta(x_{s+1}) \right) u(c_t).$$

In the limit case where the discount factor δ no longer depends on consumption or capital, both criteria collapse and we recover the classic Ramsey-Cass-Koopmans (RCK) model:

$$W(x_0, x_1, \dots) = \sum_{t=0}^{\infty} \delta^t u(c_t).$$

3. DISCOUNTING AS A FUNCTION OF CURRENT CONSUMPTION

3.1 FUNDAMENTALS

Let f , u and δ be the production, utility and discount functions respectively. The intertemporal utility function satisfies the recursive structure:

$$U(c_0, c_1, c_2, \dots) = u(c_0) + \delta(c_0)U(c_1, c_2, c_3, \dots).$$

The optimization program can be formulated as follows:

$$\max_{(c_t, x_{t+1})_{t=0}^{\infty}} \left[u(c_0) + \sum_{t=1}^{\infty} \left(\prod_{s=0}^{t-1} \delta(c_s) \right) u(c_t) \right],$$

subject to $c_t + x_{t+1} \leq f(x_t)$, $c_t \geq 0$, $x_t \geq 0$ for any t , given $x_0 \geq 0$.

Given the initial condition $x_0 \geq 0$, the accumulation path $\chi = (x_0, x_1, \dots)$ is said to be feasible from x_0 if $0 \leq x_{t+1} \leq f(x_t)$ for any t . The set of all feasible paths is denoted by $\Pi(x_0)$. A consumption sequence $(c_0, c_1, \dots)$ is feasible from $x_0 \geq 0$ if there exists $\chi \in \Pi(x_0)$ such that $0 \leq c_t \leq f(x_t) - x_{t+1}$ for any $t \geq 0$.

Let us introduce the following assumptions about the properties of production, utility and discounting.

Assumption 1.

1. The production function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, strictly concave and strictly increasing. It satisfies $f(0) = 0$, $\lim_{x \rightarrow \infty} f'(x) < 1$.
2. The function $\delta : \mathbb{R}_+ \rightarrow [0, 1]$ is continuous, nondecreasing, twice differentiable and concave with $c > 0$, $0 < \delta(c) < 1$, $\delta'(c) \geq 0$ and $\delta''(c) \leq 0$.
3. The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, strictly increasing and strictly concave. It satisfies the Inada condition $u'(0) = \infty$.
4. $u(0) = 0$.

Let us justify the double condition implied by $u(0) = 0$: the utility function is not only bounded below, but also normalised to zero at zero. We want the intertemporal utility function to satisfy the following two properties.

1. Pareto property: for any two sequences $(c_0, c_1, c_2, \dots)$ and $(c'_0, c'_1, c'_2, \dots)$ with $c_t \geq c'_t$ for every t , with at least one strict inequality, then $U(c_0, c_1, c_2, \dots) > U(c'_0, c'_1, c'_2, \dots)$.

2. $U(c_0, c_1, c_2, \dots)$ is strictly concave with respect to each argument c_t .

If the discount factor function δ is not constant, the Pareto property implies that u is bounded from below. Indeed, assume that $u(0) = -\infty$ and there are c_0 and c'_0 such that $c_0 > c'_0$ and $\delta(c_0) > \delta(c'_0)$. Choose c close enough to zero so that

$$\frac{u(c)}{1 - \delta(c)} < \frac{u(c'_0) - u(c_0)}{\delta(c_0) - \delta(c'_0)}.$$

It is easy to see that $U(c_0, c, c, \dots) < U(c'_0, c, c, \dots)$, a contradiction.

Due to the concavity of U with respect to each argument c_t , the function $U(c, 0, 0, \dots)$ is also concave with respect to c . More explicitly,

$$U(c, 0, 0, \dots) = u(c) + \delta(c) \frac{u(0)}{1 - \delta(0)}$$

is a concave function with respect to c . Let us define a new utility function $\hat{u}$:

$$\hat{u}(c) \equiv u(c) + \delta(c) \frac{u(0)}{1 - \delta(0)} - \frac{u(0)}{1 - \delta(0)}.$$

Clearly, $\hat{u}$ is also a strictly increasing function by the Pareto property and concave, with $\hat{u}(0) = 0$.

Let $\hat{U}$ be defined as:

$$\hat{U}(c_0, c_1, c_2, \dots) \equiv U(c_0, c_1, c_2, \dots) - \frac{u(0)}{1 - \delta(0)}.$$

The function $\hat{U}$ represents the same preferences as U . A simple calculus gives us:

$$\hat{U}(c_0, c_1, c_2, \dots) = \hat{u}(c_0) + \delta(c_0) \hat{U}(c_1, c_2, c_3, \dots) = \hat{u}(c_0) + \sum_{t=1}^{\infty} \left(\prod_{s=0}^{t-1} \delta(c_s) \right) \hat{u}(c_t).$$

Therefore, we can always study the problem with a utility function u such that $u(0) = 0$.

3.2 OPTIMAL PATH AND BELLMAN EQUATION

Noting that the constraints will be binding at the optimum because utility is strictly increasing, we introduce the function W defined over the set of consumption sequences as follows:

$$U(c_0, c_1, c_2, \dots) = u(c_0) + \sum_{t=1}^{\infty} \left(\prod_{s=0}^{t-1} \delta(c_s) \right) u(c_t).$$

The following lemma presents some preliminary results. Its proof is a direct application of Tychonov's Theorem (see, among others, [Stokey et al. \(1989\)](#) and [Le Van and Dana \(2003\)](#)). For each capital sequence $\chi = (x_t)_{t=0}^{\infty}$, let the corresponding consumption sequence be defined by $c_t = f(x_t) - x_{t+1}$ for any $t \geq 0$, and let W be the function defined on the set of all feasible sequences:

$$W(x_0, x_1, x_2, \dots) = U(c_0, c_1, c_2, \dots).$$

Lemma 3.1 is standard, and provides compactness and continuity properties guaranteeing the existence of an optimal path.

LEMMA 3.1. *Let x^M be the solution to $f(x) = x$. Then,*

1. *For any $\chi \in \Pi(x_0)$, $x_t \leq \max\{x_0, x^M\}$.*
2. *$\Pi(x_0)$ is compact in the product topology defined in the space of sequences χ .*
3. *W is well-defined and continuous over $\Pi(x_0)$ in the product topology.*

The initial optimal growth model becomes equivalent to the following program:

$$\max_{\chi \in \Pi(x_0)} W(\chi).$$

The existence of an optimal path is a direct consequence of the fact that $\Pi(x_0)$ is compact for the product topology defined in the space of sequences χ , and that U is continuous in this topology. The positivity of optimal consumption and capital stock paths stems from the Inada condition and the boundedness of the discount function.

The value function V is defined by

$$V(x_0) \equiv \max_{\chi \in \Pi(x_0)} W(\chi),$$

for $x_0 \geq 0$.

The utility function is bounded below, and the discounting bounds, together with the existence of a maximum sustainable capital stock, guarantee a finite value function. It can be shown that the value function is non-negative and continuous. Under these conditions, the Bellman functional equation can be established, admitting the value function V as a unique solution.

PROPOSITION 3.1. 1. *For any $x_0 \geq 0$, $\operatorname{argmax}_{\chi \in \Pi(x_0)} W(\chi)$ is a nonempty set.*

2. *The value function V is continuous and strictly increasing. It satisfies the Bellman equation*

$$V(x_0) = \max_{0 \leq y \leq f(x_0)} [u(f(x_0) - y) + \delta(f(x_0) - y)V(y)],$$

for any $x_0 \geq 0$.

3. *A sequence $\chi \in \Pi(x_0)$ is the optimal solution if and only if it satisfies the equation*

$$V(x_t) = u(f(x_t) - x_{t+1}) + \delta(f(x_t) - x_{t+1})V(x_{t+1}).$$

The optimal policy correspondence φ is defined as follows:

$$\varphi(x_0) = \operatorname{argmax}_{0 \leq y \leq f(x_0)} [u(f(x_0) - y) + \delta(f(x_0) - y)V(y)].$$

Note that the solution may not be unique. From the continuity of the value function, using the Maximum Theorem, we can easily deduce the non-emptiness and closedness of the optimal correspondence, as well as its equivalence to the optimal path. Proposition 3.2 sets out the main properties of φ and their important consequences, notably the monotonicity of optimal paths. This monotonicity is based on the supermodularity structure of the function $u(f(x) - y)$, as described in Topkis (1978) and Amir (1996, 2005).

More precisely, we prove the “increasing differences” property: for every $x < x'$ and $y < y'$, we have

$$u(f(x') - y') - u(f(x') - y) > u(f(x) - y') - u(f(x) - y),$$

when this inequality is feasible.

The proof of Proposition 3.2 is a modified version of that of Dechert and Nishimura (1983), exploiting the concavity of the discount factor function δ with respect to the consumption level. The optimal policy correspondence is nondecreasing. Therefore, any optimal path is either nondecreasing or non-increasing. In section 3.3, we will see that, starting from a non-stationary state, any optimal path is strictly monotonic. Moreover, along any optimal path, from the second period onward, the optimal policy reduces to a singleton. This echoes the well-known result of dynamic programming with a constant discount factor.

PROPOSITION 3.2. 1. φ is closed and upper hemi-continuous, and $\varphi(0) = \{0\}$.

2. If $x_0 > 0$, then $0 < x_1 < f(x_0)$ for any $x_1 \in \varphi(x_0)$.

3. A sequence $\chi \in \Pi(x_0)$ is optimal if and only if $x_{t+1} \in \varphi(x_t)$ for any $t \geq 0$.

4. Assume that $y \in \varphi(x)$ and $y' \in \varphi(x')$ with $x < x'$. Then, $y \leq y'$.

3.3 DYNAMICS

3.3.1 EULER EQUATION AND MONOTONICITY OF THE OPTIMAL PATH

We begin our long-run study by characterizing the Euler equations. Interestingly, although these equations involve the utility function u and the value function V , the optimal path remains unchanged.

PROPOSITION 3.3. Fix $x_0 > 0$ and an optimal path $\chi \in \Pi(x_0)$ starting from x_0 .

1. The path χ satisfies the Euler equation for any $t \geq 0$:

$$\begin{aligned} u'(f(x_t) - x_{t+1}) &= \delta(f(x_t) - x_{t+1})u'(f(x_{t+1}) - x_{t+2})f'(x_{t+1}) \\ &\quad - \delta'(f(x_t) - x_{t+1})V(x_{t+1}) \\ &\quad + \delta(f(x_t) - x_{t+1})f'(x_{t+1})\delta'(f(x_{t+1}) - x_{t+2})V(x_{t+2}). \end{aligned} \tag{1}$$

2. Along an optimal path, $\varphi(x_t)$ is a singleton for every $t \geq 1$, while $\varphi(x_0)$ may be multivalued.

We observe that the right-hand side of the Euler equation is strictly increasing with respect to x_{t+2} . In other words, $\varphi(x_{t+1})$ is a singleton for any $t \geq 0$. Following the same arguments as in the proofs of Theorem 6 and Corollary 4 in [Dechert and Nishimura \(1983\)](#), we conclude that the value function V is differentiable at x_t when $t \geq 1$. But, by the Monotone Differentiation Theorem, the value function is also differentiable almost everywhere.¹⁰

Furthermore, Euler's equations imply important properties of non-trivial steady states. A necessary condition for their existence is established. Moreover, if the economy starts from an initial level that does not correspond to a steady state, it can converge asymptotically to a steady state, but it can never reach it in a finite number of periods.

PROPOSITION 3.4. *Assume that x^* is a positive steady state.*

1. We have

$$\delta(f(x^*) - x^*)f'(x^*) = 1. \quad (2)$$

2. For any optimal path $(x_t)_{t=0}^\infty$, if $x_0 \neq x^*$, then $x_t \neq x^*$ for any $t \geq 0$.

The monotonicity of optimal policy correspondence guarantees that, if an optimal path does not start from a steady state, it will never reach one. Proposition 3.5 states a stronger property: if an optimal path starts from a steady state, it will remain there indefinitely.

PROPOSITION 3.5. 1. If x^* is a steady state, then $\varphi(x^*) = \{x^*\}$.

2. Any optimal path is either constant or strictly monotonic.

A natural question arises: how do we determine whether an optimal trajectory is increasing or decreasing? Lemma 3.2 provides a partial answer. It should be noted that this result is established without relying on the convexity structure of RCK models, and in particular on the concavity of the intertemporal utility function and the value function. To overcome the difficulties associated with the absence of this convexity, we develop a new approach. Details are provided in the appendix.

LEMMA 3.2. Fix any open interval $I \subset (0, x^M)$.

1. If $\delta(f(x) - x)f'(x) > 1$ for any $x \in I$, then there exists no strictly decreasing optimal path $(x_t)_{t=0}^\infty$ such that $x_t \in I$ for any $t \geq T$, for some T large enough.

¹⁰Interested readers may refer to Theorem 1.6.25 in [Tao \(2011\)](#).

2. If $\delta(f(x) - x)f'(x) < 1$ for any $x \in I$, then there exists no strictly increasing optimal path $(x_t)_{t=0}^{\infty}$ such that $x_t \in I$ for any $t \geq T$, for some T large enough.

Lemma 3.2 generalizes a well-known result: if productivity exceeds the discount rate, the economy accumulates capital, whereas, in the opposite case, the capital stock decreases.

Proposition 3.6 characterizes optimal trajectories. Part (1) states that, if, for a low value of the capital stock, productivity exceeds the discount rate, then the optimal trajectory starting from these values is upward sloping. Part (2) provides a condition for the opposite result.

PROPOSITION 3.6. 1. If $\liminf_{x \rightarrow 0} \delta(f(x) - x)f'(x) > 1$, then any optimal path converges to a positive steady state.

2. If $\delta(f(x) - x)f'(x) < 1$ for every $x > \hat{x}$ for some positive value $\hat{x}$, then any optimal path starting from $x_0 > \hat{x}$ is strictly decreasing.

A direct consequence of Proposition 3.6 is a generalization of well-known results of the RCK framework, with a constant discount rate.

3.3.2 POVERTY TRAP

Let the critical value for poverty trap be defined as a positive value x^C such that: if $x_0 < x^C$, then every optimal path from x_0 converges to the origin; if $x_0 > x^C$, then every optimal path from x_0 remains bounded away from zero. The monotonicity property of the optimal policy correspondence φ guarantees that, if x^C exists, it is unique. Lemma 3.3 provides a simple characterization of the existence of the poverty trap based on the monotonicity of optimal paths.

LEMMA 3.3. A poverty trap exists if and only if there is an optimal path that converges to the origin and an optimal path that converges to a positive steady state.

It is worth noting that, contrary to intuition, the poverty trap can arise even when every optimal path is nonincreasing. In particular, we do not exclude the possibility of a steady state x^* such that, while every optimal path starting from $x_0 < x^*$ strictly decreases and converges to the origin, every optimal path starting from $x_0 > x^*$ decreases and converges to x^* .

From Lemma 3.3, we deduce a simple characterization of the existence of the poverty trap. If there are positive capital stocks $x_0 < x'_0$ such that there is a decreasing optimal path starting from x_0 and converging to the origin, and there is an increasing optimal path starting from x'_0 , the poverty trap exists.

Lemma 3.2 helps us better understand the poverty trap. To see it more clearly, let us consider a configuration where equation (2) has exactly two positive solutions $x_s < x^s$. It is clear that $\delta(f(x) - x)f'(x) < 1$ in the interval $(0, x_s)$ and $\delta(f(x) - x)f'(x) > 1$ in (x_s, x^s) . The two values x_s and x^s are candidates for a steady state.¹¹ Clearly, if a poverty trap exists, the corresponding critical level should be smaller or equal to x^s . Otherwise, every optimal path converges to the origin.

The mechanism underlying the poverty trap here differs from that described in Dechert and Nishimura (1983) and Akao et al. (2025). Indeed, in our model, even with high productivity, low initial capital leads to a low discount rate (which, in both cases, depends on future consumption or capital) and, consequently, a low level of investment, thereby creating a vicious circle of underinvestment and a poverty trap.

Consider the case where x_s is a steady state; this occurs when, for example, φ is a continuous function that intersects the 45-degree line at two points, x_s and x^s .

Suppose the economy starts at a level $x_0 < x_s$. Then, any optimal trajectory is decreasing and converges to the origin. Otherwise, it would be increasing and converge to x_s , which would contradict point (2) of Lemma 3.2. If the economy starts at a level $x_0 \in (x_s, x^s)$, again according to Lemma 3.2, the optimal capital path increases and thus converges to x^s , the other candidate for steady-state status. If $x_0 > x^s$, it is evident that any optimal trajectory converges to x^s . Thus, x_s is an unstable steady state that turns out to be the economy's critical level. Conversely, the higher value x^s corresponds to a locally stable steady state (Figure 1).

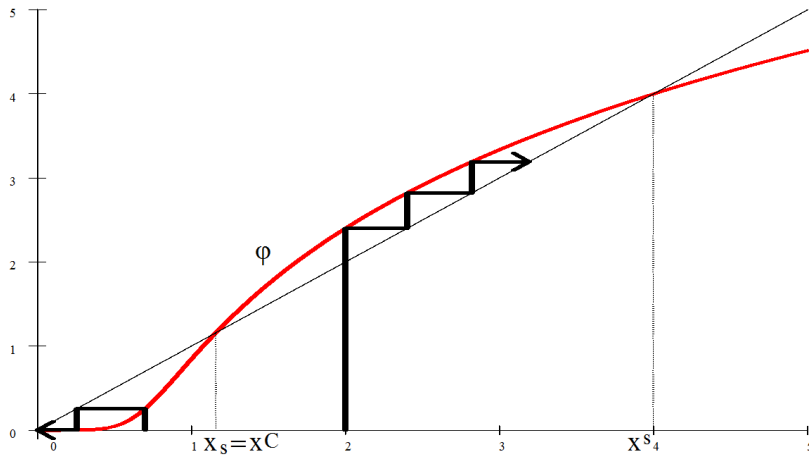


Figure 1: There are two steady states: a stable one, x^s , and an unstable one, $x_s = x^C$.

Let us now assume that x_s is not a steady state because the optimal transition is discontinuous. Assume the existence of a critical level $x^C < x^s$. Starting from $x_0 > x^C$, the economy converges to

¹¹The number of candidates for a steady state may be infinite, or even uncountable, as in the case where $\delta(f(x) - x) = 1/f'(x)$ for every x in some interval of $\mathbb{R}_+$. However, the set of parameter values for which this occurs has measure zero.

x^s . It is important to understand what happens at x^C . By definition, since x_s is not a steady state, $x^C \notin \varphi(x^C)$. The upper hemi-continuity property¹² guarantees that $\varphi(x^C)$ contains at least two distinct elements, $\underline{x}$ and $\bar{x}$, such that $\underline{x} < x^C < \bar{x}$. Consequently, starting from x^C , there exists at least one optimal trajectory converging to x^s and at least one optimal trajectory converging to the origin. We observe that the optimal choice “jumps” from a position below the 45-degree line to a position above it when the capital stock crosses x^C from below (Figure 2).¹³

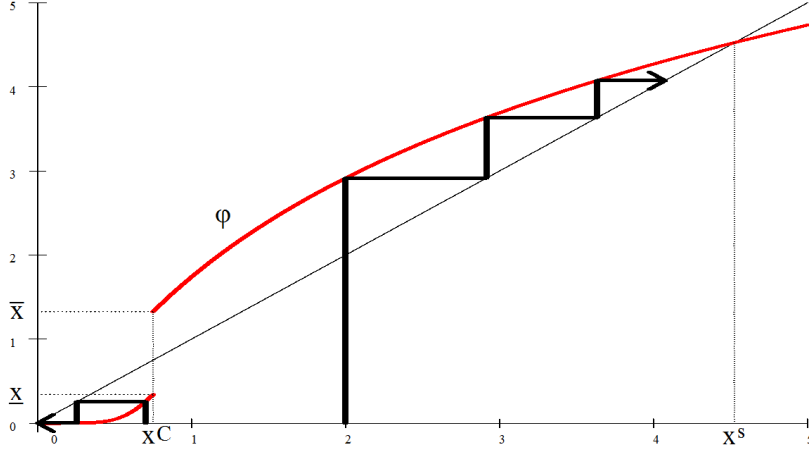


Figure 2: $\varphi(x^C)$ contains two elements $\underline{x}$ and $\bar{x}$.

Finally, we cannot rule out a third situation, in which the higher candidate x^s also represents a critical level. In this case, there is only one non-strictly decreasing path: $(x^s, x^s, x^s, \dots)$.

How can one determine whether the economy converges to the origin or to a non-trivial steady state? Lemma 3.4 provides a partial answer to this question. To simplify the notation, let $c_y \equiv f(y) - y$. It is worth noting that the two integrals in Lemma 3.4 may take infinite values.

LEMMA 3.4. *Assume that equation (2) has exactly two positive solutions $x_s < x^s$, with $\delta(f(x) - x)f'(x) < 1$ if $x < x_s$ and $\delta(f(x) - x)f'(x) > 1$ if $x \in (x_s, x^s)$.*

1. *If*

$$\int_0^{x^s} \left([\delta(c_y)f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta(c_y)} + \frac{u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} \right] - \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] \right) dy < 0,$$

then there exists $\underline{x} < x_s$ such that every optimal path starting from $x_0 < \underline{x}$ converges to the origin.

¹²By virtue of the upper hemi-continuity property, the graph of φ , that is, the set of pairs (x, y) such that $y \in \varphi(x)$, is closed.

¹³It should be noted that the question regarding the conditions under which the lower candidate x_s is greater or less than x^C remains open.

2. If

$$\int_0^{x^s} \left([\delta(c_y)f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta(c_y)} + \frac{u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} \right] - \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] \right) dy > 0,$$

then there exists $\bar{x} < x^s$ such that every optimal path starting from $x_0 > \bar{x}$ converges to x^s .

The lemma 3.4 provides a sufficient condition for the existence of a trajectory converging to the origin, as well as for that of a non-degenerate path. This result is important for establishing an example where two types of trajectories coexist, or, in other words, where a poverty trap exists. Since the conditions are formulated not only in terms of economic fundamentals but also in terms of the value function, we prefer to call this result a “lemma”.

4. EXAMPLE WITH DISCOUNTING DEPENDING ON CONSUMPTION

We present an example illustrating the existence of a poverty trap in the case of an explicit consumption-dependent discount function. More precisely, assuming high productivity at zero, we show that the poverty trap is a robust phenomenon.

Let $f(x) \equiv Ax^\alpha$, $u(c) \equiv c^\rho$ with $1/2 < \alpha < 1$ and $0 < \rho < 1$.¹⁴ We denote the solution to $f'(x) = 1$ by

$$x^G \equiv (\alpha A)^{\frac{1}{1-\alpha}}.$$

We introduce a discount factor function with hyperbolic curvature:

$$\delta^b(c) \equiv \frac{c}{b+c},$$

where $b > 0$ is a constant. The function $\delta^b(c)$ is clearly concave with respect to c .

This form captures the key properties of a discount factor: continuity, increasing monotonicity, boundedness between zero and one, and convergence to unity with the necessary asymptotic concavity. Thanks to these properties and the wide range of variation for parameter b , which allows for the modulation of concavity, the conclusions drawn from this functional form are robust.

The proof of the Lemma 4.1 is conceptually simple, but the computations involved are tedious. The single-peak property simplifies the analysis of equation (2) because, in general, given an appropriate choice of the constant b , this equation admits exactly two positive solutions. We can therefore focus on these two candidates for the status of steady state.

¹⁴Our objective is to prove the existence of a parameter configuration such that a poverty trap arises for a nonzero-measure set of parameter values. We cannot rule out the existence of a poverty trap in the case where $0 < \alpha \leq 1/2$, possibly with other concave discount functions.

Note that equation $\delta^b(f(x) - x)f'(x) = 1$ is equivalent to

$$\frac{\alpha Ax^{\alpha-1}(Ax^\alpha - x)}{b + Ax^\alpha - x} = 1.$$

LEMMA 4.1. *Function $\zeta(x) \equiv \alpha Ax^{\alpha-1}(Ax^\alpha - x) - (Ax^\alpha - x)$ is single-peaked. If $0 < b < b^M \equiv \max_{x \in [0, x^G]} \zeta(x)$, equation $\zeta(x) = b$ has exactly two solutions $x_s(b) < x^s(b)$: $x_s(b)$ is increasing with respect to b , $x^s(b)$ is decreasing. Moreover, $\delta^b(f(x) - x)f'(x) > 1$ if $x \in (x_s(b), x^s(b))$ and $\delta^b(f(x) - x)f'(x) < 1$ if $x \notin [x_s(b), x^s(b)]$.*

To avoid any confusion, we shall henceforth denote by $\mathcal{E}(b)$ the economy corresponding to the discount function δ^b . The associated intertemporal utility function will be denoted by $U^b(c_0, c_1, c_2, \dots)$ and the value function by V^b . For any feasible capital sequence $(x_t)_{t=0}^\infty$, let us also define

$$W^b(x_0, x_1, x_2, \dots) \equiv U^b(c_0, c_1, c_2, \dots),$$

with $c_t = f(x_t) - x_{t+1}$ for $t \geq 0$.

Finally, let $\mathcal{C}$ be the set of parameters b for which $\mathcal{E}(b)$ exhibits a poverty trap.

PROPOSITION 4.1. *There exist two positive values $\underline{b} < \bar{b}$ such that $(\underline{b}, \bar{b}) \subset \mathcal{C}$.*

The proof is organized into two parts. First, we demonstrate that there exists a $b \in (0, b^M)$ such that the economy $\mathcal{E}(b)$ exhibits a poverty trap. Second, we prove that $\mathcal{C}$ contains an open interval of parameter values.

The parameter b plays a key role in shaping the poverty trap. When the capital stock is low, the discount factor may prove insufficient to ensure the adequate investment needed to escape this trap. We therefore consider discount factor functions that vary monotonically with b . The lower the value of b , the higher the discount factor and, consequently, the greater the probability of escaping the low-capital zone and converging toward a positive steady state. Conversely, a high value of b corresponds to a low discount factor, implying that the economy remains more likely stuck in this trap.

According to Fisher, the most important characteristic of the discount factor function is that it is increasing. Its inverse is a strictly decreasing function capable of generating business cycles, as shown by [Drugeon \(1996, 1998\)](#) and [Vasilev \(2022\)](#). The discount factor is bounded above by one. Therefore, the simplest discount factor function is both bounded and concave. To construct an example, two natural choices for discounting present themselves: a function with hyperbolic curvature, such as $\delta^b(c) \equiv c/(b + c)$ with $b > 0$, or a function with exponential curvature, such as $\delta^b(c) \equiv 1 - e^{-bc}$ with $b > 0$.

We opt for the hyperbolic discounting function because it is easier to handle, particularly when combined with the chosen production and utility functions, which are commonly used in economic research.

Choosing an exponential curvature would yield similar results. It should be noted that with exponential curvature, $\delta^b(c)$ is an increasing function of b . A sufficiently high value of b ensures that the economy always converges to a positive steady state. Conversely, when b is too low, the economy systematically collapses. Based on the same arguments used for hyperbolic discount factors, one can demonstrate the existence of a parameter range in which the economy faces a poverty trap.

5. DISCOUNTING AS A FUNCTION OF FUTURE CAPITAL

In this section, we revisit the economy described in [Erol et al. \(2011\)](#), where the discount factor depends on future capital. We address the following problem:

$$\max \left[u(c_0) + \sum_{t=1}^{\infty} \left(\prod_{s=0}^{t-1} \delta(x_{s+1}) \right) u(c_t) \right],$$

subject to $c_t + x_{t+1} \leq f(x_t)$, $c_t \geq 0$, $x_t \geq 0$ for any t , given $x_0 \geq 0$.

Assumption 2.

1. The production function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, strictly concave and strictly increasing with the limit conditions: $f(0) = 0$, $\lim_{x \rightarrow \infty} f'(x) < 1$.
2. Function δ is differentiable with respect to x , with $0 < \delta(x) < 1$ for every $x > 0$.
3. The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is bounded below, continuous, strictly increasing and strictly concave with the Inada condition $u'(0) = \infty$.

As you can see, we impose neither a condition on the minimum utility value $u(0)$ nor a condition regarding the monotonicity (increasing or decreasing) of δ . Proposition 5.1 lists certain classical results from the dynamic programming literature.

PROPOSITION 5.1. *1. For any $x_0 \geq 0$, there exists an optimal path starting from x_0 . If $x_0 > 0$, then $0 < x_{t+1} < f(x_t)$ for any t .*

2. The value function V is continuous and strictly increasing. It satisfies the Bellman equation

$$V(x_0) = \max_{0 \leq y \leq f(x_0)} [u(f(x_0) - y) + \delta(y)V(y)],$$

for any $x_0 \geq 0$.

3. A sequence $\chi \in \Pi(x_0)$ is the optimal solution if and only if it satisfies:

$$V(x_t) = u(f(x_t) - x_{t+1}) + \delta(x_{t+1})V(x_{t+1}).$$

The optimal policy correspondence φ is defined as follows:

$$\varphi(x_0) = \operatorname{argmax}_{0 \leq y \leq f(x_0)} [u(f(x_0) - y) + \delta(y)V(y)].$$

Proposition 5.2 presents fundamental properties of the optimal policy correspondence and establishes certain important dynamic results, such as the monotonicity of optimal trajectories.

PROPOSITION 5.2. *1. φ is closed and upper hemi-continuous.*

2. A sequence $\chi \in \Pi(x_0)$ is optimal if and only if $x_{t+1} \in \varphi(x_t)$ for any $t \geq 0$.

3. Assume that $y \in \varphi(x), y' \in \varphi(x')$ with $x < x'$. Then, $y \leq y'$.

The proposition 5.3 presents the Euler equations and shows that, from time $t = 1$ onwards, any optimal path is unique. This result echoes a well-known result from Dechert and Nishimura (1983).

PROPOSITION 5.3. *Let $(x_t)_{t=0}^\infty$ be an optimal path starting from $x_0 > 0$.*

1. The Euler equation is given by:

$$u'(f(x_t) - x_{t+1}) = \delta(x_{t+1})u'(f(x_{t+1}) - x_{t+2})f'(x_{t+1}) + \delta'(x_{t+1})V(x_{t+1}). \quad (3)$$

2. Along an optimal path, $\varphi(x_t)$ is a singleton for every $t \geq 1$, while $\varphi(x_0)$ may be multivalued.

A direct consequence is that if x^* is a steady state, then $\varphi(x^*) = \{x^*\}$. Since the right-hand side is strictly increasing with respect to x_{t+2} , we obtain $\varphi(x_{t+1}) = \{x_{t+2}\}$ for any $t \geq 0$. By applying the same arguments as in the proof of Theorem 6 and Corollary 4 of Dechert and Nishimura (1983), we prove that $V(x_t)$ is differentiable at x_t for $t \geq 1$. The Monotone Differentiation Theorem guarantees that the value function $V(x)$ is differentiable almost everywhere.

Proposition 5.4 states a necessary equation for the steady state. Importantly, if the economy starts from a non-steady state, then the optimal capital path is strictly monotonic. Therefore, the economy converges either to a positive steady state or to the origin.

It should be noted that if δ is increasing and $u(c) > 0$ for all $c > 0$, the steady state is larger than the one corresponding to the Modified Golden Rule of the RCK model. The same happens in Erol et al. (2011). Conversely, if δ is decreasing, the economy reaches a lower steady state.

PROPOSITION 5.4. *Assume that $x^* > 0$ is a steady state.*

1. x^* is a solution to the following equation:

$$\delta(x^*)f'(x^*) = 1 - \frac{\delta'(x^*)u(f(x^*) - x^*)}{[1 - \delta(x^*)]u'(f(x^*) - x^*)}. \quad (4)$$

2. If $(x_t)_{t=0}^{\infty}$ is an optimal path starting from $x_0 \neq x^*$, then $x_t \neq x^*$ for any $t \geq 0$. Moreover, this sequence is strictly monotonic.

Proposition 5.5 echoes a well-known result that holds when the discount factor δ is constant. Interestingly, our new approach yields results without any convexity assumptions.

PROPOSITION 5.5. 1. Assume the existence of $\hat{x} > 0$ such that, for any $y < \hat{x}$,

$$\delta(y)f'(y) > 1 - \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]u'(f(y) - y)}.$$

Then, every optimal path converges to a strictly positive steady state.

2. Consider $x_0 > 0$ and assume that, for any $y > x_0$,

$$\delta(y)f'(y) < 1 - \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]u'(f(y) - y)}.$$

Then, every optimal path starting from x_0 is strictly decreasing.

Corollary 5.1 is a direct consequence of Proposition 5.5 and generalizes a classical result of the RCK model: the optimal sequence is increasing when productivity is high and decreasing when it is low.

COROLLARY 5.1. 1. Assume that, for y sufficiently small,

$$\delta(y)f'(y) > 1 - \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]u'(f(y) - y)}.$$

Then, every optimal path converges to a positive steady state.

2. Assume that, for $0 < y < x^M$,

$$\delta(y)f'(y) < 1 - \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]u'(f(y) - y)}.$$

Then, every optimal path converges to the origin.

Let us show why Corollary 5.1 implies Propositions 9 and 10 of Erol et al. (2011).

In their article, $u(c) > 0$ for $c > 0$ and $\delta'(y) > 0$. Let us define

$$\varepsilon \equiv \sup_{y \in (0, x^M)} \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]u'(f(y) - y)}.$$

Note that this value is smaller than the corresponding one in Erol et al. (2011). If $\sup_{y \geq 0} \delta(y)f'(0) < 1 - \varepsilon$, then, for any $y \geq 0$,

$$\delta(y)f'(y) \leq \sup_{y \geq 0} \delta(y)f'(0) < 1 - \varepsilon \leq 1 - \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]u'(f(y) - y)},$$

and any optimal path converges to the origin. This is precisely the conclusion of Proposition 9.

Focus now on Proposition 10 in [Erol et al. \(2011\)](#). The condition $\inf_{y \geq 0} \delta(y)f'(0) > 1$ implies that, for y small enough,

$$\delta(y)f'(y) > 1 - \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]u'(f(y) - y)},$$

and that every optimal path converges to a positive steady state.

Let us now turn to the main result. Proposition 5.6 states sufficient conditions for the existence of decreasing or increasing optimal paths. While these properties are traditionally easy to establish by exploiting convexity, in this article we are led to develop new techniques to overcome potential non-convexities.

PROPOSITION 5.6. *Fix x_0 that is not a steady state.*

1. *Assume that, for any $x^* > x_0$ solution to (4),*

$$\int_{x_0}^{x^*} \left([\delta(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta(y)} + \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} \right) dy < 0,$$

Then, every optimal path starting from x_0 is decreasing.

2. *Assume that, for any $x^* < x_0$ solution to (4) or $x^* = 0$,*

$$\int_{x^*}^{x_0} \left([\delta(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta(y)} + \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} \right) dy > 0,$$

Then, every optimal path starting from x_0 is increasing.

To illustrate Proposition 5.6, Corollary 5.2 examines the case where there are exactly two positive candidates for a steady state. This corollary states a condition under which the larger of these candidates constitutes a steady state, as well as a condition guaranteeing the existence of an optimal trajectory converging to the origin.

COROLLARY 5.2. *Assume that equation (4) has exactly two positive solutions $x_s < x^s$ with the following property:*

$$\delta(y)f'(y) < 1 - \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]u'(f(y) - y)}, \tag{5}$$

if $0 < y < x_s$ or $y > x^s$, and

$$\delta(y)f'(y) > 1 - \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]u'(f(y) - y)}, \tag{6}$$

if $x_s < y < x^s$.

1. *If*

$$\int_0^{x^s} \left([\delta(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta(y)} + \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} \right) dy > 0,$$

then x^s is a steady state and there exists $\bar{x} < x^s$ such that every optimal path starting from $x_0 > \bar{x}$ converges to x^s .

2. If

$$\int_0^{x^s} \left([\delta(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta(y)} + \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} \right) dy < 0,$$

then there exists $\underline{x} > 0$ such that every optimal path starting from $x_0 < \underline{x}$ converges to the origin.

Let

$$\eta(y) \equiv [\delta(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta(y)} + \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2}.$$

Conditions (5) and $\eta(y) > 0$ are equivalent, as are conditions (6) and $\eta(y) < 0$. The function $\eta(y)$ is negative on $(0, x_s)$ and positive on (x_s, x^s) . If the integral of this function over the “positive region” outweighs that over the “negative region”, then x^s is guaranteed to be a locally stable steady state. Conversely, starting from a low capital stock, the economy converges to the origin.

6. EXAMPLE WITH DISCOUNTING DEPENDING ON FUTURE CAPITAL

As before, we present an example illustrating the existence and robustness of a poverty trap in the case of an explicit discount function that now depends on future capital.

Let us define $f(x) \equiv Ax^\alpha$ and $u(c) \equiv c^\rho$ with $\alpha, \rho \in (0, 1)$. Reconsider a discount factor function with hyperbolic curvature:

$$\delta^b(x) = \frac{x}{b + x},$$

where $b > 0$ is constant.

As seen above, due to the properties of the discount factor and the scalability of parameter b , this functional form demonstrates that the poverty trap is a robust feature. Let $\mathcal{E}(b)$ be the economy associated with the discount function δ^b . The following equation is key for proving the existence of a poverty trap:

$$\delta^b(x)f'(x) = 1 - \frac{\delta^{b'}(x)u(f(x) - x)}{[1 - \delta^b(x)]u'(f(x) - x)}.$$

Let $\mathcal{C}$ be the set of parameters b for which $\mathcal{E}(b)$ exhibits a poverty trap.

PROPOSITION 6.1. *There exist two positive values $\underline{b} < \bar{b}$ such that $(\underline{b}, \bar{b}) \subset \mathcal{C}$.*

The arguments in the proof are analogous to those in section 4, but here we apply Corollary 5.2. For appropriate values of $b > 0$, equation (4) has two positive solutions $x_s(b) < x^s(b)$. By introducing the new auxiliary function:

$$\Phi(b) \equiv \int_0^{x^s(b)} \left([\delta^b(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta^b(y)} + \frac{\delta^{b'}(y)u(f(y) - y)}{[1 - \delta^b(y)]^2} \right) dy,$$

and proving the existence of a b such that $\Phi(b) > 0$, one can show that the economy follows an optimal path converging to $x^s(b)$. Similarly, there exists a b such that $\Phi(b) < 0$ and for which the economy follows an optimal path converging to the origin. Repeating the arguments of section 4, we prove that there exists a value between these two critical values such that the corresponding economy experiences a poverty trap.

7. CONCLUSION

In this article, we addressed the issue of the monotonicity of the optimal path and the existence of a poverty trap when discounting is endogenous and depends on current consumption or future capital. To obtain these results, we first demonstrated the existence of an optimal path and showed that the set of feasible sequences is compact and that the intertemporal utility is continuous in the product topology. We then proved that if the utility function is bounded below and the discounting is bounded, then the value function is finite, non-negative, and continuous. Remarkably, it is also the unique solution of the Bellman functional equation and is differentiable almost everywhere.

We also characterized the policy function, showing that the continuity of the value function implies that the optimal correspondence is non-empty, closed and monotonic. Although the policy correspondence is not necessarily unique, it is single-valued for any x_t with $t \geq 1$. We noted that the differentiability of the Bellman equation almost everywhere relies on the fact that the policy correspondence is a singleton almost everywhere from $t = 1$ onwards.

Having established these technical yet important results, we demonstrated the monotonicity of the optimal policy correspondence; this ensures that if an optimal trajectory does not start from a steady state, it is strictly monotonic. The existence of a poverty trap is linked to the existence of a critical capital level such that, if the initial condition is below (or above) it, the optimal trajectory converges to the origin (or to a positive steady state). The monotonicity property of the optimal policy correspondence is essential to guarantee the uniqueness of this critical level, when it exists.

However, since establishing the general conditions for the existence of a threshold in the presence of endogenous discounting is far from straightforward, we supplemented our theoretical analysis with two examples corresponding to cases where discounting depends on current consumption or future capital. We demonstrated that, in both cases, a poverty trap exists for a nonzero-measure subset of parameter values. These examples suggest that the poverty trap is a robust phenomenon in economies with endogenous discounting.

We leave for future research the examination of the case where discounting depends on current capital,

as [Borissov et al. \(2025\)](#) have already done, albeit in a different context (intertemporal utility defined as a weighted average of period utilities), while noting that, according to their results, the dynamic properties remain qualitatively similar to those of the case where discounting depends on current consumption.

APPENDIX

PROOF OF LEMMA 3.1

See the proof of Lemma 2 in [Le Van and Morhaim \(2002\)](#). □

PROOF OF PROPOSITION 3.1

Apply the same arguments as in the proof of Proposition 3.4.1 in [Le Van and Dana \(2003\)](#).¹⁵ □

PROOF OF PROPOSITION 3.2

The Maximum Theorem implies (i), while Proposition 3.1 implies (ii) and (iii).

Let us prove part (iv): for any $x < x'$ and $y \in \varphi(x)$, $y' \in \varphi(x')$, we have $y \leq y'$.

Assume the contrary, that is $y > y'$. We obtain

$$\begin{aligned} u(f(x) - y) + \delta(f(x) - y) V(y) &\geq u(f(x) - y') + \delta(f(x) - y') V(y'), \\ u(f(x') - y') + \delta(f(x') - y') V(y') &\geq u(f(x') - y) + \delta(f(x') - y) V(y). \end{aligned}$$

Summing these inequalities, we get

$$\begin{aligned} u(f(x) - y) + u(f(x') - y') + \delta(f(x) - y) V(y) + \delta(f(x') - y') V(y') \\ \geq u(f(x) - y') + u(f(x') - y) + \delta(f(x) - y') V(y') + \delta(f(x') - y) V(y) \end{aligned}$$

Thus, using the same argument as in [Dechert and Nishimura \(1983\)](#), with $x < x'$ and $y > y'$, we find

$$\delta(f(x) - y) V(y) + \delta(f(x') - y') V(y') > \delta(f(x) - y') V(y') + \delta(f(x') - y) V(y).$$

This inequality implies

$$V(y') [\delta(f(x') - y') - \delta(f(x) - y')] > V(y) [\delta(f(x') - y) - \delta(f(x) - y)]$$

¹⁵Interested readers may refer to the articles by [Stahl \(2017\)](#) and [Rincón-Zapatero \(2020\)](#) for the case involving uncertainty.

or, equivalently,

$$V(y') \int_x^{x'} \delta'(f(z) - y') f'(z) dz > V(y) \int_x^{x'} \delta'(f(z) - y) f'(z) dz,$$

leading to a contradiction with the concavity property of δ . $\square$

PROOF OF PROPOSITION 3.3

Part (1). The partial derivative of $U(c_0, c_1, c_2, \dots)$ with respect to x_{t+1} is equal to zero because $0 < x_{t+1} < f(x_t)$ for any $t \geq 0$. Computing the derivative with respect to x_{t+1} , we obtain:

$$\begin{aligned} u'(f(x_t) - x_{t+1}) &= -\delta'(f(x_t) - x_{t+1})u(f(x_{t+1}) - x_{t+2}) \\ &\quad + \delta(f(x_t) - x_{t+1})u'(f(x_{t+1}) - x_{t+2})f'(x_{t+1}) \\ &\quad - \delta'(f(x_t) - x_{t+1})\delta(f(x_{t+1}) - x_{t+2})V(x_{t+2}) \\ &\quad + \delta(f(x_t) - x_{t+1})\delta'(f(x_{t+1}) - x_{t+2})f'(x_{t+1})V(x_{t+2}) \\ &= \delta(f(x_t) - x_{t+1})u'(f(x_{t+1}) - x_{t+2})f'(x_{t+1}) - \delta'(f(x_t) - x_{t+1})V(x_{t+1}) \\ &\quad + \delta(f(x_t) - x_{t+1})f'(x_{t+1})\delta'(f(x_{t+1}) - x_{t+2})V(x_{t+2}). \end{aligned}$$

Part (2). The right-hand side of the Euler equation is strictly increasing in x_{t+2} . Hence, $\varphi(x_{t+1})$ is a singleton for any $t \geq 0$.

Applying the same arguments as in the proofs of Theorem 6 and Corollary 4 in [Dechert and Nishimura \(1983\)](#), the value function V is differentiable at x_{t+1} , for any $t \geq 0$. $\square$

PROOF OF PROPOSITION 3.4

(i) Assume that $x^* > 0$ is a steady state. Let $c^* = f(x^*) - x^*$ and observe that

$$V(x^*) = \frac{u(c^*)}{1 - \delta(c^*)}.$$

Using Euler equation, we find

$$\begin{aligned} u'(c^*) &= -\delta'(c^*)u(c^*) + \delta(c^*)u'(c^*)f'(x^*) \\ &\quad - \delta'(c^*)\delta(c^*)\frac{u(c^*)}{1 - \delta(c^*)} + \delta(c^*)\delta'(c^*)f'(x^*)\frac{u(c^*)}{1 - \delta(c^*)}. \end{aligned}$$

and, thus,

$$u'(c^*)[1 - \delta(c^*)f'(x^*)] = -\frac{\delta'(c^*)u(c^*)}{1 - \delta(c^*)}[1 - \delta(c^*)f'(x^*)].$$

Recalling that $u(c^*) > 0$, we get $\delta(f(x^*) - x^*)f'(x^*) = 1$.

(ii) Assume the contrary. Then, there exists some $t \geq 0$ such that $x_t \neq x^*$, and $x_{t+1} = x^*$. The sequence $(x_t, x^*, x^*, \dots)$ is an optimal path starting from x_t .

Consider the case $x_t < x^*$. Recalling that $\delta(f(x^*) - x^*)f'(x^*) = 1$ and $u(c) > 0$ for any $c > 0$, and using Euler equation, we obtain:

$$\begin{aligned}
u'(f(x_t) - x^*) &= -\delta'(f(x_t) - x^*)u(f(x^*) - x^*) + \delta(f(x_t) - x^*)u'(f(x^*) - x^*)f'(x^*) \\
&\quad - \delta'(f(x_t) - x^*)\delta(f(x^*) - x^*)V(x^*) + \delta(f(x_t) - x^*)\delta'(f(x^*) - x^*)f'(x^*)V(x^*) \\
&< -\delta'(f(x^*) - x^*)u(f(x^*) - x^*) + \delta(f(x^*) - x^*)u'(f(x^*) - x^*)f'(x^*) \\
&\quad - \delta'(f(x^*) - x^*)\delta(f(x^*) - x^*)\frac{u(f(x^*) - x^*)}{1 - \delta(f(x^*) - x^*)} \\
&\quad + \delta(f(x^*) - x^*)\delta'(f(x^*) - x^*)f'(x^*)\frac{u(f(x^*) - x^*)}{1 - \delta(f(x^*) - x^*)} \\
&= u'(f(x^*) - x^*),
\end{aligned}$$

a contradiction.

In the case $x_t > x^*$, applying the same arguments, but with reversed inequalities, also leads to a contradiction. $\square$

PROOF OF PROPOSITION 3.5

Part (1). Since x^* is a steady state, the sequence $(x^*, x^*, x^*, \dots)$ is an optimal path starting from x^* . By Proposition 3.3, $\varphi(x^*)$ is a singleton.

Part (2). Consider $x_0 > 0$ that is not a steady state. By Proposition 3.4, there is no t such that x_t is a steady state. This implies $x_{t+1} \neq x_t$ for every t , otherwise x_t will be a steady state. If $x_0 < x_1$, since $x_1 \in \varphi(x_0)$ and $x_2 \in \varphi(x_1)$, Proposition 3.2 implies that $x_1 < x_2$. By induction, we get $x_t < x_{t+1}$ for any t .

If $x_0 > x_1$, using the same arguments, we find that the path $(x_t)_{t=0}^\infty$ is strictly decreasing.

The monotonicity of $(x_t)_{t=0}^\infty$ entails that this sequence has a limit. The hemi-continuity property of the optimal policy correspondence ensures that this limit is a steady state. $\square$

PROOF OF LEMMA 3.2

Part (1). Assume the contrary. Then there exists an open interval $I \subset (0, x^M)$ such that for any $x \in I$, $\delta(f(x) - x)f'(x) > 1$, and a strictly decreasing optimal path $(x_t)_{t \geq 0}$ such that $x_t \in I$ for every $t \geq T$ for some T large enough.

Since $(x_T, x_T, \dots)$ is feasible,

$$V(x_T) \geq \frac{u(f(x_T) - x_T)}{1 - \delta(f(x_T) - x_T)}.$$

The following inequality holds for any t :

$$\frac{u(f(x_t) - x_t)}{1 - \delta(f(x_t) - x_t)} \geq u(f(x_t) - x_{t+1}) + \delta(f(x_t) - x_{t+1}) \frac{u(f(x_{t+1}) - x_{t+1})}{1 - \delta(f(x_{t+1}) - x_{t+1})}.$$

Indeed, considering the following function

$$\Psi(y) \equiv u(f(x_t) - y) + \delta(f(x_t) - y) \frac{u(f(y) - y)}{1 - \delta(f(y) - y)},$$

we can prove that $\Psi'(y) > 0$ in the interval (x_{t+1}, x_t) . For the sake of simplicity, let $c_y = f(y) - y$, with $dc_y/dy = f'(y) - 1$.

Recalling that $u(c_y) > 0$ and $f'(y) > 1$, for $y \in (x_{t+1}, x_t)$, we have

$$\begin{aligned} \Psi'(y) &\equiv -u'(f(x_t) - y) - \delta'(f(x_t) - y) \frac{u(c_y)}{1 - \delta(c_y)} \\ &\quad + \delta(f(x_t) - y) \frac{[1 - \delta(c_y)]u'(c_y) + u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} [f'(y) - 1] \\ &> -u'(c_y) - \delta'(c_y) \frac{u(c_y)}{1 - \delta(c_y)} + \delta(c_y) \frac{[1 - \delta(c_y)]u'(c_y) + u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} [f'(y) - 1] \\ &= u'(c_y) \left[-1 + \frac{\delta(c_y)[f'(y) - 1]}{1 - \delta(c_y)} \right] + \frac{\delta'(c_y)u(c_y)}{1 - \delta(c_y)} \left[-1 + \frac{\delta(c_y)[f'(y) - 1]}{1 - \delta(c_y)} \right] \\ &= \frac{\delta(c_y)f'(y) - 1}{1 - \delta(c_y)} \left[u'(c_y) + \frac{\delta'(c_y)u(c_y)}{1 - \delta(c_y)} \right] > 0. \end{aligned}$$

The function Ψ is strictly increasing in the interval (x_{t+1}, x_t) , which implies that $\Psi(x_t) > \Psi(x_{t+1})$.

Thus, the inequality holds for $t \geq T$, and

$$\begin{aligned} \frac{u(f(x_T) - x_T)}{1 - \delta(f(x_T) - x_T)} &> u(f(x_T) - x_{T+1}) + \delta(f(x_T) - x_{T+1}) \frac{u(f(x_{T+1}) - x_{T+1})}{1 - \delta(f(x_{T+1}) - x_{T+1})} \\ &> u(f(x_T) - x_{T+1}) + \delta(f(x_T) - x_{T+1}) u(f(x_{T+1} - x_{T+2})) \\ &\quad + \delta(f(x_T) - x_{T+1}) \delta(f(x_{T+1}) - x_{T+2}) \frac{u(f(x_{T+2}) - x_{T+2})}{1 - \delta(f(x_{T+2}) - x_{T+2})} \\ &> \dots \geq V(x_T), \end{aligned}$$

a contradiction.

Part (2). We arrive at a contradiction by following exactly the same arguments as before, but with reversed inequalities and a decreasing function Ψ on the interval (x_t, x_{t+1}) (under the assumption that the optimal trajectory is strictly increasing). $\square$

PROOF OF PROPOSITION 3.6

Part (1). Since $\liminf_{x \rightarrow 0} \delta(f(x) - x)f'(x) > 1$, there exists $z > 0$ small enough such that $\delta(f(x) - x)f'(x) > 1$ for any $x < z$. The convergence of $(x_t)_{t \geq 0}$ to the origin implies the existence of a period T such that $x_t \in (0, z)$ for any $t \geq T$, a contradiction with Lemma 3.2.

Part (2). Assume the contrary: there exists an increasing optimal path $(x_t)_{t=0}^{\infty}$ starting from $x_0 > \hat{x}$. Since $\liminf_{x \rightarrow 0} \delta(f(x) - x)f'(x) < 1$ in this interval, this path contains a non-steady state element. Proposition 3.5 implies that it is strictly decreasing, and we arrive at a contradiction with the Lemma 3.2. $\square$

PROOF OF LEMMA 3.3

It is clear that, if the critical level x^C exists, there are two sequences satisfying the properties mentioned in the statement of this lemma. We can use the definition of x^C as well as the monotonicity property of optimal paths.

Now, let's assume the existence of an optimal path $(\underline{x}_t)_{t=0}^{\infty}$ converging to zero, as well as a sequence $(\bar{x}_t)_{t=0}^{\infty}$ converging to a positive steady state. Let x^C be the infimum of the set of capital levels x_0 for which there exists an optimal path starting from x_0 that is bounded away from zero.

The existence of a trajectory converging to zero implies that $x^C > 0$. Indeed, if $x^C = 0$, the definition of x^C and the non decreasing property of φ would imply that any optimal path starting from $x_0 > 0$ is bounded away from zero, which is a contradiction.

Since we have chosen x^C in this way, if $x_0 < x^C$, any optimal path starting from x_0 will converge to the origin. Conversely, if $x_0 > x^C$, Proposition 3.2 guarantees that any optimal path starting from x_0 will be bounded away from zero. $\square$

PROOF OF LEMMA 3.4

Before proving parts (1) and (2), let us observe that, starting from $x_0 \neq x_s$, the economy never converges to x_s . Let us assume the contrary: x_s is a steady state.

Consider the case $x_0 < x_s$. This means that the optimal path $(x_t)_{t=0}^{\infty}$ strictly increases towards x_s . Recalling that, in the interval $(0, x_s)$, $\delta(f(x) - x)f'(x) < 1$ and using the same arguments as in the proof of Proposition 3.6, we come to a contradiction.

In the case $x_s < x_0 < x^s$, a similar argument leads also to a contradiction. Hence, starting from a non-steady state, the economy either converges to x^s , or to the origin.

Part (1). Fix $\underline{x} < x_s$ close enough to the origin such that

$$\int_{x_0}^{x^s} [\delta(c_y)f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta(c_y)} + \frac{u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} \right] dy < \int_{x_0}^{x^s} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] dy,$$

for every $x_0 < \underline{x}$.¹⁶

Let $(x_t)_{t=0}^\infty$ be the optimal path starting from $x_0 < \underline{x}$. Assume that it is increasing and converging to a positive steady state. Clearly, this steady state is x^s . For any t ,

$$\begin{aligned} V(x_{t+1}) - V(x_t) &\geq u(f(x_{t+1}) - x_{t+1}) + \delta(f(x_{t+1}) - x_{t+1})V(x_{t+1}) \\ &\quad - u(f(x_t) - x_{t+1}) - \delta(f(x_t) - x_{t+1})V(x_{t+1}) \\ &= \int_{x_t}^{x_{t+1}} u'(f(y) - x_{t+1})f'(y)dy + V(x_{t+1}) \int_{x_t}^{x_{t+1}} \delta'(f(y) - x_{t+1})f'(y)dy \\ &\geq \int_{x_t}^{x_{t+1}} u'(f(y) - y)f'(y)dy + \int_{x_t}^{x_{t+1}} \delta'(f(y) - y)f'(y)V(y)dy \\ &= \int_{x_t}^{x_{t+1}} u'(f(y) - y)f'(y)dy + \int_{x_t}^{x_{t+1}} \delta'(f(y) - y)f'(y) \frac{u(c_y)}{1 - \delta(c_y)} dy \\ &\quad + \int_{x_t}^{x_{t+1}} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] dy. \end{aligned}$$

Therefore,

$$\begin{aligned} V(x^s) - V(x_0) &= \sum_{t=0}^{\infty} [V(x_{t+1}) - V(x_t)] \\ &\geq \int_{x_0}^{x^s} u'(f(y) - y)f'(y)dy + \int_{x_0}^{x^s} \delta'(f(y) - y)f'(y) \frac{u(c_y)}{1 - \delta(c_y)} dy \\ &\quad + \int_{x_0}^{x^s} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] dy \end{aligned}$$

Consider now the left-hand side:

$$\begin{aligned} V(x^s) - V(x_0) &\leq \frac{u(f(x^s) - x^s)}{1 - \delta(f(x^s) - x^s)} - \frac{u(f(x_0) - x_0)}{1 - \delta(f(x_0) - x_0)} \\ &= \int_{x_0}^{x^s} \frac{[1 - \delta(c_y)]u'(c_y) + u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} [f'(y) - 1] dy \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{x_0}^{x^s} \frac{[1 - \delta(c_y)]u'(c_y) + u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} [f'(y) - 1] dy \\ &\geq \int_{x_0}^{x^s} u'(f(y) - y)f'(y)dy + \int_{x_0}^{x^s} \delta'(f(y) - y)f'(y) \frac{u(c_y)}{1 - \delta(c_y)} dy \\ &\quad + \int_{x_0}^{x^s} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] dy. \end{aligned}$$

¹⁶The integrand represents the change in utility resulting from an increase of one unit in capital. Note that this differs from the notion of “net gain of investment” introduced in [Kamihigashi and Roy \(2006, 2007\)](#).

This implies

$$\int_{x_0}^{x^s} [\delta(c_y)f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta(c_y)} + \frac{u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} \right] dy \geq \int_{x_0}^{x^s} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] dy,$$

a contradiction.

Part (2). Fix $\bar{x}$ close enough to x^s such that

$$\int_0^{x_0} [\delta(c_y)f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta(c_y)} + \frac{u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} \right] dy \geq \int_0^{x_0} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] dy$$

for any $x_0 > \bar{x}$.

Let $(x_t)_{t=0}^\infty$ be the optimal path starting from $x_0 < \bar{x}$. Assume that it is decreasing and converging to the origin. For any t ,

$$\begin{aligned} V(x_t) - V(x_{t+1}) &\leq u(f(x_t) - x_{t+1}) + \delta(f(x_t) - x_{t+1})V(x_{t+1}) \\ &\quad - u(f(x_{t+1}) - x_{t+1}) - \delta(f(x_{t+1}) - x_{t+1})V(x_{t+1}) \\ &= \int_{x_{t+1}}^{x_t} u'(f(y) - x_{t+1})f'(y)dy + V(x_{t+1}) \int_{x_{t+1}}^{x_t} \delta'(f(y) - x_{t+1})f'(y)dy \\ &\leq \int_{x_{t+1}}^{x_t} u'(f(y) - y)f'(y)dy + \int_{x_{t+1}}^{x_t} \delta'(f(y) - y)f'(y)V(y)dy \\ &= \int_{x_{t+1}}^{x_t} u'(f(y) - y)f'(y)dy + \int_{x_{t+1}}^{x_t} \delta'(f(y) - y)f'(y) \frac{u(c_y)}{1 - \delta(c_y)} dy \\ &\quad + \int_{x_{t+1}}^{x_t} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] dy. \end{aligned}$$

Therefore,

$$\begin{aligned} V(x_0) - V(0) &= \sum_{t=0}^{\infty} [V(x_t) - V(x_{t+1})] \\ &\leq \int_0^{x_0} u'(f(y) - y)f'(y)dy + \int_0^{x_0} \delta'(f(y) - y)f'(y) \frac{u(c_y)}{1 - \delta(c_y)} dy \\ &\quad + \int_0^{x_0} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] dy. \end{aligned}$$

Consider now the left-hand side:

$$\begin{aligned} V(x_0) - V(0) &\geq \frac{u(f(x_0) - x_0)}{1 - \delta(f(x_0) - x_0)} - \frac{u(f(0) - 0)}{1 - \delta(f(0) - 0)} \\ &= \int_0^{x_0} \frac{[1 - \delta(c_y)]u'(c_y) + u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} [f'(y) - 1] dy. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_0^{x_0} \frac{[1 - \delta(c_y)]u'(c_y) + u(c_y)\delta'(c_y)}{[1 - \delta(c_y)]^2} [f'(y) - 1] dy \\ &\leq \int_0^{x_0} u'(f(y) - y)f'(y)dy + \int_0^{x_0} \delta'(f(y) - y)f'(y) \frac{u(c_y)}{1 - \delta(c_y)} dy \\ &\quad + \int_0^{x_0} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] dy. \end{aligned}$$

Therefore,

$$\int_0^{x_0} [\delta(c_y) f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta(c_y)} + \frac{u(c_y) \delta'(c_y)}{[1 - \delta(c_y)]^2} \right] dy \leq \int_0^{x_0} \delta'(c_y) f'(y) \left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)} \right] dy,$$

a contradiction. $\square$

PROOF OF LEMMA 4.1

We have

$$\delta^b(f(x) - x) f'(x) = \frac{\alpha A x^{\alpha-1} (A x^\alpha - x)}{b + A x^\alpha - x} = \frac{\alpha A x^{2\alpha-1} (A - x^{1-\alpha})}{b + A x^\alpha - x}.$$

Fixing $\alpha > 1/2$, we obtain $\lim_{x \rightarrow 0} \delta^b(f(x) - x) f'(x) = 0$.

Equation $\delta^b(f(x) - x) f'(x) = 1$ is equivalent to

$$\frac{\alpha A x^{\alpha-1} (A x^\alpha - x)}{b + A x^\alpha - x} = 1.$$

Now, let us study the function $\zeta(x) = \alpha A x^{\alpha-1} (A x^\alpha - x) - (A x^\alpha - x)$. Observe that $\zeta(x) = [f(x) - x] [f'(x) - 1]$.

Equation $\delta(f(x) - x) f'(x) = 1$ is equivalent to $\zeta(x) = b$.

We will show that the function ζ is single-peaked in the interval $[0, x^G]$: there exists $\hat{x} \in (0, x^G)$ such that ζ is increasing in $[0, \hat{x})$ and decreasing in $(\hat{x}, x^G]$. The advantage of working with single-peaked functions is that, by suitably choosing the parameter b , the equation $\delta(f(x) - x) f'(x) = 1$ admits exactly two solutions, meaning that there are exactly two candidates for the status of steady state.

Recalling that $\zeta(0) = \zeta(x^G) = 0$ and letting $\hat{x} \in \arg \max_{0 \leq x \leq x^G} \zeta(x)$, we can prove that $\hat{x}$ is the unique solution to $\zeta'(x) = 0$ in $(0, x^G)$.

Indeed, the equation $\zeta'(x) = 0$ is equivalent to

$$\xi(x) \equiv [f'(x) - 1]^2 + [f(x) - x] f''(x) = 0.$$

Let's show that $\xi(x) = 0$ has a unique solution. Focus first on its derivative:

$$\begin{aligned} \xi'(x) &= 2 [f'(x) - 1] f''(x) + [f'(x) - 1] f'''(x) + [f(x) - x] f''''(x) \\ &= 3(\alpha A x^{\alpha-1} - 1)\alpha(\alpha - 1)A x^{\alpha-2} + (A x^\alpha - x)\alpha(\alpha - 1)(\alpha - 2)A x^{\alpha-3} \\ &= \alpha(\alpha - 1)A x^{\alpha-3} [3(\alpha A x^\alpha - x) + (\alpha - 2)(A x^\alpha - x)] \\ &= -2\alpha(1 - \alpha)A x^{\alpha-2} \left[(2\alpha - 1)A x^{\alpha-1} - \frac{1 + \alpha}{2} \right]. \end{aligned}$$

Clearly, the equation $(2\alpha - 1)A x^{\alpha-1} - (1 + \alpha)/2 = 0$ has a unique positive solution:

$$\tilde{x} = \left[\frac{2(2\alpha - 1)}{1 + \alpha} A \right]^{\frac{1}{1-\alpha}} < (\alpha A)^{\frac{1}{1-\alpha}} = x^G,$$

since $0 < \alpha < 1$.

Note that $\xi'(x) < 0$ if $x \in (0, \tilde{x})$, and $\xi'(x) > 0$ if $x \in (\tilde{x}, x^G)$. This implies $\xi(\tilde{x}) < \xi(x^G) < 0$.

Importantly, because of this inequality, $\xi(x) = 0$ has no solution in the interval $(\tilde{x}, x^G)$.

Combining the monotonicity of ξ in the interval $(0, \tilde{x})$ with $\lim_{x \rightarrow 0} \xi(x) = \infty$, the equation $\xi(x) = 0$ admits a unique solution in $(0, \tilde{x})$.

Thus, the equation $\zeta'(x) = 0$ has unique solution in the interval $(0, x^G)$, say $\hat{x}$. The uniqueness of this solution ensures that the function ζ is increasing in $(0, \hat{x})$ and decreasing in $(\hat{x}, x^G)$. Hence, ζ is a single-peaked function.

It is easy to see that $\zeta(0) = \zeta(x^G) = 0$ and $\delta^b(f(x) - x)f'(x) = 1$ if and only if $\zeta(x) = b$.

Let $b^M \equiv \max_{x \in [0, x^G]} \zeta(x)$ and $\hat{x} \equiv \arg \max_{x \in [0, x^G]} \zeta(x)$.

If $b > b^M$, the equation $\zeta(x) = b$ has no solution. Since ζ is a single-peaked function, if $0 < b < b^M$, equation $\zeta(x) = b$ has exactly two solutions $x_s(b) < x^s(b)$, with the property that $\zeta(x) < b$ if $0 < x < x_s(b)$ or $x > x^s(b)$, and $\zeta(x) > b$ if $x_s(b) < x < x^s(b)$. Moreover, while $x_s(b)$ is increasing with respect to b , $x^s(b)$ is decreasing. More precisely, we have

$$\begin{aligned} \lim_{b \rightarrow 0} x_s(b) &= 0 \text{ and } \lim_{b \rightarrow 0} x^s(b) = x^G, \\ \lim_{b \rightarrow b^M} x_s(b) &= \hat{x} \text{ and } \lim_{b \rightarrow b^M} x^s(b) = \hat{x}. \end{aligned}$$

Remark that $\delta^b(f(x) - x)f'(x) < 1$ if $x < x_s(b)$ or $x > x^s(b)$, and $\delta^b(f(x) - x)f'(x) > 1$ if $x_s(b) < x < x^s(b)$. □

PROOF OF PROPOSITION 4.1

The proof of Proposition 4.1 is lengthy and requires preliminary arguments. For the sake of clarity, we therefore present them in a separate lemma.

LEMMA 7.1. 1. Fix $x_0 > 0$. Let $b > 0$ and a sequence of parameters $(b_n)_{n=0}^\infty$ that converges to b .

For each n , let $(x_t(n))_{t=0}^\infty$ be an optimal path of the economy $\mathcal{E}(b_n)$ starting from x_0 . If, for any n , $(x_t(n))_{t=0}^\infty$ is nondecreasing, then the economy $\mathcal{E}(b)$ has a nondecreasing optimal path starting from x_0 .

2. Fix $x_0 > 0$. Let $b > 0$ and a sequence of parameters $(b_n)_{n=0}^\infty$ that converges to b . For each n , let

$(x_t(n))_{t=0}^\infty$ be an optimal path of the economy $\mathcal{E}(b_n)$ starting from x_0 . If, for any n , $(x_t(n))_{t=0}^\infty$ is nonincreasing, then the economy $\mathcal{E}(b)$ has a nonincreasing optimal path starting from x_0 .

3. There exists $\hat{b} > 0$ such that for any $b \in (0, \hat{b})$, the economy $\mathcal{E}(b)$ exhibits a strictly increasing optimal path.

Proof of Lemma 7.1

Part 1. The compactness of the set $\Pi(x_0)$ with respect to the product topology entails the existence of a subsequence $(b_{n_k})_{k=0}^\infty$ such that the sequence of optimal paths $((x_t(b_{n_k}))_{t=0}^\infty)_{k=0}^\infty$ converges to a sequence $(\tilde{x}_t)_{t=0}^\infty$ in this topology. Clearly, $(\tilde{x}_t)_{t=0}^\infty$ is nondecreasing.

Take now any feasible path $(x_t)_{t=0}^\infty \in \Pi(x_0)$ and notice that

$$W^{b_{n_k}}(x_0, x_1(b_{n_k}), x_2(b_{n_k}), \dots) \geq W^{b_{n_k}}(x_0, x_1, x_2, \dots).$$

Let k go to infinity to obtain

$$W^b(x_0, \tilde{x}_1, \tilde{x}_2, \dots) \geq W^b(x_0, x_1, x_2, \dots).$$

Since $(x_t)_{t=0}^\infty$ is arbitrary, the sequence $(\tilde{x}_t)_{t=0}^\infty$ is an optimal paths of the economy $\mathcal{E}(b)$ starting from x_0 .

Part 2. We apply exactly the same arguments to prove that the economy $\mathcal{E}(b)$ has a nonincreasing optimal path starting from x_0 .

Part 3. Assume the contrary. Then, there exists a decreasing sequence $(b_n)_{n=0}^\infty$ converging to zero such that, for any n , every optimal path of the economy $\mathcal{E}(b_n)$ is nonincreasing.

We observe that, for $b = b_n$ for some $n \geq 0$,

$$V^b(y) < \frac{u(f(y))}{1 - \delta^b(f(y))}, \quad (7)$$

for any $y \in (0, x^G)$. Indeed, an optimal path $(y_t)_{t=0}^\infty$ starting from y is nonincreasing. Thus, $y_t \leq y$, $c_t = f(y_t) - y_{t+1} < f(y)$ for any t and

$$V(y) < u(f(y)) + \delta^b(f(y))u(f(y)) + \delta^{2b}(f(y))^2 u(f(y)) + \dots = \frac{u(f(y))}{1 - \delta^b(f(y))}.$$

We will prove the existence of $B > 0$ such that

$$\int_0^{x^s(b)} \delta^{b'(c_y)} f'(y) \left[\frac{u(f(y))}{1 - \delta^b(f(y))} - \frac{u(f(y) - y)}{1 - \delta^b(f(y) - y)} \right] dy < B, \quad (8)$$

for $b = b_n$ for some n . We have

$$\begin{aligned} & \frac{u(f(y))}{u(f(y) - y)} \frac{1 - \delta^b(f(y) - y)}{1 - \delta^b(f(y))} \\ &= \frac{u(f(y))}{u(f(y) - y)} \frac{b + f(y)}{b + f(y) - y} < \frac{u(f(y))}{u(f(y) - y)} \frac{f(y)}{f(y) - y} \\ &= \frac{1}{\left[1 - \frac{y}{f(y)}\right]^{\rho+1}} < \frac{1}{\left[1 - \frac{x^G}{f(x^G)}\right]^{\rho+1}}. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_0^{x^s(b)} \delta^{b'}(c_y) f'(y) \left[\frac{u(f(y))}{1 - \delta^b(f(y))} - \frac{u(f(y) - y)}{1 - \delta^b(f(y) - y)} \right] dy \\ & < \hat{B} \int_0^{x^s(b)} \delta^{b'}(c_y) f'(y) \frac{u(f(y) - y)}{1 - \delta^b(f(y) - y)} dy, \end{aligned} \quad (9)$$

where

$$\hat{B} \equiv \frac{1}{\left[1 - \frac{x^G}{f(x^G)}\right]^{\rho+1}} - 1.$$

Moreover,

$$\begin{aligned} & \hat{B} \int_0^{x^s(b)} \delta^{b'}(c_y) f'(y) \frac{u(f(y) - y)}{1 - \delta^b(f(y) - y)} dy \\ &= \hat{B} \int_0^{x^s(b)} \frac{b}{(b + c_y)^2} f'(y) u(f(y) - y) \frac{b + c_y}{b} dy \\ &= \hat{B} \int_0^{x^s(b)} \frac{f'(y) u(f(y) - y)}{b + c_y} dy \leq \hat{B} \int_0^{x^G} \frac{f'(y) u(f(y) - y)}{b + c_y} dy \\ &\leq \hat{B} \int_0^{x^G} \frac{f'(y) u(f(y) - y)}{f(y) - y} dy = \frac{\hat{B}}{\rho} \int_0^{x^G} f'(y) u'(f(y) - y) dy \\ &= \frac{\hat{B}}{\rho} \int_0^{x^G} [f'(y) - 1] u'(f(y) - y) dy + \frac{\hat{B}}{\rho} \int_0^{x^G} u'(f(y) - y) dy \\ &= \frac{\hat{B}}{\rho} u(f(x^G) - x^G) + \frac{\hat{B}}{\rho} \int_0^{x^G} u'(c_y) dy. \end{aligned}$$

Fix $\hat{y} \in (0, x^G)$ such that, if $y < \hat{y}$, we have $f(y) - y > y$, which implies $u'(c_y) < u'(y)$.

We observe that

$$\begin{aligned} \int_0^{x^G} u'(c_y) dy &= \int_0^{\hat{y}} u'(c_y) dy + \int_{\hat{y}}^{x^G} u'(c_y) dy \\ &< \int_0^{\hat{y}} u'(y) dy + u'(c_{\hat{y}}) \int_{\hat{y}}^{x^G} dy = u(\hat{y}) + u'(c_{\hat{y}}) (x^G - \hat{y}). \end{aligned}$$

We obtain:

$$\begin{aligned} \hat{B} \int_0^{x^s(b)} \delta^{b'}(c_y) f'(y) \frac{u(f(y) - y)}{1 - \delta^b(f(y) - y)} dy &< \frac{\hat{B}}{\rho} u(f(x^G) - x^G) + \frac{\hat{B}}{\rho} \int_0^{x^G} u'(c_y) dy \\ &< \frac{\hat{B}}{\rho} u(f(x^G) - x^G) + \frac{\hat{B}}{\rho} [u(\hat{y}) + u'(c_{\hat{y}})(x^G - \hat{y})] \\ &\equiv B. \end{aligned}$$

According to (7), (8) and (9), if $b = b_n$ for some n , we have

$$\int_0^{x^s(b)} \delta^{b'}(c_y) f'(y) \left[V^b(y) - \frac{u(f(y) - y)}{1 - \delta^b(f(y) - y)} \right] dy < B. \quad (10)$$

Now, let us prove that

$$\lim_{b \rightarrow 0} \int_0^{x^s(b)} [\delta^b(c_y) f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta^b(c_y)} + \frac{u(c_y) \delta^{b'}(c_y)}{[1 - \delta^b(c_y)]^2} \right] dy = 0. \quad (11)$$

Recall that, for any b , when $y < x_s(b)$, $\delta^b(c_y)f'(y) < 1$. Moreover, $\lim_{b \rightarrow 0} x_s(b) = 0$.

Fix $\varepsilon \in (0, 1)$. For b sufficiently small, we have $x_s(b) < \hat{y}$ and

$$\delta^b(c_y) < \frac{1}{f'(y)} < \frac{1}{f'(x_s(b))} < \varepsilon$$

with $y < x_s(b)$. Recall also that $c_y > y$ for $y < x_s(b)$, since $x_s(b) < \hat{y}$.

For b small enough,

$$\begin{aligned} \int_0^{x_s(b)} \frac{u'(c_y)}{1 - \delta^b(c_y)} dy &< \int_0^{x_s(b)} \frac{u'(c_y)}{1 - \varepsilon} dy = \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} u'(c_y) dy \\ &< \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} u'(y) dy = \frac{u(x_s(b))}{1 - \varepsilon}. \end{aligned}$$

Hence,

$$\lim_{b \rightarrow 0} \int_0^{x_s(b)} \frac{u'(c_y)}{1 - \delta^b(c_y)} dy = 0. \quad (12)$$

In addition, for small values of b ,

$$\begin{aligned} &\int_0^{x_s(b)} \frac{u(c_y)\delta^{b'}(c_y)}{[1 - \delta^b(c_y)]^2} dy \\ &< \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} \frac{u(c_y)\delta^{b'}(c_y)}{1 - \delta^b(c_y)} dy = \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} \frac{u(c_y) \frac{b}{(b+c_y)^2}}{\frac{b}{b+c_y}} dy \\ &= \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} \frac{u(c_y)}{b + c_y} dy \leq \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} \frac{u(c_y)}{c_y} dy \\ &= \frac{1}{(1 - \varepsilon)\rho} \int_0^{x_s(b)} u'(c_y) dy < \frac{1}{(1 - \varepsilon)\rho} \int_0^{x_s(b)} u'(y) dy < \frac{u(x_s(b))}{(1 - \varepsilon)\rho}. \end{aligned}$$

Hence,

$$\lim_{b \rightarrow 0} \int_0^{x_s(b)} \frac{u(c_y)\delta^{b'}(c_y)}{[1 - \delta^b(c_y)]^2} dy = 0. \quad (13)$$

Combining (12) and (13) with $\delta^b(c_y)f'(y) < 1$ for $y < x_s(b)$, we obtain (11).

Now, let us prove that

$$\lim_{b \rightarrow 0} \int_{x_s(b)}^{x^s(b)} [\delta^b(c_y)f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta^b(c_y)} + \frac{u(c_y)\delta^{b'}(c_y)}{[1 - \delta^b(c_y)]^2} \right] dy = \infty. \quad (14)$$

Indeed, fixing $0 < x_* < x^* < x^G$ and $\hat{b}$ small enough such that, for $b < \hat{b}$, we have

$$0 < x_s(b) < x_* < x^* < x^s(b) < x^G,$$

we obtain

$$\begin{aligned} &\int_{x_s(b)}^{x^s(b)} [\delta^b(c_y)f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta^b(c_y)} + \frac{u(c_y)\delta^{b'}(c_y)}{[1 - \delta^b(c_y)]^2} \right] dy \\ &> \int_{x_*}^{x^*} [\delta^b(c_y)f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta^b(c_y)} + \frac{u(c_y)\delta^{b'}(c_y)}{[1 - \delta^b(c_y)]^2} \right] dy \\ &> \int_{x_*}^{x^*} [\delta^{\hat{b}}(c_y)f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta^{\hat{b}}(c_y)} + \frac{u(c_y)\delta^{b'}(c_y)}{[1 - \delta^{\hat{b}}(c_y)]^2} \right] dy. \end{aligned}$$

For every $y \in (x_*, x^*)$, $\lim_{b \rightarrow 0} \delta^b(c_y) = 1$. Therefore,

$$\lim_{b \rightarrow 0} \int_{x_*}^{x^*} [\delta^b(c_y) f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta^b(c_y)} + \frac{u(c_y) \delta^{b'}(c_y)}{[1 - \delta^b(c_y)]^2} \right] dy = \infty.$$

and we get (14).

Using (10), (11) and (14), we find $\lim_{n \rightarrow \infty} \Phi(b_n) = \infty$.

Hence, for n sufficiently large, $\Phi(b_n) > 0$. By Lemma 3.4, there exists $\bar{x}(b_n) < x^s(b_n)$ such that any optimal path starting from $x_0 > \bar{x}(b_n)$ converges to $x^s(b_n)$: a contradiction with the hypothesis that every optimal path of $\mathcal{E}(b_n)$ is nonincreasing. Thus, Part 3 is proven. $\square$

Let us now return to the proof of Proposition 4.1.

First, we will prove the existence of b such that $\mathcal{E}(b)$ has a poverty trap. Assume the contrary. Then, for any fixed b , either every optimal path converges to the origin, or every optimal path converges to a positive steady state.

Although we cannot calculate the explicit critical values of the fundamentals, we can demonstrate the existence of values that give rise to a poverty trap.

In this respect, we introduce an auxiliary function:

$$\begin{aligned} \Phi(b) \equiv & \int_0^{x^s(b)} [\delta^b(c_y) f'(y) - 1] \left[\frac{u'(c_y)}{1 - \delta^b(c_y)} + \frac{u(c_y) \delta^{b'}(c_y)}{[1 - \delta^b(c_y)]^2} \right] dy \\ & - \int_0^{x^s(b)} \delta^{b'}(c_y) f'(y) \left[V^b(y) - \frac{u(c_y)}{1 - \delta^b(c_y)} \right] dy. \end{aligned} \quad (15)$$

When $b = b^M$, both $x_s(b)$ and $x^s(b)$ are equal to $\hat{x}$. In this case, $\delta^b(c_y) f'(y) \leq 1$ for any y . Then, $\Phi(b^M) < 0$. By Lemma 3.4, for b close enough to b^M , there exists $\underline{x}(b) > 0$ such that every optimal path starting from $x_0 < \underline{x}(b)$ converges to the origin. Since $\mathcal{E}(b)$ has no poverty trap, every optimal path of these economies converges to the origin for any value of x_0 .

We summarize the results we have obtained so far, assuming no poverty trap for any $b \in (0, b^M)$ as follows. First, there exists b such that every optimal path in the economy $\mathcal{E}(b)$ converges to a positive steady state. Second, $\Phi(b^M) < 0$ and, for any b close enough to b^M , every optimal path of the economy $\mathcal{E}(b)$ converges to the origin.

Let us now take a closer look at the second result: if we fix b close enough to b^M , then, for any $b' \in (b, b^M)$, every optimal path of the economy $\mathcal{E}(b')$ converges to the origin.

Let b^* be the infimum of values b such that, for any $b < b' < b^M$, any optimal path of the economy $\mathcal{E}(b')$ converges to the origin. As a consequence of point (3) in Lemma 7.1, $b^* > 0$.

Fix $r > 0$ such that $0 < b^* - r < b^* + r < b^M$ and $x_0 < x_s(b^* - r)$. By the definition of b^* , there exists a sequence $(\bar{b}_n)_{n=0}^\infty$ converging to b^* with $\bar{b}_n \in (b^* - r, b^*)$ for any n such that, starting from x_0 , any

optimal path $(x_t(\bar{b}_n))_{t=0}^{\infty}$ of the economy $\mathcal{E}(\bar{b}_n)$ converges to a positive steady state. There exists also a sequence $(\underline{b}_n)_{n=0}^{\infty}$ with $\underline{b}_n \in (b^*, b^* + r)$ converging to b^* such that, starting from x_0 , any optimal path $(x_t(\underline{b}_n))_{t=0}^{\infty}$ of the economy $\mathcal{E}(\underline{b}_n)$ converges to the origin.

Since x_0 is smaller than any possible steady state of $\mathcal{E}(\bar{b}_n)$ and $\mathcal{E}(\underline{b}_n)$, the optimal path is strictly increasing in the former case, and strictly decreasing in the latter. By Lemma 7.1, starting from x_0 , the economy $\mathcal{E}(b^*)$ has a nonincreasing optimal path $(\underline{x}_t)_{t=0}^{\infty}$, and a nondecreasing optimal path $(\bar{x}_t)_{t=0}^{\infty}$, both starting from x_0 .

According to Proposition 3.5, since x_0 is smaller than any positive candidate for a steady state of the economy $\mathcal{E}(b^*)$, these sequences are strictly monotonic. The first sequence is strictly decreasing and converges to the origin. The second sequence is strictly increasing. Therefore, x_0 is the critical level of the economy $\mathcal{E}(b^*)$. Recalling that we have chosen x_0 arbitrarily less than $x_s(b^* - r)$, we reach a contradiction: each $x_0 < x_s(b^* - r)$ is a critical level of $\mathcal{E}(b^*)$.

This contradiction comes from the hypothesis that, for any $b \in (0, b^M)$, the economy $\mathcal{E}(b)$ has no poverty trap. Therefore, there exists some b such that the economy $\mathcal{E}(b)$ exhibits a poverty trap. In other words, $\mathcal{C} \neq \emptyset$.

Now, let us prove the following claim: there exists $b \in \mathcal{C}$ and $\varepsilon > 0$ such that, for any $b' \in (b - \varepsilon, b)$, the economy $\mathcal{E}(b')$ has a poverty trap.

Suppose the statement is false and choose $b \in \mathcal{C}$ as well as an $x_0 > 0$ that is less than both the critical level of $\mathcal{E}(b)$ and $x_s(b)$. Observe that there exists $\varepsilon > 0$ small enough such that, for $b' \in (b - \varepsilon, b)$, every optimal path of $\mathcal{E}(b')$ starting from x_0 is nonincreasing. Indeed, the contrary implies the existence of an increasing sequence $(b_n)_{n=0}^{\infty}$ that converges to b , and strictly increasing sequences $(x_t(n))_{t=0}^{\infty}$ that are optimal path of the economy $\mathcal{E}(b_n)$ starting from x_0 . By Lemma 7.1, this entails the existence of an optimal path of the economy $\mathcal{E}(b)$ starting from x_0 that is nondecreasing, a contradiction with the choice of x_0 .

Hence, there exists $\varepsilon > 0$ small enough such that for $b' \in (b - \varepsilon, b)$, every optimal path of $\mathcal{E}(b')$ starting from x_0 is nonincreasing. Under the hypothesis that the claim is false, there exists a sequence $(b_n)_{n=0}^{\infty}$ converging to b , with $b_n \in (b - \varepsilon, b)$ for any n , such that $\mathcal{E}(b_n)$ has no poverty trap and each optimal path of these economies starting from x_0 is nonincreasing. Thus, each optimal path of these economies is decreasing and converges to the origin. According to Lemma 7.1, for any $x_0 > 0$, the economy $\mathcal{E}(b)$ has an optimal path starting from x_0 that is nonincreasing.

We will prove that every optimal path of $\mathcal{E}(b)$ is nonincreasing.¹⁷ Assume the contrary: there exists

¹⁷The critical level is $x^s(b)$: optimal paths starting from $x_0 < x^s(b)$ converge to the origin, whereas those starting from $x_0 > x^s(b)$ converge to $x^s(b)$.

$\tilde{x}_0 > 0$ and an optimal path starting from $\tilde{x}_0$ that is strictly increasing and converges to x^* . By the monotonicity of the optimal policy correspondence, there exists a strictly increasing optimal path that converges to x^* from $x'_0 \in (\tilde{x}_0, x^*)$. Moreover, $\tilde{x}_0$ is not a steady state and $\tilde{x}_0 < x^s(b)$, the largest steady-state candidate. The level x^* is either $x_s(b)$ or $x^s(b)$.

Consider a nonincreasing optimal path starting from $\tilde{x}_0$. Since $\tilde{x}_0$ is not a steady state, by Proposition 3.5, this sequence is strictly decreasing. If this sequence converges to $x_s(b)$, any member belongs to the interval $(x_s(b), x^s(b))$ with $\delta(f(x) - x)f'(x) > 1$, in contradiction with Lemma 3.2.

Hence, the sequence $(\tilde{x}_t)_{t=0}^\infty$ converges to the origin. In other words, $\tilde{x}_0$ is a critical level of $\mathcal{E}(b)$. Since the argument leading to this conclusion can be applied for any $x'_0 \in (\tilde{x}_0, x^*)$, we arrive at a contradiction.

Therefore, under the assumption that the claim is false, we conclude that, for any $b \in \mathcal{C}$, every optimal path of economy $\mathcal{E}(b)$ is nonincreasing. Lemma 7.1 ensures that $b^m \equiv \inf \mathcal{C} > 0$.

Let $(\underline{b}_n)_{n=0}^\infty$ be a sequence of parameters in $\mathcal{C}$ converging to b^m . We do not need the strict monotonicity of the sequence: if $b^m \in \mathcal{C}$, we can simply fix $\underline{b}_n \equiv b^m$ for any n . Since every optimal path of $\mathcal{E}(\underline{b}_n)$ is nonincreasing, by Lemma 7.1, for every $x_0 > 0$, there exists a nonincreasing optimal path of $\mathcal{E}(b^m)$ starting from x_0 .

Let b_* the infimum of values b with the following property: for each $x_0 > 0$ and any b' such that $b \leq b' \leq b^m$, there exists an optimal path of $\mathcal{E}(b')$ starting from x_0 which is nonincreasing.

We observe that $b_* > 0$. This is clear if $b_* = b^m$. Focus instead on the case $b_* < b^m$. Then, for any $b \in (b_*, b^m)$ and any $0 < x_0 < x_s(b)$, there exists an optimal path of $\mathcal{E}(b)$ starting from x_0 which is nonincreasing. The choice of x_0 ensures that this path converges to the origin. By the definition of b^m , the economy $\mathcal{E}(b)$ has no poverty trap. Hence, every optimal path of $\mathcal{E}(b)$ is decreasing and converges to the origin. Since this is true for any $b \in (b_*, b^m)$, by Lemma 7.1, we conclude that $b_* > 0$.

By the choice of b_* and Lemma 7.1, for any $x_0 < x_s(b_*)$, there exists a nonincreasing optimal path starting from x_0 . The choice of x_0 guarantees that this path converges to the origin.

Choose $r \in (0, b_*)$ and $0 < x_0 < x_s(b_* - r)$. By the definition of b_* , there exists a strictly increasing sequence $(\bar{b}_n)_{n=0}^\infty$ with $\bar{b}_n \in (b_* - r, b_*)$ for any n and $\lim_{n \rightarrow \infty} \bar{b}_n = b_*$, such that the economy $\mathcal{E}(b_n)$ has no poverty trap. Since $x_0 < x_s(b_n)$, starting from x_0 , any optimal path $(x_t(\bar{b}_n))_{t=0}^\infty$ of the economy $\mathcal{E}(\bar{b}_n)$ is strictly increasing. Hence, there exists a nondecreasing optimal path of $\mathcal{E}(b_*)$ starting from x_0 . In other words, starting from every $x_0 < x_s(b_* - r)$, there exists a nondecreasing optimal path. Recall that there exists also a decreasing optimal path converging to the origin. This implies that x_0 is a critical level. Since the choice of $x_0 < x_s(b_* - r)$ was arbitrary, we arrive at a contradiction.

The claim is true and there exist $b \in \mathcal{C}$ and $\varepsilon > 0$ such that $(b - \varepsilon, b) \subset \mathcal{C}$. Simply choose $\underline{b} \equiv b - \varepsilon$

and $\bar{b} \equiv b$. □

PROOF OF PROPOSITION 5.1

The proof is standard. We follow exactly the same arguments used in the proofs of Propositions 2, 3, 4 and Theorems 1, 2 in [Le Van and Morhaim \(2002\)](#). □

PROOF OF PROPOSITION 5.2

The proofs of Parts (1) and (2) are also standards. For part (3), we use the supermodularity of the function $u(f(x) - y)$. We observe that

$$\frac{\partial^2}{\partial x \partial y} u(f(x) - y) = -u''(f(x) - y) f'(x) > 0.$$

Assume that $y' < y$. We have

$$\begin{aligned} u(f(x) - y) + \delta(y)V(y) &\geq u(f(x) - y') + \delta(y')V(y'), \\ u(f(x') - y') + \delta(y')V(y') &\geq u(f(x') - y) + \delta(y)V(y). \end{aligned}$$

Summing these inequalities, we get

$$u(f(x) - y) + u(f(x') - y') \geq u(f(x) - y') + u(f(x') - y),$$

a contradiction with the supermodularity property of $u(f(x) - y)$. □

PROOF OF PROPOSITION 5.3

Part (1). We remind that $0 < x_{t+1} < f(x_t)$ for any $t \geq 0$. Taking the derivative of the intertemporal utility function with respect to x_{t+1} , we obtain

$$\begin{aligned} &u'(f(x_t) - x_{t+1}) \\ &= \delta'(x_{t+1})u(f(x_{t+1}) - x_{t+2}) + \delta(x_{t+1})u'(f(x_{t+1}) - x_{t+2})f'(x_{t+1}) \\ &+ \delta'(x_{t+1})\delta(x_{t+2})V(x_{t+2}) = \delta(x_{t+1})u'(f(x_{t+1}) - x_{t+2})f'(x_{t+1}) + \delta'(x_{t+1})V(x_{t+1}). \end{aligned}$$

Part (2). We note that the right-hand side of Euler equation (3) is strictly increasing with respect to x_{t+2} . Hence, for $t \geq 1$, $\varphi(x_t)$ is a singleton. □

PROOF OF PROPOSITION 5.4

Replacing x_t , x_{t+1} and x_{t+2} by x^* in the Euler equation (3), we obtain Part (1). To prove part (2), let $x_{t+1} = x^*$. Replacing x_{t+1}, x_{t+2} by x^* in the Euler equation gives

$$u'(f(x_t) - x^*) = \delta(x^*)u'(f(x^*) - x^*)f'(x^*) + \delta'(x^*)V(x^*).$$

This implies $x_t = x^*$. Hence, if $x_t \neq x^*$, the next state $x_{t+1} \neq x^*$.

The monotonicity property is a direct consequence of Proposition 5.2. □

PROOF OF PROPOSITION 5.5

Before proving the proposition, we introduce an auxiliary function $\psi(y)$:

$$\psi(y) \equiv [\delta(y)f'(y) - 1]u'(f(y) - y) + \frac{\delta'(y)u(f(y) - y)}{1 - \delta(y)}.$$

Part (1). Clearly, $\psi(y) > 0$ if and only if

$$\delta(y)f'(y) > 1 - \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]u'(f(y) - y)}.$$

We have $\psi(y) > 0$ for $0 < y < \hat{x}$. Assume the existence of an optimal path $(x_t)_{t=0}^\infty$ starting from $x_0 > 0$ and converging to zero. Monotonicity ensures that this sequence is strictly decreasing. There exists some T such that $x_t < \hat{x}$ for any $t \geq T$.

We will prove that, if $t \geq T$,

$$\frac{u(f(x_t) - x_t)}{1 - \delta(x_t)} > u(f(x_t) - x_{t+1}) + \delta(x_{t+1})\frac{u(f(x_{t+1}) - x_{t+1})}{1 - \delta(x_{t+1})}.$$

Define

$$\zeta(y) \equiv u(f(x_t) - y) + \delta(y)\frac{u(f(y) - y)}{1 - \delta(y)}.$$

ζ is strictly increasing in (x_{t+1}, x_t) . Indeed,

$$\begin{aligned} \zeta'(y) &= -u'(f(x_t) - y) + \delta'(y)\frac{u(f(y) - y)}{1 - \delta(y)} \\ &\quad + \delta(y)\frac{[1 - \delta(y)]u'(f(y) - y)[f'(y) - 1] + \delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} \\ &> -u'(f(y) - y) + \delta'(y)\frac{u(f(y) - y)}{1 - \delta(y)} \\ &\quad + \delta(y)\frac{[1 - \delta(y)]u'(f(y) - y)[f'(y) - 1] + \delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} \\ &= \frac{\psi(y)}{1 - \delta(y)} > 0. \end{aligned}$$

Hence, ζ is strictly increasing in (x_{t+1}, x_t) and we have $\zeta(x_t) > \zeta(x_{t+1})$, which implies

$$\begin{aligned} \frac{u(f(x_T) - x_T)}{1 - \delta(f(x_T) - x_T)} &> u(f(x_T) - x_{T+1}) + \delta(x_{T+1})\frac{u(f(x_{T+1}) - x_{T+1})}{1 - \delta(x_{T+1})} \\ &> u(f(x_T) - x_{T+1}) + \delta(x_{T+1})u(f(x_{T+1}) - x_{T+2}) \\ &\quad + \delta(x_{T+1})\delta(x_{T+2})\frac{u(f(x_{T+2}) - x_{T+2})}{1 - \delta(x_{T+2})} > \dots \geq V(x_T), \end{aligned}$$

a contradiction.

Part (2). Using similar arguments, we can prove that, if the optimal path $(x_t)_{t=0}^\infty$ is increasing, the function ζ is decreasing in (x_t, x_{t+1}) , which leads to a contradiction. $\square$

PROOF OF COROLLARY 5.1

Apply the proof of Proposition 5.5. $\square$

PROOF OF PROPOSITION 5.6

Part (1). Assume that there exists an increasing optimal path $(x_t)_{t=0}^\infty$ starting from x_0 with limit x^* . For any $t \geq 0$, we have

$$\begin{aligned} V(x_{t+1}) - V(x_t) &\geq u(f(x_{t+1}) - x_{t+1}) + \delta(x_{t+1})V(x_{t+1}) - u(f(x_t) - x_{t+1}) - \delta(x_{t+1})V(x_{t+1}) \\ &= u(f(x_{t+1}) - x_{t+1}) - u(f(x_t) - x_{t+1}) = \int_{x_t}^{x_{t+1}} u'(f(y) - x_{t+1})f'(y)dy \\ &\geq \int_{x_t}^{x_{t+1}} u'(f(y) - y)f'(y)dy. \end{aligned}$$

This implies

$$\begin{aligned} V(x^*) - V(x_0) &= \sum_{t=0}^{\infty} [V(x_{t+1}) - V(x_t)] \geq \sum_{t=0}^{\infty} \int_{x_t}^{x_{t+1}} u'(f(y) - y)f'(y)dy \\ &= \int_{x_0}^{x^*} u'(f(y) - y)f'(y)dy. \end{aligned}$$

On the other hand,

$$\begin{aligned} V(x^*) - V(x_0) &\leq \frac{u(f(x^*) - x^*)}{1 - \delta(x^*)} - \frac{u(f(x_0) - x_0)}{1 - \delta(x_0)} \\ &= \int_{x_0}^{x^*} \frac{[1 - \delta(y)]u'(f(y) - y)[f'(y) - 1] + \delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} dy. \end{aligned}$$

This implies

$$\int_{x_0}^{x^*} \frac{[1 - \delta(y)]u'(f(y) - y)[f'(y) - 1] + \delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} dy \geq \int_{x_0}^{x^*} u'(f(y) - y)f'(y)dy.$$

Taking the left-hand side minus the right-hand side, we get

$$\int_{x_0}^{x^*} \left([\delta(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta(y)} + \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} \right) dy \geq 0,$$

a contradiction.

Part (2). Using similar arguments, we can prove that assuming a decreasing optimal path starting from x_0 leads to a contradiction. $\square$

PROOF OF COROLLARY 5.2

Part (1). Choose $\bar{x} < x^s$ close enough to x^s such that

$$\int_0^{\bar{x}} \left([\delta(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta(y)} + \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} \right) dy > 0,$$

and apply Proposition 5.6.

Part (2). Choose $\underline{x} < x_s$ close enough to 0 such that

$$\int_{\underline{x}}^{x^s} \left([\delta(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta(y)} + \frac{\delta'(y)u(f(y) - y)}{[1 - \delta(y)]^2} \right) dy < 0,$$

and apply Proposition 5.6. □

PROOF OF PROPOSITION 6.1

We consider the function $\zeta(x) \equiv (1 + \rho\alpha)Ax^\alpha - (1 + \rho)x$. Clearly, (4) is equivalent to $\zeta(x) = \rho b$. Let

$$\hat{x} \equiv \left(\frac{1 + \rho\alpha}{1 + \rho} A \right)^{\frac{1}{1-\alpha}}.$$

The function ζ is strictly concave, with $\zeta(0) = \zeta(\hat{x}) = 0$. Define

$$b^M \equiv \frac{1}{\rho} \max_{x \in [0, \hat{x}]} \zeta(x).$$

If $b > b^M$, then $\zeta(x) < \rho b$ for any $x \geq 0$. If $0 < b < b^M$, then the equation $\zeta(x) = \rho b$ has exactly two positive solutions: $x_s(b) < x^s(b)$. The first one is strictly increasing with respect to b , while the second one is strictly decreasing. We obtain $\lim_{b \rightarrow 0} x_s(b) = 0$ and $\lim_{b \rightarrow 0} x^s(b) = \hat{x}$.

Let us denote the economy associated to the discount function δ^b by $\mathcal{E}(b)$, and the corresponding intertemporal utility function by $W^b(x_0, x_1, x_2, \dots)$.

Reminding that, if $b = b^M$, for any $y \in [0, \hat{x}]$, we have

$$\delta^b(y)f'(y) \leq 1 - \frac{\delta^{b'}(y)u(f(y) - y)}{[1 - \delta^b(y)]u'(f(y) - y)},$$

which is equivalent to

$$[\delta^b(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta^b(y)} + \frac{\delta^{b'}(y)u(f(y) - y)}{[1 - \delta^b(y)]^2} \leq 0.$$

Consider

$$\Phi(b) \equiv \int_0^{x^s(b)} \left([\delta^b(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta^b(y)} + \frac{\delta^{b'}(y)u(f(y) - y)}{[1 - \delta^b(y)]^2} \right) dy.$$

Clearly, $\Phi(b^M) < 0$.

At this stage of the proof, we need to introduce a preparatory Lemma.

LEMMA 7.2. 1. Fix $x_0 > 0$. Consider $b > 0$ and a sequence of parameters $(b_n)_{n=0}^\infty$ that converges to b . For each n , let $(x_t(n))_{t=0}^\infty$ be an optimal path of the economy $\mathcal{E}(b_n)$ starting from x_0 . If, for any n , $(x_t(n))_{t=0}^\infty$ is nondecreasing, then the economy $\mathcal{E}(b)$ has a nondecreasing optimal path starting from x_0 .

2. Fix $x_0 > 0$. Consider $b > 0$ and a sequence of parameters $(b_n)_{n=0}^\infty$ that converges to b . For each n , let $(x_t(n))_{t=0}^\infty$ be an optimal path of the economy $\mathcal{E}(b_n)$ starting from x_0 . If, for any n , $(x_t(n))_{t=0}^\infty$ is nonincreasing, then the economy $\mathcal{E}(b)$ has a nonincreasing optimal path starting from x_0 .

3. For $b > 0$ close enough to zero, we have $\Phi(b) > 0$.

Proof of Lemma 7.2

Part (1) and (2): the proof follows exactly the same arguments as in the proof of Lemma 7.1.

We consider Part (3). We will prove that, for b close enough to zero, $\Phi(b) > 0$.

Fix $\varepsilon \in (0, 1)$ and $\hat{y}$ such that, if $y < \hat{y}$, we have $f(y) - y > y$. For b small enough, $x_s(b) < \hat{y}$ and $\delta^b(y) < 1/f'(y) < \varepsilon$ for any $y < x_s(b)$.

For small b , we have

$$\begin{aligned} \int_0^{x_s(b)} \frac{u'(f(y) - y)}{1 - \delta^b(y)} dy &< \int_0^{x_s(b)} \frac{u'(f(y) - y)}{1 - \varepsilon} dy \\ &= \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} u'(f(y) - y) dy \\ &< \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} u'(y) dy = \frac{u(x_s(b))}{1 - \varepsilon}. \end{aligned}$$

and, hence,

$$\lim_{b \rightarrow 0} \int_0^{x_s(b)} \frac{u'(f(y) - y)}{1 - \delta^b(y)} dy = 0. \quad (16)$$

Moreover,

$$\begin{aligned} &\int_0^{x_s(b)} \frac{\delta^{b'}(y) u(f(y) - y)}{[1 - \delta^b(y)]^2} dy \\ &< \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} \frac{\delta^{b'}(y) u(f(y) - y)}{1 - \delta^b(y)} dy = \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} \frac{\frac{b}{(b+y)^2} u(f(y) - y)}{\frac{b}{b+y}} dy \\ &= \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} \frac{u(f(y) - y)}{b + y} dy \leq \frac{1}{1 - \varepsilon} \int_0^{x_s(b)} \frac{u(f(y))}{y} dy \\ &= \frac{A^\rho}{1 - \varepsilon} \int_0^{x_s(b)} y^{\rho\alpha - 1} dy = \frac{A^\rho}{(1 - \varepsilon)\rho\alpha} \int_0^{x_s(b)} (y^{\rho\alpha})' dy = \frac{A^\rho [x_s(b)]^{\rho\alpha}}{(1 - \varepsilon)\rho\alpha}. \end{aligned}$$

Thus,

$$\lim_{b \rightarrow 0} \int_0^{x_s(b)} \frac{\delta^{b'}(y) u(f(y) - y)}{[1 - \delta^b(y)]^2} dy = 0. \quad (17)$$

The limits in (16) and (17), jointly with $\delta^b(y)f'(y) < 1$ for $y < x_s(b)$, imply

$$\lim_{b \rightarrow 0} \int_0^{x_s(b)} \left([\delta^b(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta^b(y)} + \frac{\delta^{b'}(y)u(f(y) - y)}{[1 - \delta^b(y)]^2} \right) dy = 0. \quad (18)$$

We can now show that

$$\lim_{b \rightarrow 0} \int_{x_s(b)}^{x^s(b)} \left([\delta^b(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta^b(y)} + \frac{\delta^{b'}(y)u(f(y) - y)}{[1 - \delta^b(y)]^2} \right) dy = \infty. \quad (19)$$

Fix $0 < x_* < x^* < x^G$ (solution to $f'(x) = 1$) and $\hat{b}$ small enough such that, for $b < \hat{b}$,

$$0 < x_s(b) < x_* < x^* < x^s(b) < \hat{x},$$

and $\delta^{\hat{b}}(y)f'(y) > 1$ for any $x_* \leq y \leq x^*$. We obtain

$$\begin{aligned} & \int_{x_s(b)}^{x^s(b)} \left([\delta^b(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta^b(y)} + \frac{\delta^{b'}(y)u(f(y) - y)}{[1 - \delta^b(y)]^2} \right) dy \\ & > \int_{x_*}^{x^*} \left([\delta^b(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta^b(y)} + \frac{\delta^{b'}(y)u(f(y) - y)}{[1 - \delta^b(y)]^2} \right) dy \\ & > \int_{x_*}^{x^*} [\delta^{\hat{b}}(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta^b(y)} dy. \end{aligned}$$

For any $y \in (x_*, x^*)$, $\lim_{b \rightarrow 0} \delta^b(y) = 1$. Hence,

$$\lim_{b \rightarrow 0} \int_{x_*}^{x^*} [\delta^{\hat{b}}(y)f'(y) - 1] \frac{u'(f(y) - y)}{1 - \delta^b(y)} dy = \infty.$$

From (18) and (19), we have $\lim_{b \rightarrow 0} \Phi(b) = \infty$. Thus, there exists b small enough such that $\Phi(b) > 0$. $\square$

We can now return to the proof of Proposition 6.1.

First, we show the existence of b such that the economy $\mathcal{E}(b)$ has a poverty trap.

Assume the contrary. Remind that, if b is close enough to b^M , we have $\Phi(b) < 0$. By Proposition 5.6, there are optimal paths converging to the origin. Then, any optimal path of $\mathcal{E}(b)$ converges to the origin, otherwise a poverty trap exists.

Conversely, if b is close enough to zero, we find $\Phi(b) > 0$. By Proposition 5.6, there are optimal paths converging to a positive steady state. Then, every optimal path of $\mathcal{E}(b)$ converges to a positive steady state.

Let b^* be the infimum of values b with the following property: for any $b < b' < b^M$, every optimal path of the economy $\mathcal{E}(b')$ converges to the origin. By Lemma 7.2, $b^* > 0$.

From now on, we can apply the arguments in section 4, taking care to replace Lemma 7.1 by Lemma 7.2, and obtain a contradiction. These arguments also imply the existence of $\underline{b} < \bar{b}$ such that, for any $b \in (\underline{b}, \bar{b})$, the economy $\mathcal{E}(b)$ experiences a poverty trap. $\square$

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