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## Abstract

We consider a general Ramsey model with endogenous discounting, depending on current consumption or future capital, study the monotonicity properties of the optimal path, and provide a new narrative for the existence of a poverty trap, alternative to the literature on convex-concave production functions. We prove the continuity and differentiability properties of the value function, as well as the monotonicity of the policy correspondence, which in turn entails the strict monotonicity of the optimal path. Importantly, the existence of a poverty trap relies on the existence of a critical level of capital such that, if the initial condition is lower, the optimal path converges to the origin, while, if it is higher, this path converges to a positive steady state. Since it is impossible to compute this critical level under endogenous discounting when the discount factor is a general function of current consumption or future capital, in both these cases, we complement the theoretical analysis with robust corresponding examples and, showing that a poverty trap exists for a nonzero-measure set of parameter values, we demonstrate that the poverty trap is a pervasive feature under endogenous discounting.

**Keywords**: Endogenous discounting, dynamic programming, Ramsey-Cass-Koopmans model.

**JEL** classification numbers: C61, D11, O4.

## 1. Introduction

Discounting is key to shape individual decision, drive capital accumulation and general equilibrium dynamics. In order to put current and future generations on an equal footing, Ramsey (1928) assumes that there is no discounting. Phelps (1961), Cass (1965) and Koopmans (1965) draw on Fisher (1930)'s approach to introduce an exogenous discount rate and revise Ramsey's main conclusion to obtain a modified golden rule. Fisher (1930) is also relevant to endogenous discounting, conjecturing that wealth increases the individual's level of patience and, consequently, reversing the causal nexus of Ramsey (1928)'s conjecture, which refers to an economy with exogenous discounting, but heterogeneous agents, and stipulates that the most patient agent becomes the richest in the long run. Uzawa (1968) was the first to model discounting on the basis of past consumption experience. Since then, an extensive literature has developed on the dynamic properties and time inconsistency of economies where discounting is a function of other variables, namely consumption or wealth (see Vasilev (2022), among others).

Our aim is to adopt a general approach when discounting depends on consumption experience or wealth in order to draw robust conclusions about the dynamic properties of the optimal path, such as monotonicity or the existence of a poverty trap.

We study optimal growth models in which the discount factor may not be constant, but may depend on either consumption or the level of capital.

We consider a standard period utility $u$ depending solely on consumption and a standard production function $f$ in a discrete-time optimal growth model. Given a feasible capital path $(x_t)_{t=0}^{\infty}$, the corresponding consumption sequence is determined by $c_t = f(x_t) - x_{t+1}$.

In what follows, we define intertemporal utility recursively and consider two specifications of discounting.

(1) First, we assume that intertemporal utility depends only on the consumption path:

$$U(c_0, c_1, c_2, \ldots) = u(c_0) + \delta(c_0)U(c_1, c_2, c_3, \ldots).$$

The intertemporal utility of the consumption path $(c_0, c_1, \ldots)$ is defined as a sum of the period utility $u\,(c_0)$ and the discounted utility derived from future consumption, determined by the sequence $(c_1, c_2, \ldots)$.

(2) Alternatively, we assume that, instead of current capital, the discount factor depends on future capital. The intertemporal utility generated by $(x_t)_{t=0}^{\infty}$ is given

by:

$$W(x_0, x_1, \ldots) = u\left(f(x_0) - x_1\right) + \delta(x_1) W(x_1, x_2, \ldots).$$

We will take a closer look at these criteria later. By developing these recursive forms to infinity, under appropriate boundedness properties ensuring the convergence of infinite sums, we find the following.

(1) Discounting based on consumption:

$$U(c_0, c_1, \ldots) = u(c_0) + \sum_{t=1}^{\infty} u(c_t) \prod_{s=0}^{t-1} \delta(c_s).$$

(2) Discounting based on future capital with $c_t = f(x_t) - x_{t+1}$:

$$W(x_0, x_1, \ldots) = u(c_0) + \sum_{t=1}^{\infty} u(c_t) \prod_{s=0}^{t-1} \delta(x_{s+1}).$$

In the limit case where the discount factor $\delta$ no longer depends on consumption or capital, both criteria collapse and we recover the classical Ramsey-Cass-Koopmans model:

$$W(x_0, x_1, \ldots) = \sum_{t=0}^{\infty} \delta^t u(c_t).$$

Our paper is close to Borissov et al. (2025). However, they consider an intertemporal utility defined as a recursive convex combination of current and future utility generating a weighted average of current and futures period utilities as value. In this respect they remains close to Chambers and Echenique (2018), but, differently, as in our paper, they consider endogenous discounting. More precisely, their discounting depends positively on current consumption or capital and, negatively, on future capital.

Here, we focus only on two cases, that is on a function of current consumption or future capital. We assume that, in the case of the first criterion, functions $\delta(c_t)$ is increasing, while, in the second case, $\delta(x_{t+1})$ is supposed to be either decreasing or, differently from Borissov et al. (2025), increasing. In the first case, we capture Fisher (1930)'s intuition that the present weighs more when the agent is poorer. In the second case, we do not impose restrictions on the sign of $\delta'(x_{t+1})$: forward-looking discounting can be increasing or decreasing. We just consider a strictly increasing function in the example.

The main added value of our contribution rests on the proofs of monotonicity of the optimal path and of the existence of a poverty trap.

The central question of our contribution is: since the discount factor is endogenous, can an economy starting from a low level of capital stock converge to a better state, or will it remain trapped at a low level despite high initial productivity? The logic of a pessimistic scenario is simple: when the economic agent is poor, the discount factor is low, which leads to future underinvestment and, thus, creates a vicious circle that traps the economy in poverty.

In the literature, such a vicious circle arises when a low capital stock has a low productivity, which is the case in presence of fixed costs or Allee effects. The economy therefore exhibits a critical level of capital stock, generally referred to as Dechert-Nishimura-Skiba point, with the property that, below this level, the economy collapses and, above it, it can converge to a positive state.[1]

In this paper, we focus on the endogenous discount factor as an alternative explanation for the existence of a poverty trap. We are faced with difficulties that do not appear in the literature on convex-concave production functions, where the existence of a critical level and a poverty trap relies on the interplay between the (constant) discount factor, productivity at the inflection point[2] and maximum average production.[3] The analysis becomes difficult or even impossible when the discount factor is no longer constant. In this case, it is not easy to find the general conditions for the existence of a poverty trap. To overcome this theoretical obstacle, we specify the fundamentals in two examples where the critical level delimiting the poverty trap appears to be a pervasive feature, in the sense that the trap exists for an infinite number of parameter values.

In many respects, these examples remain sufficiently general to capture robust properties. Firstly, the production function is Cobb-Douglas, meaning that productivity at the origin is infinite, while the utility function has a constant elasticity of intertem-

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[1] Interested readers are referred to the classic contributions by Clark (1971), Skiba (1978), Majumdar and Mitra (1982) and Dechert and Nishimura (1983). They are also referred to a recent review by Akao et al. (2025). Hung et al. (2009) also show the existence of a critical level, but in the context of a production function aggregating two different technologies.

[2] The level where the production function changes its shape from convex to concave.

[3] See Dechert and Nishimura (1983) and Le Van and Dana (2003) for low, high and intermediate discounting conditions. The poverty trap appears in the case of high or intermediate discounting.

poral substitution. Secondly, the discount factor function is simply hyperbolic, and the existence of a poverty trap depends only on the value of its curvature.

More precisely, we prove the monotonicity of the optimal policy correspondence, which guarantees that, if an optimal path does not start from a steady state, it must be strictly monotonic. The existence of a poverty trap is linked to the existence of a critical level of capital such that, if the initial condition is below (above), the optimal path converges towards the origin (towards a positive steady state). The monotonicity property of the optimal policy correspondence guarantees that, if a critical level exists, it is unique.

We cannot conclude without referring to what is probably the closest paper to ours, that is Erol et al. (2011), where the sum of discounted factors differs from one and discounting is endogenous. In their contribution, the discount factor $\beta(x_{t+1}) \in (0,1)$ depends on future capital (see their Proposition 4) with $\beta'(x_{t+1}) > 0$ (the rich are more patient). As in our paper, they define the intertemporal utility as the sum of current utility and a discounted future utility. However, our paper differs from theirs in many respects. In the first part, we consider also a discounting which, in the spirit of Uzawa (1968), depends on consumption as a proxy of wealth or consumption habits. In the second part, we focus on future capital, but we remain more general, because our theoretical results hold in both cases of increasing or decreasing discounting ($\beta'(x_{t+1}) \lesseqgtr 0$). Importantly, we find a monotonicity property and, so, rule out cycles also in the less favorable case $\beta'(x_{t+1}) > 0$ (notice that there is no room for cycles in Borissov et al. (2025) when $\beta'(x_{t+1}) < 0$). Finally, Erol et al. (2011) show the monotonicity property of the optimal path and the existence of a poverty trap only through numerical simulations, while we prove that through a robust analytical example with very standard production, utility and discounting functions.

In the first part, we study the case where discounting is a function of current consumption and we complement the theoretical results with a simple but robust example in order to prove the existence of a poverty trap more explicitly. In the second part, we consider the case where discounting depends on future capital and, similarly to what we obtain in the first part, we show the existence of a poverty trap more explicitly. All proofs are gathered in the Appendix.

# 2. Discounting as a function of current consumption

## 2.1 Fundamentals

Let $f$, $u$ and $\delta$ be the production, utility and discount functions respectively. The intertemporal utility function satisfies the recursive structure :

$$U(c_0, c_1, c_2, \ldots) = u(c_0) + \delta(c_0)U(c_1, c_2, c_3, \ldots).$$

The optimization program can be formulated as follows

$$\max_{(c_t, x_{t+1})_{t=0}^{\infty}} \left[ u(c_0) + \sum_{t=1}^{\infty} u(c_t) \prod_{s=0}^{t-1} \delta(c_s) \right],$$

subject to $c_t + x_{t+1} \leq f(x_t)$, $c_t \geq 0$, $x_t \geq 0$ for any $t$, given $x_0 \geq 0$.

Given the initial condition $x_0 \geq 0$, the accumulation path $\chi = (x_0, x_1, \ldots)$ is said to be feasible from $x_0$ if $0 \leq x_{t+1} \leq f(x_t)$ for any $t$. The set of all feasible paths is denoted by $\Pi(x_0)$. Under the increasing monotonicity of $f$, if $x_0 < x_0'$, then $\Pi(x_0) \subset \Pi(x_0')$. A consumption sequence $(c_0, c_1, \ldots)$ is feasible from $x_0 \geq 0$ if there exists $\chi \in \Pi(x_0)$ such that $0 \leq c_t \leq f(x_t) - x_{t+1}$ for any $t \geq 0$.

Let us introduce the following assumptions about the properties of production, utility and discounting.

**Assumption 1.**

1. The production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly concave and strictly increasing, and it satisfies $f(0) = 0$, $\lim_{x\to\infty} f'(x) < 1$.

2. The function $\delta$ is continuous, non-decreasing, twice differentiable, and concave: for any $c > 0$, $\delta'(c) \geq 0$ and $\delta''(c) \leq 0$.

3. The utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing and strictly concave, and it satisfies the Inada condition $u'(0) = \infty$.

4. $u(0) = 0$.

Let us justify the double condition in $u(0) = 0$: the utility function is bounded from below and the value at zero is normalized to zero. We want the intertemporal utility function to satisfy the following two properties.

1. Pareto property: for any two sequences $(c_0, c_1, c_2, \ldots)$ and $(c'_0, c'_1, c'_2, \ldots)$ with $c_t \geq c'_t$ for every $t$, with at least one strict inequality, then $U(c_0, c_1, c_2, \ldots) > U(c'_0, c'_1, c'_2, \ldots)$.

2. $U(c_0, c_1, c_2, \ldots)$ is strictly concave with respect to each argument $c_t$.

If the discount factor function $\delta$ is not constant, the Pareto property implies that $u$ is bounded from below. Indeed, assume that $u(0) = -\infty$ and there are $c_0 > c'_0$ such that $\delta(c_0) > \delta(c'_0)$. Choose $c$ close enough to zero so that

$$\frac{u(c)}{1-\delta(c)} < \frac{u(c'_0) - u(c_0)}{\delta(c_0) - \delta(c'_0)}.$$

It is easy to check that $U(c_0, c, c, \ldots) < U(c'_0, c, c, \ldots)$, a contradiction.

By the concavity of $U$ with respect to each argument $c_t$, the function $U(c, 0, 0, \ldots)$ is also concave with respect to $c$. More explicitly,

$$U(c, 0, 0, \ldots) = u(c) + \delta(c)\frac{u(0)}{1-\delta(0)}$$

is a concave function with respect to $c$. Let us define a new utility function $\hat{u}$:

$$\hat{u}(c) \equiv u(c) + \delta(c)\frac{u(0)}{1-\delta(0)} - \frac{u(0)}{1-\delta(0)}.$$

Clearly, $\hat{u}$ is also a strictly increasing function (by the Pareto property), concave, with $\hat{u}(0) = 0$. Let $\hat{U}$ be defined as:

$$\hat{U}(c_0, c_1, c_2, \ldots) \equiv U(c_0, c_1, c_2, \ldots) - \frac{u(0)}{1-\delta(0)}.$$

The function $\hat{U}$ represents the same preferences as $U$. A simple calculus gives us:

$$\hat{U}(c_0, c_1, c_2, \ldots) = \hat{u}(c_0) + \delta(c_0)\hat{U}(c_1, c_2, c_3, \ldots) = \hat{u}(c_0) + \sum_{t=1}^{\infty} \hat{u}(c_t) \prod_{s=0}^{t-1} \delta(c_s).$$

Therefore, we can always study the problem with a utility function $u$ such that $u(0) = 0$.

## 2.2 Optimal path and Bellman equation

Noting that the constraints will be binding at the optimum because utility is strictly increasing and the discount function is strictly positive, we introduce the function $W$ defined over the set of consumption sequences as follows:

$$U(c_0, c_1, c_2 \ldots) = u(c_0) + \sum_{t=1}^{\infty} u(c_t) \prod_{s=0}^{t-1} \delta(c_s).$$

The following lemma presents some preliminary results. Its proof is a direct application of Tychonov's Theorem (see, among others, Stokey et al. (1989) and Le Van and Dana (2003)). For each sequence of capital $\chi = \left(x_t\right)_{t=0}^{\infty}$, let the corresponding consumption sequence be defined as $c_t = f(x_t) - x_{t+1}$ for any $t \geq 0$, and $W$ be the function defined on the set of feasible sequences:

$$W(x_0, x_1, x_2, \dots) = U(c_0, c_1, c_2, \dots).$$

Lemma 2.1 is standard, and provides compactness and continuity properties guaranteeing the existence of an optimal path.

Lemma **2.1.** *Let $x^M$ be the solution to $f(x) = x$. Then,*

1. *For any $\chi \in \Pi(x_0)$, we have $x_t \leq \max\left\{x_0, x^M\right\}$.*
2. *$\Pi(x_0)$ is compact in the product topology defined in the space of sequences $\chi$.*
3. *$W$ is well-defined and it is continuous over $\Pi(x_0)$ in the product topology.*

The initial optimal growth model becomes equivalent to the following program:

$$\max_{\chi \in \Pi(x_0)} W(\chi).$$

The existence of an optimal path is a direct consequence of the fact that $\Pi(x_0)$ is compact for the product topology defined in the space of sequences $\chi$, and that $U$ is continuous in this topology. The positivity of optimal consumption and capital stock paths follows from the Inada condition and the boundedness of the discount function.

The value function $V$ is defined by

$$V(x_0) \equiv \max_{\chi \in \Pi(x_0)} W(\chi),$$

for $x_0 \geq 0$.

The utility function is bounded from below, and the bounds of discounting, together with the existence of a maximum sustainable capital stock, guarantee a finite value function. It can be shown that the value function is non-negative and continuous. Under these conditions, the Bellman functional equation can be established, which accepts the value function $V$ as a unique solution.

**Proposition 2.1.** 1. *For any* $x_0 \geq 0$, $\operatorname{argmax}_{\chi \in \Pi(x_0)} W(\chi)$ *is a nonempty set.*

2. *The value function* $V$ *is continuous and strictly increasing. It satisfies the Bellman equation*

$$V(x_0) = \max_{0 \leq y \leq f(x_0)} \left[ u\big(f(x_0) - y\big) + \delta\big(f(x_0) - y\big) V(y) \right],$$

*for any* $x_0 \geq 0$.

3. *A sequence* $\chi \in \Pi(x_0)$ *is the optimal solution if and only if it satisfies the equation*

$$V(x_t) = u\big(f(x_t) - x_{t+1}\big) + \delta\big(f(x_t) - x_{t+1}\big) V(x_{t+1}).$$

The optimal policy correspondence, $\varphi : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}_+)$, is defined as follows:

$$\varphi(x_0) = \arg \max_{0 \leq y \leq f(x_0)} \left[ u\left(f(x_0) - y\right) + \delta\left(f(x_0) - y\right) V(y) \right].$$

Note that the solution may not be unique. The non-emptiness and closedness of the optimal correspondence, as well as its equivalence to the optimal path, can be easily deduced from the continuity of the value function using a standard application of the Maximum Theorem. Proposition 2.2 states the main properties of $\varphi$ with important consequences thereafter, such as the monotonicity of optimal paths.

**Proposition 2.2.** 1. $\varphi$ *is closed and upper hemi-continuous and* $\varphi(0) = \{0\}$.

2. *If* $x_0 > 0$, *then* $0 < x_1 < f(x_0)$ *for any* $x_1 \in \varphi(x_0)$.

3. *A sequence* $\chi \in \Pi(x_0)$ *is optimal if and only if* $x_{t+1} \in \varphi(x_t)$ *for any* $t \geq 0$.

4. *Assume that* $y \in \varphi(x), y' \in \varphi(x')$ *with* $x < x'$, *then* $y \leq y'$.

## 2.3 Dynamics

### 2.3.1 Euler equation and monotonicity of the optimal path

We begin our study of the long run by characterizing the Euler equations. Interestingly, although these equations involve the utility function $u$ and the value function $V$, the optimal path remains unchanged.

**Proposition 2.3.** *Fix* $x_0 > 0$ *and an optimal path* $\chi \in \Pi(x_0)$ *starting from* $x_0$.

1. *The path $\chi$ satisfies the Euler equation for any $t \geq 0$:*

$$\begin{aligned} u'\big(f(x_t) - x_{t+1}\big) &= \delta\big(f(x_t) - x_{t+1}\big)u'\big(f(x_{t+1}) - x_{t+2}\big)f'(x_{t+1}) \\ &\quad - \delta'\big(f(x_t) - x_{t+1}\big)V(x_{t+1}) \\ &\quad + \delta\big(f(x_t) - x_{t+1}\big)f'(x_{t+1})\delta'\big(f(x_{t+1}) - x_{t+2}\big)V(x_{t+2}). \end{aligned} \tag{1}$$

2. *For any $t \geq 0$, $\varphi(x_{t+1})$ is a singleton and the value function $V$ is differentiable at $x_{t+1}$.*

Observe that the right-hand side of the Euler equation is strictly increasing with respect to $x_{t+2}$. In other words, $\varphi(x_{t+1})$ is a singleton for any $t \geq 1$. Following the same arguments as in the proofs of Theorem 6 and Corollary 4 in Dechert and Nishimura (1983), we conclude that the value function $V$ is differentiable at $x_t$ when $t \geq 1$. But, by Monotone Differentiation Theorem,[4] the value function is also differentiable almost everywhere.

In addition, the Euler equations entail important properties of non-trivial steady states. A necessary condition for their existence is established. Moreover, if the economy starts from an initial level that does not correspond to a steady state, it can asymptotically converge to a steady state, but it can never reach it in a finite number of periods.

PROPOSITION **2.4.** *Assume that $x^*$ is a positive steady state.*

1. *We have*

$$\delta\big(f(x^*) - x^*\big)f'(x^*) = 1. \tag{2}$$

2. *For any optimal path $\big(x_t\big)_{t=0}^{\infty}$, if $x_0 \neq x^*$, then $x_t \neq x^*$ for any $t \geq 0$.*

The monotonicity of the optimal policy correspondence guarantees that, if an optimal path does not start from a steady state, it would never reach one. Proposition 2.5 states a stronger property: if an optimal path starts from a steady state, it will remain there indefinitely.

PROPOSITION **2.5.** 1. *If $x^*$ is a steady state, then $\varphi(x^*) = \{x^*\}$.*

[4] Interested readers are referred to Theorem 1.6.25 in Tao (2011).

2. *Any optimal path is either constant or strictly monotonic.*

A natural question arises about how we can determine whether an optimal path is increasing or decreasing. Lemma 2.2 provides a partial answer to this question. It should be noted that this result is established without relying on the convexity structure of RCK models, particularly the concavity of the intertemporal utility function and the value function. To overcome the difficulties associated with the absence of such convexity, we develop a new approach to solve the problem. Details are provided in the Appendix.

Lemma **2.2.** *Fix any open interval* $I \subset \big(0, x^M\big)$.

1. *If* $\delta\big(f(x) - x\big) f'(x) > 1$ *for any* $x \in I$, *then there exists no strictly decreasing optimal path* $(x_t)_{t=0}^{\infty}$ *such that* $x_t \in I$ *for any* $t \geq T$, *for some* $T$ *large enough.*
2. *If* $\delta\big(f(x) - x\big) f'(x) < 1$ *for any* $x \in I$, *then there exists no strictly increasing optimal path* $(x_t)_{t=0}^{\infty}$ *such that* $x_t \in I$ *for any* $t \geq T$, *for some* $T$ *large enough.*

Lemma 2.2 generalizes a well-known result: if productivity exceeds the discount rate, the economy accumulates capital, whereas, in the opposite case, the capital stock shrinks.

Proposition 2.6 provides a characterization of optimal paths. Part (1) states that, if, for a small value of the capital stock, productivity exceeds the discount rate, then the optimal path from these values is increasing. Part (2) provides a condition for an inverse result.

Proposition **2.6.**

1. *If* $\liminf_{x \to 0} \delta\big(f(x) - x\big) f'(x) > 1$, *then any optimal path converges to a positive steady state.*
2. *If* $\delta\big(f(x) - x\big) f'(x) < 1$ *for every* $x > \hat{x}$ *for some positive value* $\hat{x}$, *then any optimal path starting from* $x_0 > \hat{x}$ *is strictly decreasing.*

A direct consequence of Proposition 2.6 is a generalization of well-known results of the Ramsey-Cass-Koopmans (RCK) framework, with a constant discount rate.

### 2.3.2 Poverty trap

Let the critical value for poverty trap be defined as a positive value $x^C$ such that: if $x_0 < x^C$, then every optimal path from $x_0$ converges to the origin and, if $x_0 > x^C$, then every optimal path from $x_0$ remains bounded away from zero. The monotonicity property of the optimal policy correspondence $\varphi$ guarantees that, if $x^C$ exists, it is unique. Lemma 2.3 provides a simple characterization of the existence of the poverty trap based on the monotonicity of optimal paths.

LEMMA **2.3.** *A poverty trap exists if and only if there is an optimal path that converges to the origin and an optimal path that converges to a positive steady state.*

Note that, contrary to intuition, poverty trap can occur in a situation where every optimal path is non-increasing. In particular, we do not rule out the possibility of a steady state $x^*$ such that, while every optimal path starting from $x_0 < x^*$ strictly decreases and converges to the origin, every optimal path starting from $x_0 > x^*$ decreases and converges to $x^*$.

From Lemma 2.3, we deduce a simple characterization of the existence of the poverty trap. If there are positive capital stocks $x_0 < x_0'$ such that there is a decreasing optimal path starting from $x_0$ and converging to the origin, and there is an increasing optimal path starting from $x_0'$, the poverty trap exists.

Lemma 2.2 helps us better understand the poverty trap. To see it more clearly, let us consider a configuration where equation (2) has exactly two positive solution $x_s < x^s$. It is clear that $\delta\big(f(x)-x\big)f'(x) < 1$ in the interval $(0, x_s)$ and $\delta\left(f(x)-x\right)f'(x) > 1$ in $(x_s, x^s)$. The two values $x_s$ and $x^s$ are candidates for a steady state.[5] Clearly, if a poverty trap exists, the corresponding critical level should be smaller or equal to $x^s$. Otherwise every optimal path converges to the origin.

Consider the case where $x_s$ is a steady state, which is the case when, for instance, $\varphi$ is a continuous function crossing twice the 45-degrees line at $x_s$ and $x^s$.

Assume that the economy starts at $x_0 < x_s$. Then, every optimal path is decreasing and converging to the origin. Otherwise it should be increasing and converging to $x_s$, a contradiction with part (2) of Lemma 2.2. If the economy starts at $x_0 \in (x_s, x^s)$, always by Lemma 2.2, the optimal capital path increases and therefore converges to $x^s$, the remaining candidate for steady state. If $x_0 > x^s$, clearly, every optimal path converges to $x^s$. Hence, $x_s$ is an unstable steady state and it turns out to be

[5] The number of candidates for a steady state may be infinite, or even uncountable, as in the case where $\delta\big(f(x)-x\big) = 1/f'(x)$ for every $x$ in some interval of $\mathbb{R}_+$. However, the set of parameter values for which this occurs has measure zero.

the critical level of the economy. Conversely, the higher value $x^s$ is a locally stable steady state.

Suppose now that $x_s$ is not a steady state because the optimal transition is discontinuous. Assume the existence of a critical level $x^C < x^s$. Starting from $x_0 > x^C$, the economy converges to $x^s$. It is important to understand what happens at $x^C$. By definition, since $x_s$ is not a steady state, $x^C \notin \varphi\left(x^C\right)$. The property of upper hemi-continuity[6] ensures that $\varphi$ contains at least two different elements $\underline{x}$ and $\overline{x}$ such that $\underline{x} < x^C < \overline{x}$. Therefore, starting from $x^C$, there is at least one optimal path converging to $x^s$, and at least one optimal path converging to the origin. We see that optimal choice "jumps" from below to above the 45-degrees line, when the capital stock crosses $x^C$ from below.

Finally, we cannot exclude a third situation, where the higher candidate $x^s$ represents also a critical level. In this case, there is only one non strictly decreasing path: $(x^s, x^s, x^s, \ldots)$.

How could we know whether the economy converges to the origin or to a non-trivial steady state? Lemma 2.4 provides a partial answer to this question. For ease of notation, let $c_y \equiv f(y) - y$.

Lemma **2.4.** *Assume that equation (2) has exactly two positive solutions $x_s < x^s$, with $\delta\big(f(x) - x\big)f'(x) < 1$ if $x < x_s$ and $\delta\big(f(x) - x\big)f'(x) > 1$ if $x \in (x_s, x^s)$.*

1. *If*

$$\int_0^{x^s} \left[\delta(c_y)f'(y) - 1\right]\left[\frac{u'(c_y)}{1-\delta(c_y)} + \frac{u(c_y)\delta'(c_y)}{\left[1-\delta(c_y)\right]^2}\right]dy$$
$$< \int_{x_0}^{x^s} \delta'(c_y)f'(y)\left[V(y) - \frac{u(c_y)}{1-\delta(c_y)}\right]dy,$$

   *then there exists $\underline{x} < x_s$ such that every optimal path starting from $x_0 < \underline{x}$ converges to the origin.*

2. *If*

$$\int_0^{x^s} \left[\delta(c_y)f'(y) - 1\right]\left[\frac{u'(c_y)}{1-\delta(c_y)} + \frac{u(c_y)\delta'(c_y)}{\left[1-\delta(c_y)\right]^2}\right]dy$$
$$> \int_0^{x^s} \delta'(c_y)f'(y)\left[V(y) - \frac{u(c_y)}{1-\delta(c_y)}\right]dy,$$

[6] Under the upper hemi-continuity property, the graph of $\varphi$, that is the set of $(x, y)$ such that $y \in \varphi(x)$, is closed.

*then there exists $\overline{x} < x^s$ such that every optimal path starting from $x_0 > \overline{x}$ converges to $x^s$.*

Lemma 2.4 provides a sufficient condition for the existence of a convergent trajectory towards the origin, as well as the existence of a non-degenerate path. This is important to establish an example where two types of trajectory coexist or, in other words, where a poverty trap exists. As the conditions are stated not only in terms of economic fundamentals, but also in terms of value function, we prefer to call this result a lemma.

# 3. Example of hyperbolic discounting with consumption

We give an example of high productivity at zero, where there is a poverty trap.

Let $f(x) \equiv Ax^\alpha$, $u(c) \equiv c^\rho$ with $1/2 < \alpha < 1$ and $0 < \rho < 1$. We denote the solution to $f'(x) = 1$ by

$$x^G \equiv (\alpha A)^{\frac{1}{1-\alpha}}.$$

We introduce a hyperbolic discounting:

$$\delta^b(c) \equiv \frac{c}{b+c},$$

where $b > 0$ is a constant. Function $\delta^b(c)$ is clearly concave with respect to $c$.

This form captures the key properties of a discount factor: continuity, increasing monotonicity, boundedness between zero and one, and convergence to one with the necessary asymptotic concavity. Thanks to these properties and the wide range of the parameter $b$, which allows concavity to be modulated, the conclusions we will obtain with this functional form are robust: by continuity in the function space, they will hold in a neighborhood of our hyperbolic specification.

The proof of Lemma 3.1 is conceptually simple, but its computations are cumbersome. The single-peak property simplifies the analysis of the equation (1) since, in general, with an appropriate choice of the constant $b$, this equation has exactly two positive solutions. We can therefore focus on these two candidate steady states.

Observe that equation $\delta^b(f(x)-x)f'(x)=1$ is equivalent to

$$\frac{\alpha A x^{\alpha-1}\big(Ax^\alpha - x\big)}{b + Ax^\alpha - x} = 1.$$

Lemma **3.1.** *Function* $\zeta(x) \equiv \alpha A x^{\alpha-1}\big(Ax^\alpha - x\big) - \big(Ax^\alpha - x\big)$ *is single-peaked. If* $0 < b < b^M \equiv \max_{x\in[0,x^G]} \zeta(x)$*, equation* $\zeta(x) = b$ *has exactly two solutions* $x_s(b) < x^s(b)$*:* $x_s(b)$ *is increasing with respect to* $b$*,* $x^s(b)$ *is decreasing. Moreover,* $\delta^b\big(f(x)-x\big)f'(x) > 1$ *if* $x \in (x_s(b), x^s(b))$ *and* $\delta^b\big(f(x)-x\big)f'(x) < 1$ *if* $x \notin [x_s(b), x^s(b)]$.

From now on, to avoid any confusion, we will call $\mathcal{E}(b)$ the economy corresponding to the discount function $\delta^b$. The corresponding intertemporal utility function will be denoted by $U^b(c_0, c_1, c_2, \ldots)$ and the value function by $V^b$. For each feasible capital sequence $\big(x_t\big)_{t=0}^{\infty}$, let also

$$W^b(x_0, x_1, x_2, \ldots) \equiv U^b(c_0, c_1, c_2, \ldots),$$

with $c_t = f(x_t) - x_{t+1}$ for $t \geq 0$.

Finally, let $\mathcal{C}$ be the set of parameters $b$ such that $\mathcal{E}(b)$ exhibits a poverty trap.

Proposition **3.1.** *There exist two positive values* $\underline{b} < \overline{b}$ *such that* $(\underline{b}, \overline{b}) \subset \mathcal{C}$.

The proof is articulated in two parts. First, we prove that there exists $b \in \big(0, b^M\big)$ such that the economy $\mathcal{E}(b)$ has a poverty trap. Second, we prove that $\mathcal{C}$ contains an open interval of parameters.

## 4. Discounting as a function of future capital

In this section, we reconsider the economy in Erol et al. (2011), where the discount factor depends on future capital. We address the following problem:

$$\max \left[ u(c_0) + \sum_{t=1}^{\infty} u(c_t) \prod_{s=0}^{t-1} \delta(x_{s+1}) \right],$$

subject to $c_t + x_{t+1} \leq f(x_t)$, $c_t \geq 0$, $x_t \geq 0$ for any $t$, given $x_0 \geq 0$.

**Assumption 2.**

1. The production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly concave and strictly increasing with the limit conditions: $f(0) = 0$, $\lim_{x\to\infty} f'(x) < 1$.

2. Function $\delta$ is differentiable with respect to $x$, with $0 < \delta(x) < 1$ for any $x \geq 0$.

3. The utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded from below, continuous, strictly increasing and strictly concave with the Inada condition $u'(0) = \infty$.

As you can see, we impose neither a condition on the minimum value $u(0)$ of utility, nor a condition about decreasing or increasing monotonicity of $\delta$. Proposition 4.1 lists some standard results from the dynamic programming literature.

PROPOSITION **4.1.** *1. For any $x_0 \geq 0$, there exists an optimal path starting from $x_0$. If $x_0 > 0$, then $0 < x_{t+1} < f(x_t)$ for any $t$.*

*2. The value function $V$ is continuous and strictly increasing. It satisfies the Bellman equation*

$$V(x_0) = \max_{0\leq y\leq f(x_0)} \left[u\big(f(x_0) - y\big) + \delta(y)V(y)\right],$$

*for any $x_0 \geq 0$.*

*3. A sequence $\chi \in \Pi(x_0)$ is the optimal solution if and only if it satisfies:*

$$V(x_t) = u\big(f(x_t) - x_{t+1}\big) + \delta(x_{t+1})V(x_{t+1}).$$

The optimal policy correspondence $\varphi : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}_+)$ is defined as follows:

$$\varphi(x_0) = \arg \max_{0\leq y\leq f(x_0)} \left[u\left(f(x_0) - y\right) + \delta(y)V(y)\right].$$

Proposition 4.2 presents fundamental properties of the optimal policy correspondence and grounds some important dynamical results such as the monotonicity of the optimal paths.

PROPOSITION **4.2.** *1. $\varphi$ is closed and upper hemi-continuous.*

*2. A sequence $\chi \in \Pi(x_0)$ is optimal if and only if $x_{t+1} \in \varphi(x_t)$ for any $t \geq 0$.*

*3. Assume that $y \in \varphi(x), y' \in \varphi(x')$ with $x < x'$. Then, $y \leq y'$.*

Proposition 4.3 presents the Euler equations and shows that, from date $t = 1$ on, every optimal path is unique. This finding echoes a well-known result in Dechert and Nishimura (1983).

**Proposition 4.3.** *Let $\left(x_t\right)_{t=0}^{\infty}$ be an optimal path starting from $x_0 > 0$.*

1. *The Euler equation is given by:*

$$u'\big(f(x_t)-x_{t+1}\big) = \delta(x_{t+1})u'\big(f(x_{t+1})-x_{t+2}\big)f'(x_{t+1})+\delta'(x_{t+1})V(x_{t+1}). \quad (3)$$

2. *For $t \geq 1$, the optimal policy correspondence is a function ($\varphi(x_t)$ is a singleton).*

A direct consequence is that, if $x^*$ is a steady state, then $\varphi(x^*) = \{x^*\}$. Since the right-hand side is strictly increasing with respect to $x_{t+2}$, we obtain $\varphi(x_t) = \{x_{t+1}\}$ for any $t \geq 0$. Applying the same arguments as in the proof of Theorem 6 and Corollary 4 in Dechert and Nishimura (1983), we prove that $V(x_t)$ is differentiable at $x_t$ for $t \geq 1$. The Monotone Differentiation Theorem ensures that the value function $V(x)$ is differentiable almost everywhere.

Proposition 4.4 states a necessary equation for the steady state. Importantly, if the economy starts from a non-steady state, then the optimal path of capital is strictly monotonic. Therefore, the economy converges either to a positive steady state or to the origin.

It is worth noting that, if $\delta$ is increasing and $u(c) > 0$ for any $c > 0$, the steady state is larger than the Modified Golden Rule of the Ramsey-Cass-Koopmans model. The same happens in Erol et al. (2011). Conversely, if $\delta$ is decreasing, the economy experiences a smaller steady state.

**Proposition 4.4.** *Assume that $x^* > 0$ is a steady state.*

1. *$x^*$ is a solution to the following equation:*

$$\delta(x^*)f'(x^*) = 1 - \frac{\delta'(x^*)u\big(f(x^*)-x^*\big)}{\big[1-\delta(x^*)\big]u'\big(f(x^*)-x^*\big)}. \quad (4)$$

2. *If $\left(x_t\right)_{t=0}^{\infty}$ is an optimal path starting from $x_0 \neq x^*$, then $x_t \neq x^*$ for any $t \geq 0$. Moreover, this sequence is strictly monotonic.*

Proposition 4.5 echoes a well-known result holding when the discount factor $\delta$ is constant. Interestingly, our new approach leads to results without any convexity assumption.

Proposition **4.5.** 1. *Assume the existence of* $\hat{x} > 0$ *such that, for any* $y < \hat{x}$,

$$\delta(y)f'(y) > 1 - \frac{\delta'(y)u\big(f(y)-y\big)}{\big[1-\delta(y)\big]u'\big(f(y)-y\big)}.$$

*Then, every optimal path converges to a strictly positive steady state.*

2. *Consider* $x_0 > 0$ *and assume that, for any* $y > x_0$,

$$\delta(y)f'(y) < 1 - \frac{\delta'(y)u\big(f(y)-y\big)}{\big[1-\delta(y)\big]u'\big(f(y)-y\big)}.$$

*Then, every optimal path starting from* $x_0$ *is strictly decreasing.*

Corollary 4.1 is a direct consequence of Proposition 4.5 and generalizes a classical result of the Ramsey-Cass-Koopmans model: the optimal sequence is increasing when the productivity is high; decreasing when the productivity is low.

Corollary 4.1. 1. *Assume that, for* $y$ *sufficiently small,*

$$\delta(y)f'(y) > 1 - \frac{\delta'(y)u\big(f(y)-y\big)}{\big[1-\delta(y)\big]u'\big(f(y)-y\big)}.$$

*Then, every optimal path converges to a positive steady state.*

2. *Assume that, for* $0 < y < x^M$,

$$\delta(y)f'(y) < 1 - \frac{\delta'(y)u\big(f(y)-y\big)}{\big[1-\delta(y)\big]u'\big(f(y)-y\big)}.$$

*Then, every optimal path converges to the origin.*

Let us show why Corollary 4.1 implies Propositions 9 and 10 in Erol et al. (2011). In their article, $u(c) > 0$ for $c > 0$ and $\delta'(y) > 0$. Let us define

$$\varepsilon \equiv \sup_{y\in(0,x^M)} \frac{\delta'(y)u\big(f(y)-y\big)}{\big[1-\delta(y)\big]u'\big(f(y)-y\big)}.$$

Note that this value is smaller than the corresponding one in Erol et al. (2011). If $\sup_{y\geq 0}\delta(y)f'(0) < 1-\varepsilon$, then, for any $y \geq 0$,

$$\delta(y)f'(y) \leq \sup_{y\geq 0}\delta(y)f'(0) < 1-\varepsilon \leq 1 - \frac{\delta'(y)u\big(f(y)-y\big)}{\big[1-\delta(y)\big]u'\big(f(y)-y\big)},$$

and any optimal path converges to the origin. This is precisely the conclusions of Proposition 9.

Focus now on Proposition 10 in Erol et al. (2011). The condition $\inf_{y \geq 0} \delta(y) f'(0) > 1$ implies that, for $y$ small enough,

$$\delta(y) f'(y) > 1 - \frac{\delta'(y) u\big(f(y) - y\big)}{\big[1 - \delta(y)\big] u'\big(f(y) - y\big)},$$

and that every optimal path converges to a positive steady state.

Let us now turn to the main result. Proposition 4.6 states sufficient conditions for the existence of decreasing or increasing optimal paths. While traditionally, these properties are easy to derive by exploiting the convexity properties, in our paper, we are forced to develop new techniques to overcome potential non-convexities.

PROPOSITION **4.6.** *Fix $x_0$ that is not a steady state.*

1. *Assume that, for any $x^* > x_0$ solution to (4),*

$$\int_{x_0}^{x^*} \left( \big[\delta(y) f'(y) - 1\big] \frac{u'\big(f(y) - y\big)}{1 - \delta(y)} + \frac{\delta'(y) u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]^2} \right) dy < 0,$$

   *Then, every optimal path starting from $x_0$ is decreasing.*

2. *Assume that, for any $x^* < x_0$ solution to (4) or $x^* = 0$,*

$$\int_{x_0}^{x^*} \left( \big[\delta(y) f'(y) - 1\big] \frac{u'\big(f(y) - y\big)}{1 - \delta(y)} + \frac{\delta'(y) u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]^2} \right) dy > 0,$$

   *Then, every optimal path starting from $x_0$ is increasing.*

To illustrate Proposition 4.6, Corollary 4.2 considers the properties of a normal context where there are exactly two positive steady-state candidates. The Corollary provides a condition under which the larger candidate is a steady state, and a condition under which there is an optimal path converging to the origin.

COROLLARY 4.2. *Assume that equation (4) has exactly two positive solutions $x_s < x^s$ with the following property:*

$$\delta(y) f'(y) < 1 - \frac{\delta'(y) u\big(f(y) - y\big)}{\big[1 - \delta(y)\big] u'\big(f(y) - y\big)}, \tag{5}$$

*if $0 < y < x_s$ or $y > x^s$, and*

$$\delta(y) f'(y) > 1 - \frac{\delta'(y) u\big(f(y) - y\big)}{\big[1 - \delta(y)\big] u'\big(f(y) - y\big)}, \tag{6}$$

*if $x_s < y < x^s$.*

1. *If*

$$\int_0^{x^s} \left( \big[\delta(y) f'(y) - 1\big] \frac{u'\big(f(y) - y\big)}{1 - \delta(y)} + \frac{\delta'(y) u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]^2} \right) dy > 0,$$

*then $x^s$ is a steady state and there exists $\overline{x} < x^s$ such that every optimal path starting from $x_0 > \overline{x}$ converges to $x^s$.*

2. *If*

$$\int_0^{x^s} \left( \big[\delta(y) f'(y) - 1\big] \frac{u'\big(f(y) - y\big)}{1 - \delta(y)} + \frac{\delta'(y) u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]^2} \right) dy < 0,$$

*then there exists $\underline{x} > 0$ such that every optimal path starting from $x_0 < \underline{x}$ converges to the origin.*

Let

$$\eta(y) \equiv \big[\delta(y) f'(y) - 1\big] \frac{u'\big(f(y) - y\big)}{1 - \delta(y)} + \frac{\delta'(y) u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]^2}.$$

Conditions (5) and $\eta(y) > 0$ are equivalent, as are conditions (6) and $\eta(y) < 0$. The function $\eta(y)$ is negative in $(0, x_s)$ and positive in $(x_s, x^s)$. If the integral of this function in the "positive region" dominates that in the "negative one", we are assured that $x^s$ is a locally stable steady state. Conversely, starting with low capital stock, the economy converges to the origin.

# 5. Example of hyperbolic discounting with capital

Let us define $f(x) \equiv Ax^\alpha$ and $u(c) \equiv c^\rho$ with $\alpha, \rho \in (0,1)$. Reconsider a hyperbolic discounting:

$$\delta^b(x) = \frac{x}{b + x},$$

where $b > 0$ is constant.

As seen above, because of the properties of a discount factor and scalability of the parameter $b$, this functional form leads to robust results and demonstrates that the poverty trap is a pervasive feature under endogenous discounting.

Let $\mathcal{E}(b)$ be the economy associated to the discount function $\delta^b$. The following equation is key to prove the existence of a poverty trap:

$$\delta^b(x)f'(x) = 1 - \frac{\delta^{b\prime}(x)u\big(f(x)-x\big)}{\big[1-\delta^b(x)\big]u'\big(f(x)-x\big)}.$$

Let $\mathcal{C}$ be the set of parameters $b$ such that $\mathcal{E}(b)$ exhibits a poverty trap.

PROPOSITION **5.1.** *There exist two positive values* $\underline{b} < \overline{b}$ *such that* $(\underline{b}, \overline{b}) \subset \mathcal{C}$.

The arguments of the proof are similar to those of Section 3, but now we apply Lemma 4.2. For appropriate values of $b > 0$, equation (4) has two positive solutions $x_s(b) < x^s(b)$. By introducing the new auxiliary function:

$$\Phi(b) \equiv \int_0^{x^s(b)} \left( \big[\delta^b(y)f'(y) - 1\big] \frac{u'\big(f(y)-y\big)}{1-\delta^b(y)} + \frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{\big[1-\delta^b(y)\big]^2} \right) dy,$$

and proving the existence of $b$ such that $\Phi(b) > 0$, we can show that the economy has an optimal path converging to $x^s(b)$. Similarly, there exists $b$ such that $\Phi(b) < 0$ and the economy has an optimal path converging to the origin. Repeating the arguments of Section 3, we prove that there exists a value between these two critical values such that the corresponding economy experiences a poverty trap.

## 6. Conclusion

In this paper, we have addressed the issue of the monotonicity of the optimal path and of the existence of a poverty trap when discounting is endogenous and depends on current consumption or future capital.

In order to obtain this results, preliminarily, we have proven the existence of an optimal path and shown that the set of feasible sequences is compact and the intertemporal utility is continuous in the product topology. Then, we have proven that, if the utility function is bounded from below and discounting is bounded, then the value function is finite, non-negative and continuous. Interestingly, it is also the unique solution of the Bellman functional equation and is differentiable almost everywhere.

We have also provided a characterization of the policy function, showing that the continuity of the value function implies the non-emptiness and closedness of the optimal correspondence, and that the optimal policy correspondence is monotonic. Even if the policy function may not be unique, it is a singleton for any $x_t$ for $t \geq 1$. We have remarked that the differentiability of the Bellman equation almost everywhere rests on the fact that the policy correspondence is singleton almost everywhere from $t = 1$ on.

After obtaining these technical but important results, we have shown the monotonicity of the optimal policy correspondence, which ensures that, if an optimal path does not start from a steady state, it must be strictly monotonic. The existence of a poverty trap is linked to the existence of a critical level of capital such that, if the initial condition is below (above), the optimal path converges towards the origin (towards a positive steady state). The monotonicity property of the optimal policy correspondence is key to guarantee that, if a critical level exists, it is unique.

However, since it is far from easy to give general conditions for the existence of a critical level under endogenous discounting, we have complemented our theoretical analysis with two examples corresponding to the cases where discounting depends, respectively, on current consumption or future capital and proven explicitly that, in both cases, a poverty trap exists for a nonzero-measure set of values in the parameter space. Because of the robustness of these examples, it appears that the poverty trap is far from being a rare event in economies with endogenous discounting.

We leave for future research the examination of the case where discounting depends on current capital, as already done by Borissov et al. (2025) but in a different context (intertemporal utility as a weighted average of period utilities), while observing that, according to their results, the dynamic properties remain qualitatively similar to those of the case where discounting depends on current consumption.

# 7. Appendix

## 7.1 Proof of Lemma 2.1

See the proof of Lemma 2 in Le Van and Morhaim (2002). QED

## 7.2 Proof of Proposition 2.1

Use the same arguments as in the proof of Proposition 3.4.1 in Le Van and Dana (2003). QED

## 7.3 Proof of Proposition 2.2

Parts $(i)$, $(ii)$ and $(iv)$ come directly from Proposition 2.1. The Maximum Theorem implies $(iii)$.

Let us prove part $(v)$: for any $x < x'$ and $y \in \varphi(x)$, $y' \in \varphi(x')$, we have $y \leq y'$.

Assume the contrary, that is $y > y'$. We obtain

$$u\big(f(x) - y\big) + \delta\,(f(x) - y)\,V(y) \geq u\big(f(x) - y'\big) + \delta\,(f(x) - y')\,V(y'),$$
$$u\big(f(x') - y'\big) + \delta\,(f(x') - y')\,V(y') \geq u\big(f(x') - y\big) + \delta\,(f(x') - y)\,V(y).$$

Summing these inequalities, we get

$$u\big(f(x) - y\big) + u\big(f(x') - y'\big) + \delta\,(f(x) - y)\,V(y) + \delta\,(f(x') - y')\,V(y')$$
$$\geq u\big(f(x) - y'\big) + u\big(f(x') - y\big) + \delta\,(f(x) - y')\,V(y') + \delta\,(f(x') - y)\,V(y)$$

Thus, using the same argument as in Dechert and Nishimura (1983),[7] with $x < x'$ and $y > y'$, we find

$$\delta\big(f(x) - y\big)V(y) + \delta\big(f(x') - y'\big)V(y') > \delta\big(f(x) - y'\big)V(y') + \delta\big(f(x') - y\big)V(y).$$

This inequality implies

$$V(y')\left[\delta\big(f(x') - y'\big) - \delta\big(f(x) - y'\big)\right] > V(y)\left[\delta\big(f(x') - y\big) - \delta\big(f(x) - y\big)\right]$$

or, equivalently,

$$V(y') \int_x^{x'} \delta'\big(f(z) - y'\big) f'(z) dz > V(y) \int_x^{x'} \delta'\big(f(z) - y\big) f'(z) dz,$$

leading to a contradiction with the concavity property of $\delta$. QED

[7] The argument is based on the strict supermodularity property: for $x < x'$ and $y < y'$ such that $0 < y < f(x)$ and $0 < y' < f(x')$, we have $u\big(f(x)-y\big)+u\big(f(x')-y'\big) > u\big(f(x)-y'\big)+u\big(f(x')-y\big)$. See Amir (1996, 2005) for the properties of supermodular functions.

## 7.4 Proof of Proposition 2.3

Part (1). The partial derivative of $U(c_0, c_1, c_2, \dots)$ with respect to $x_{t+1}$ is equal to zero because $0 < x_{t+1} < f(x_t)$ for any $t \geq 0$. Computing the derivative with respect to $x_{t+1}$, we obtain:

$$\begin{aligned}
u'\big(f(x_t) - x_{t+1}\big) &= -\delta'\big(f(x_t) - x_{t+1}\big)u\big(f(x_{t+1}) - x_{t+2}\big) \\
&\quad + \delta\big(f(x_t) - x_{t+1}\big)u'\big(f(x_{t+1}) - x_{t+2}\big)f'(x_{t+1}) \\
&\quad - \delta'\big(f(x_t) - x_{t+1}\big)\delta\big(f(x_{t+1}) - x_{t+2}\big)V(x_{t+2}) \\
&\quad + \delta\big(f(x_t) - x_{t+1}\big)\delta'\big(f(x_{t+1}) - x_{t+2}\big)f'(x_{t+1})V(x_{t+2}) \\
&= \delta\big(f(x_t) - x_{t+1}\big)u'\big(f(x_{t+1}) - x_{t+2}\big)f'(x_{t+1}) - \delta'\big(f(x_t) - x_{t+1}\big)V(x_{t+1}) \\
&\quad + \delta\big(f(x_t) - x_{t+1}\big)f'(x_{t+1})\delta'\big(f(x_{t+1}) - x_{t+2}\big)V(x_{t+2}).
\end{aligned}$$

Part (2). The right-hand side of the Euler equation is strictly increasing with respect to $x_{t+2}$. Hence, $\varphi(x_{t+1})$ is a singleton for any $t \geq 1$.

Using the same arguments as in the proofs of Theorem 6 and Corollary 4 in Dechert and Nishimura (1983), the value function $V$ is differentiable at $x_{t+1}$, for any $t \geq 0$. QED

## 7.5 Proof of Proposition 2.4

$(i)$ Assume that $x^* > 0$ is a steady state. Let $c^* = f(x^*) - x^*$. Observe that

$$V(x^*) = \frac{u(c^*)}{1 - \delta(c^*)}.$$

Using Euler equation, we find

$$\begin{aligned}
u'(c^*) &= -\delta'(c^*)u(c^*) + \delta(c^*)u'(c^*)f'(x^*) \\
&\quad - \delta'(c^*)\delta(c^*)\frac{u(c^*)}{1 - \delta(c^*)} + \delta(c^*)\delta'(c^*)f'(x^*)\frac{u(c^*)}{1 - \delta(c^*)}.
\end{aligned}$$

This implies

$$u'(c^*)\big[1 - \delta(c^*)f'(x^*)\big] = -\frac{\delta'(c^*)u(c^*)}{1 - \delta(c^*)}\big[1 - \delta(c^*)f'(x^*)\big].$$

Recalling that $u(c^*) > 0$, we get $\delta\big(f(x^*) - x^*\big)f'(x^*) = 1$.

$(ii)$ Assume the contrary. Then, there exists some $t \geq 0$ such that $x_t \neq x^*$, and $x_{t+1} = x^*$. The sequence $(x_t, x^*, x^*, \dots)$ is an optimal path starting from $x_t$.

Consider the case $x_t < x^*$. Recalling that $\delta\big(f(x^*) - x^*\big)f'(x^*) = 1$ and $u(c) > 0$ for any $c > 0$, and using Euler equation, we obtain:

$$\begin{aligned}
u'\big(f(x_t) - x^*\big) &= -\delta'\big(f(x_t) - x^*\big)u\big(f(x^*) - x^*\big) + \delta\big(f(x_t) - x^*\big)u'\big(f(x^*) - x^*\big)f'(x^*) \\
&\quad - \delta'\big(f(x_t) - x^*\big)\delta\big(f(x^*) - x^*\big)V(x^*) + \delta\big(f(x_t) - x^*\big)\delta'\big(f(x^*) - x^*\big)f'(x^*)V(x^*) \\
&< -\delta'\big(f(x^*) - x^*\big)u\big(f(x^*) - x^*\big) + \delta\big(f(x^*) - x^*\big)u'\big(f(x^*) - x^*\big)f'(x^*) \\
&\quad - \delta'\big(f(x^*) - x^*\big)\delta\big(f(x^*) - x^*\big)\frac{u\big(f(x^*) - x^*\big)}{1 - \delta\big(f(x^*) - x^*\big)} \\
&\quad + \delta\big(f(x^*) - x^*\big)\delta'\big(f(x^*) - x^*\big)f'(x^*)\frac{u\big(f(x^*) - x^*\big)}{1 - \delta\big(f(x^*) - x^*\big)} \\
&= u'\big(f(x^*) - x^*\big),
\end{aligned}$$

a contradiction.

In the case $x_t > x^*$, applying the same arguments, but with reversed inequalities, also leads to a contradiction. QED

## 7.6 Proof of Proposition 2.5

Part (1). Since $x^*$ is a steady state, the sequence $(x^*, x^*, x^*, \ldots)$ is an optimal path starting from $x^*$. By Proposition 2.3, $\varphi(x^*)$ is a singleton.

Part (2). Consider $x_0 > 0$ that is not a steady state. By Proposition 2.4, there is no $t$ such that $x_t$ is a steady state. This implies $x_{t+1} \neq x_t$ for every $t$, otherwise $x_t$ will be a steady state. If $x_0 < x_1$, since $x_1 \in \varphi(x_0)$ and $x_2 \in \varphi(x_1)$, Proposition 2.2 implies that $x_1 < x_2$. By induction, we get $x_t < x_{t+1}$ for any $t$.

If $x_0 > x_1$, using the same arguments, we find that the path $(x_t)_{t=0}^{\infty}$ is strictly decreasing.

The monotonicity of $(x_t)_{t=0}^{\infty}$ entails that this sequence has a limit. The hemi-continuity property of the optimal policy correspondence ensures that this limit is a steady state. QED

## 7.7 Proof of Lemma 2.2

Part (1). Assume the contrary. Then there exists an open interval $I \subset \big(0, x^M\big)$ such that for any $x \in I$, $\delta\big(f(x) - x\big)f'(x) > 1$, and a strictly decreasing optimal path $(x_t)_{t\geq 0}$ such that $x_t \in I$ for every $t \geq T$ from some $T$ large enough.

Since $(x_T, x_T, \ldots)$ is feasible,

$$V(x_T) \geq \frac{u\big(f(x_T) - x_T\big)}{1 - \delta\big(f(x_T) - x_T\big)}.$$

The following inequality holds for any $t$:

$$\frac{u\big(f(x_t) - x_t\big)}{1 - \delta\big(f(x_t) - x_t\big)} \geq u\big(f(x_t) - x_{t+1}\big) + \delta\big(f(x_t) - x_{t+1}\big) \frac{u\big(f(x_{t+1}) - x_{t+1}\big)}{1 - \delta\big(f(x_{t+1}) - x_{t+1}\big)}.$$

Indeed, considering the following function

$$\Psi(y) \equiv u\big(f(x_t) - y\big) + \delta\big(f(x_t) - y\big) \frac{u\big(f(y) - y\big)}{1 - \delta\big(f(y) - y\big)},$$

we can prove that $\Psi'(y) > 0$ in the interval $(x_{t+1}, x_t)$. For the sake of simplicity, let $c_y = f(y) - y$, with $dc_y/dy = f'(y) - 1$.

Recalling that $u(c_y) > 0$ and $f'(y) > 1$, for $y \in (x_{t+1}, x_t)$, we have

$$\begin{aligned}
\Psi'(y) &\equiv -u'\big(f(x_t) - y\big) - \delta'\big(f(x_t) - y\big) \frac{u(c_y)}{1 - \delta(c_y)} \\
&\quad + \delta\big(f(x_t) - y\big) \frac{\big[1 - \delta(c_y)\big] u'(c_y) + u(c_y)\delta'(c_y)}{\big[1 - \delta(c_{y)}\big]^2} \big[f'(y) - 1\big] \\
&> -u'(c_y) - \delta'(c_y) \frac{u(c_y)}{1 - \delta(c_y)} + \delta(c_y) \frac{\big[1 - \delta(c_y)\big] u'(c_y) + u(c_y)\delta'(c_y)}{\big[1 - \delta(c_{y)}\big]^2} \big[f'(y) - 1\big] \\
&= u'(c_y) \left[-1 + \frac{\delta(c_y)\big[f'(y) - 1\big]}{1 - \delta(c_y)}\right] + \frac{\delta'(c_y) u(c_y)}{1 - \delta(c_y)} \left[-1 + \frac{\delta(c_y)\big[f'(y) - 1\big]}{1 - \delta(c_y)}\right] \\
&= \frac{\delta(c_y) f'(y) - 1}{1 - \delta(c_y)} \left[u'(c_y) + \frac{\delta'(c_y) u(c_y)}{1 - \delta(c_y)}\right] > 0.
\end{aligned}$$

The function $\Psi$ is strictly increasing in the interval $(x_{t+1}, x_t)$, which implies that $\Psi(x_t) > \Psi(x_{t+1})$. Thus, the inequality holds for $t \geq T$, and

$$\begin{aligned}
\frac{u\big(f(x_T) - x_T\big)}{1 - \delta\big(f(x_T) - x_T\big)} &> u\big(f(x_T) - x_{T+1}\big) + \delta\big(f(x_T) - x_{T+1}\big) \frac{u\big(f(x_{T+1}) - x_{T+1}\big)}{1 - \delta\big(f(x_{T+1}) - x_{T+1}\big)} \\
&> u\big(f(x_T) - x_{T+1}\big) + \delta\big(f(x_T) - x_{T+1}\big) u\big(f(x_{T+1} - x_{T+2}\big) \\
&\quad + \delta\big(f(x_T) - x_{T+1}\big) \delta\big(f(x_{T+1}) - x_{T+2}\big) \frac{u\big(f(x_{T+2}) - x_{T+2}\big)}{1 - \delta\big(f(x_{T+2}) - x_{T+2}\big)} \\
&> \ldots \geq V(x_T),
\end{aligned}$$

a contradiction.

Part (2). We arrive at a contradiction, following exactly the same arguments as before, but with inverse inequalities and a decreasing function $\Psi$ in the interval $(x_t, x_{t+1})$ (under the hypothesis that the optimal path is strictly increasing). QED

## 7.8 Proof of Proposition 2.6

Part (1). Since $\liminf_{x\to 0}\delta\big(f(x)-x\big)f'(x) > 1$, there exists $z > 0$ small enough such that $\delta\big(f(x)-x\big)f'(x) > 1$ for any $x < z$. The convergence of $\big(x_t\big)_{t\geq 0}$ to the origin implies the existence of a period $T$ such that $x_t \in (0, z)$ for any $t \geq T$, a contradiction with Lemma 2.2.

Part (2). Assume the contrary: there exists an increasing optimal path $\big(x_t\big)_{t=0}^{\infty}$ starting from $x_0 > \hat{x}$. Since $\liminf_{x\to 0}\delta\big(f(x)-x\big)f'(x) < 1$ in this interval, this path contains a non-steady state element. Proposition 2.5 implies that it is strictly decreasing, and we come to a contradiction with Lemma 2.2. QED

## 7.9 Proof of Lemma 2.3

Clearly, if the poverty trap $x^C$ exists, there are two sequences that satisfy the properties mentioned in the statement of this lemma. We can use the definition of $x^C$ and the monotonicity property of optimal paths.

Now, let's assume the existence of an optimal path $\big(\underline{x}_t\big)_{t=0}^{\infty}$ converging to zero, and a sequence $\big(\overline{x}_t\big)_{t=0}^{\infty}$ converging to a positive steady state. Let $x^C$ be the infimum of the set of capital levels $x_0$ such that there exists an optimal path starting from $x_0$, bounded away from zero.

The existence of a path converging to zero implies $x^C > 0$. Indeed, if $x^C = 0$, by the definition of $x^C$, any optimal path starting from $x_0 > 0$ should be bounded away from zero, a contradiction.

Since we have chosen $x^C$ in this way, if $x_0 < x^C$, any optimal path starting from $x_0$ will converge to the origin. On the other hand, if $x_0 > x^C$, Proposition 2.2 guarantees that any optimal path starting from $x_0$ will be bounded away from zero. QED

## 7.10 Proof of Lemma 2.4

Before proving parts (1) and (2), let us observe that, starting from $x_0 \neq x_s$, the economy never converges to $x_s$. Let us assume the contrary: $x_s$ is a steady state.

Consider the case $x_0 < x_s$. This means that the optimal path $\big(x_t\big)_{t=0}^{\infty}$ strictly increasese towards $x_s$. Recalling that, in the interval $(0, x_s)$, $\delta\big(f(x)-x\big)f'(x) < 1$ and using the same arguments as in the proof of Proposition 2.6, we come to a contradiction.

In the case $x_s < x_0 < x^s$, a similar argument leads also to a contradiction. Hence, starting from a non-steady state, the economy either converges to $x^s$, or to the origin.

Part (1). Fix $\underline{x} < x_s$ close enough to the origin such that

$$\int_{x_0}^{x^s} \big[\delta(c_y)f'(y){-}1\big] \left[\frac{u'(c_y)}{1-\delta(c_y)} + \frac{u(c_y)\delta'(c_y)}{\big[1-\delta(c_y)\big]^2}\right] dy < \int_{x_0}^{x^s} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1-\delta(c_y)}\right] dy,$$

for every $x_0 < \underline{x}$.

Let $\big(x_t\big)_{t=0}^{\infty}$ be the optimal path starting from $x_0 < \underline{x}$. Assume that it is increasing and converging to a positive steady state. Clearly, this steady state is $x^s$. For any $t$,

$$\begin{aligned}
V(x_{t+1}) - V(x_t) &\geq u\big(f(x_{t+1}) - x_{t+1}\big) + \delta\big(f(x_{t+1}) - x_{t+1}\big)V(x_{t+1}) \\
&\quad - u\big(f(x_t) - x_{t+1}\big) - \delta\big(f(x_t) - x_{t+1}\big)V(x_{t+1}) \\
&= \int_{x_t}^{x_{t+1}} u'\big(f(y) - x_{t+1}\big)f'(y)dy + V(x_{t+1}) \int_{x_t}^{x_{t+1}} \delta'\big(f(y) - x_{t+1}\big)f'(y)dy \\
&\geq \int_{x_t}^{x_{t+1}} u'\big(f(y) - y\big)f'(y)dy + \int_{x_t}^{x_{t+1}} \delta'\big(f(y) - y\big)f'(y)V(y)dy \\
&= \int_{x_t}^{x_{t+1}} u'\big(f(y) - y\big)f'(y)dy + \int_{x_t}^{x_{t+1}} \delta'\big(f(y) - y\big)f'(y)\frac{u(c_y)}{1-\delta(c_y)}dy \\
&\quad + \int_{x_t}^{x_{t+1}} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1-\delta(c_y)}\right] dy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V(x^s) - V(x_0) &= \sum_{t=0}^{\infty}\big[V(x_{t+1}) - V(x_t)\big] \\
&\geq \int_{x_0}^{x^s} u'\big(f(y) - y\big)f'(y)dy + \int_{x_0}^{x^s} \delta'\big(f(y) - y\big)f'(y)\frac{u(c_y)}{1-\delta(c_y)}dy. \\
&\quad + \int_{x_0}^{x^s} \delta'(c_y)f'(y) \left[V(y) - \frac{u(c_y)}{1-\delta(c_y)}\right] dy
\end{aligned}$$

Consider now the left-hand side:

$$\begin{aligned}
V(x^s) - V(x_0) &\leq \frac{u\big(f(x^s) - x^s\big)}{1-\delta\big(f(x^s) - x^s\big)} - \frac{u\big(f(x_0) - x_0\big)}{1-\delta\big(f(x_0) - x_0\big)} \\
&= \int_{x_0}^{x^s} \frac{\big[1-\delta(c_y)\big]u'(c_y) + u(c_y)\delta'(c_y)}{\big[1-\delta(c_y)\big]^2}\big[f'(y) - 1\big]dy
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{x_0}^{x^s} \frac{\big[1-\delta(c_y)\big]u'(c_y)+u(c_y)\delta'(c_y)}{\big[1-\delta(c_y)\big]^2}\big[f'(y)-1\big]dy \\
&\geq \int_{x_0}^{x^s} u'\big(f(y)-y\big)f'(y)dy + \int_{x_0}^{x^s} \delta'\big(f(y)-y\big)f'(y)\frac{u(c_y)}{1-\delta(c_y)}dy \\
&+ \int_{x_0}^{x^s} \delta'(c_y)f'(y)\left[V(y)-\frac{u(c_y)}{1-\delta(c_y)}\right]dy.
\end{aligned}$$

This implies

$$\int_{x_0}^{x^s}\big[\delta(c_y)f'(y)-1\big]\left[\frac{u'(c_y)}{1-\delta(c_y)}+\frac{u(c_y)\delta'(c_y)}{\big[1-\delta(c_y)\big]^2}\right]dy \geq \int_{x_0}^{x^s}\delta'(c_y)f'(y)\left[V(y)-\frac{u(c_y)}{1-\delta(c_y)}\right]dy,$$

a contradiction.

Part (2). Fix $\overline{x}$ close enough to $x^s$ such that

$$\int_{0}^{x_0}\big[\delta(c_y)f'(y)-1\big]\left[\frac{u'(c_y)}{1-\delta(c_y)}+\frac{u(c_y)\delta'(c_y)}{\big[1-\delta(c_y)\big]^2}\right]dy \geq \int_{0}^{x_0}\delta'(c_y)f'(y)\left[V(y)-\frac{u(c_y)}{1-\delta(c_y)}\right]dy,$$

for any $x_0 > \overline{x}$.

Let $\big(x_t\big)_{t=0}^{\infty}$ be the optimal path starting from $x_0 < \overline{x}$. Assume that it is decreasing and converging to the origin. For any $t$,

$$\begin{aligned}
V(x_t)-V(x_{t+1}) &\leq u\big(f(x_t)-x_{t+1}\big)+\delta\big(f(x_t)-x_{t+1}\big)V(x_{t+1}) \\
&\quad - u\big(f(x_{t+1})-x_{t+1}\big)-\delta\big(f(x_{t+1})-x_{t+1}\big)V(x_{t+1}) \\
&= \int_{x_{t+1}}^{x_t} u'\big(f(y)-x_{t+1}\big)f'(y)dy + V(x_{t+1})\int_{x_{t+1}}^{x_t}\delta'\big(f(y)-x_{t+1}\big)f'(y)dy \\
&\leq \int_{x_{t+1}}^{x_t} u'\big(f(y)-y\big)f'(y)dy + \int_{x_{t+1}}^{x_t}\delta'\big(f(y)-y\big)f'(y)V(y)dy \\
&= \int_{x_{t+1}}^{x_t} u'\big(f(y)-y\big)f'(y)dy + \int_{x_{t+1}}^{x_t}\delta'\big(f(y)-y\big)f'(y)\frac{u(c_y)}{1-\delta(c_y)}dy \\
&\quad + \int_{x_{t+1}}^{x_t}\delta'(c_y)f'(y)\left[V(y)-\frac{u(c_y)}{1-\delta(c_y)}\right]dy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V(x_0)-V(0) &= \sum_{t=0}^{\infty}\big[V(x_t)-V(x_{t+1})\big] \\
&\leq \int_0^{x_0} u'\big(f(y)-y\big)f'(y)dy + \int_0^{x_0}\delta'\big(f(y)-y\big)f'(y)\frac{u(c_y)}{1-\delta(c_y)}dy. \\
&\quad + \int_0^{x_0}\delta'(c_y)f'(y)\left[V(y)-\frac{u(c_y)}{1-\delta(c_y)}\right]dy.
\end{aligned}$$

Consider now the left-hand side:

$$V(x_0) - V(0) \geq \frac{u\big(f(x_0) - x_0\big)}{1 - \delta\big(f(x_0) - x_0\big)} - \frac{u\big(f(0) - 0\big)}{1 - \delta\big(f(0) - 0\big)}$$
$$= \int_0^{x_0} \frac{\big[1 - \delta(c_y)\big]u'(c_y) + u(c_y)\delta'(c_y)}{\big[1 - \delta(c_y)\big]^2}\big[f'(y) - 1\big]dy.$$

Hence,

$$\int_0^{x_0} \frac{\big[1 - \delta(c_y)\big]u'(c_y) + u(c_y)\delta'(c_y)}{\big[1 - \delta(c_y)\big]^2}\big[f'(y) - 1\big]dy$$
$$\leq \int_0^{x_0} u'\big(f(y) - y\big)f'(y)dy + \int_0^{x_0} \delta'\big(f(y) - y\big)f'(y)\frac{u(c_y)}{1 - \delta(c_y)}dy$$
$$+ \int_0^{x_0} \delta'(c_y)f'(y)\left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)}\right]dy.$$

Therefore,

$$\int_0^{x_0} \big[\delta(c_y)f'(y) - 1\big]\left[\frac{u'(c_y)}{1 - \delta(c_y)} + \frac{u(c_y)\delta'(c_y)}{\big[1 - \delta(c_y)\big]^2}\right]dy \leq \int_0^{x_0} \delta'(c_y)f'(y)\left[V(y) - \frac{u(c_y)}{1 - \delta(c_y)}\right]dy,$$

a contradiction. QED

## 7.11 Proof of Lemma 3.1

We have

$$\delta^b\big(f(x) - x\big)f'(x) = \frac{\alpha A x^{\alpha-1}\big(Ax^\alpha - x\big)}{b + Ax^\alpha - x} = \frac{\alpha A x^{2\alpha-1}\big(A - x^{1-\alpha}\big)}{b + Ax^\alpha - x}.$$

Fixing $\alpha > 1/2$, we obtain $\lim_{x\to 0} \delta^b\big(f(x) - x\big)f'(x) = 0$.

Equation $\delta^b\left(f(x) - x\right)f'(x) = 1$ is equivalent to

$$\frac{\alpha A x^{\alpha-1}\big(Ax^\alpha - x\big)}{b + Ax^\alpha - x} = 1.$$

Now, let's study the function $\zeta(x) = \alpha A x^{\alpha-1}\big(Ax^\alpha - x\big) - \big(Ax^\alpha - x\big)$. Observe that $\zeta(x) = \big[f(x) - x\big]\big[f'(x) - 1\big]$.

Equation $\delta\big(f(x) - x\big)f'(x) = 1$ is equivalent to $\zeta(x) = b$.

We will show that the function $\zeta$ is single-peaked in the interval $\big[0, x^G\big]$: there exists $\hat{x} \in \big(0, x^G\big)$ such that $\zeta$ is increasing in $[0, \hat{x})$ and decreasing in $\big(\hat{x}, x^G\big]$. The advantage of working with single-peaked functions is that, with an appropriate choice of the parameter $b$, the equation $\delta\big(f(x) - x\big)f'(x) = 1$ has exactly two solutions, meaning that there are exactly two steady-state candidates.

Recalling that $\zeta(0) = \zeta\left(x^G\right) = 0$ and letting $\hat{x} \in \arg\max_{0 \leq x \leq x^G} \zeta(x)$, we can prove that $\hat{x}$ is the unique solution to $\zeta'(x) = 0$ in $\left(0, x^G\right)$.

Indeed, the equation $\zeta'(x) = 0$ is equivalent to

$$\xi(x) \equiv \left[f'(x) - 1\right]^2 + \left[f(x) - x\right] f''(x) = 0.$$

Let's show that $\xi(x) = 0$ has a unique solution. Focus first on its derivative:

$$\begin{aligned}
\xi'(x) &= 2\left[f'(x) - 1\right] f''(x) + \left[f'(x) - 1\right] f''(x) + \left[f(x) - x\right] f'''(x) \\
&= 3\big(\alpha A x^{\alpha-1} - 1\big)\alpha(\alpha-1)Ax^{\alpha-2} + \big(Ax^\alpha - x\big)\alpha(\alpha-1)(\alpha-2)Ax^{\alpha-3} \\
&= \alpha(\alpha-1)Ax^{\alpha-3}\big[3\big(\alpha A x^\alpha - x\big) + (\alpha-2)\big(Ax^\alpha - x\big)\big] \\
&= -2\alpha(1-\alpha)Ax^{\alpha-2}\left[(2\alpha-1)Ax^{\alpha-1} - \frac{1+\alpha}{2}\right].
\end{aligned}$$

Clearly, the equation $(2\alpha-1)Ax^{\alpha-1} - (1+\alpha)/2 = 0$ has a unique positive solution:

$$\tilde{x} = \left[\frac{2(2\alpha-1)}{1+\alpha}A\right]^{\frac{1}{1-\alpha}} < (\alpha A)^{\frac{1}{1-\alpha}} = x^G,$$

since $0 < \alpha < 1$.

Note that $\xi'(x) < 0$ if $x \in (0, \tilde{x})$, and $\xi'(x) > 0$ if $x \in \left(\tilde{x}, x^G\right)$. This implies $\xi(\tilde{x}) < \xi\left(x^G\right) < 0$.

Importantly, because of this inequality, $\xi(x) = 0$ has no solution in the interval $\left(\tilde{x}, x^G\right)$.

Combining the monotonicity of $\xi$ in the interval $(0, \tilde{x})$ with $\lim_{x\to 0} \xi(x) = \infty$, the equation $\xi(x) = 0$ has a unique solution in $(0, \tilde{x})$.

Thus, the equation $\zeta'(x) = 0$ has unique solution in the interval $\left(0, x^G\right)$, say $\hat{x}$. The uniqueness of this solution ensures that the function $\zeta$ is increasing in $(0, \hat{x})$ and decreasing in $\left(\hat{x}, x^G\right)$. Hence, $\zeta$ is a single-peaked function.

It is easy to see that $\zeta(0) = \zeta\left(x^G\right) = 0$ and $\delta^b\left(f(x) - x\right) f'(x) = 1$ if and only if $\zeta(x) = b$.

Let $b^M \equiv \max_{x \in [0, x^G]} \zeta(x)$ and $\hat{x} \equiv \arg\max_{x \in [0, x^G]} \zeta(x)$.

If $b > b^M$, the equation $\zeta(x) = b$ has no solution. Since $\zeta$ is a single-peaked function, if $0 < b < b^M$, equation $\zeta(x) = b$ has exactly two solutions $x_s(b) < x^s(b)$, with the property that $\zeta(x) < b$ if $0 < x < x_s(b)$ or $x > x^s(b)$, and $\zeta(x) > b$ if $x_s(b) < x < x^s(b)$. Moreover, while $x_s(b)$ is increasing with respect to $b$, $x^s(b)$ is decreasing. More precisely, we have

$$\lim_{b\to 0} x_s(b) = 0 \text{ and } \lim_{b\to 0} x^s(b) = x^G,$$
$$\lim_{b\to b^M} x_s(b) = \hat{x} \text{ and } \lim_{b\to b^M} x^s(b) = \hat{x}.$$

Remark that $\delta^b\big(f(x)-x\big)f'(x) < 1$ if $x < x_s(b)$ or $x > x^s(b)$, and $\delta^b\big(f(x)-x\big)f'(x) > 1$ if $x_s(b) < x < x^s(b)$. QED

## 7.12 Proof of Proposition 3.1

The proof of Proposition 3.1 is long, and require some repetitive arguments. Therefore we present them in a preparatory lemma.

**Lemma 7.1.** 1. *Fix $x_0 > 0$. Let $b > 0$ and a sequence of parameters $(b_n)_{n=0}^{\infty}$ that converges to $b$. For each $n$, let $\big(x_t(n)\big)_{t=0}^{\infty}$ be an optimal path of the economy $\mathcal{E}(b_n)$ starting from $x_0$. If, for any $n$, $\big(x_t(n)\big)_{t=0}^{\infty}$ is non-decreasing, then the economy $\mathcal{E}(b)$ has a non-decreasing optimal path starting from $x_0$.*

2. *Fix $x_0 > 0$. Let $b > 0$ and a sequence of parameters $(b_n)_{n=0}^{\infty}$ that converges to $b$. For each $n$, let $\big(x_t(n)\big)_{t=0}^{\infty}$ be an optimal path of the economy $\mathcal{E}(b_n)$ starting from $x_0$. If, for any $n$, $\big(x_t(n)\big)_{t=0}^{\infty}$ is non-increasing, then the economy $\mathcal{E}(b)$ has a non-increasing optimal path starting from $x_0$.*

3. *There exists $\hat{b} > 0$ such that for any $b \in (0, \hat{b})$, the economy $\mathcal{E}(b)$ exhibits a strictly increasing optimal path.*

**Proof of Lemma 7.1**

Part 1. The compactness of the set $\Pi(x_0)$ with respect to the product topology entails the existence of a subsequence $\big(b_{n_k}\big)_{k=0}^{\infty}$ such that the sequence of optimal paths $\big((x_t(b_{n_k}))_{t=0}^{\infty}\big)_{k=0}^{\infty}$ converges to a sequence $\big(\tilde{x}_t\big)_{t=0}^{\infty}$ in this topology. Clearly, $\big(\tilde{x}_t\big)_{t=0}^{\infty}$ is non-decreasing.

Take now any feasible path $\big(x_t\big)_{t=0}^{\infty} \in \Pi(x_0)$ and notice that

$$W^{b_{n_k}}(x_0, x_1(b_{n_k}), x_2(b_{n_k}), \dots) \;\; \geq \;\; W^{b_{n_k}}(x_0, x_1, x_2, \dots).$$

Let $k$ goes to infinity to obtain

$$W^{b}(x_0, \tilde{x}_1, \tilde{x}_2, \dots) \;\; \geq \;\; W^{b}(x_0, x_1, x_2, \dots).$$

Since $(x_t)_{t=0}^{\infty}$ is arbitrary, the sequence $(\tilde{x}_t)_{t=0}^{\infty}$ is an optimal paths of the economy $\mathcal{E}(b)$ starting from $x_0$.

Part 2. We use exactly the same arguments to prove that the economy $\mathcal{E}(b)$ has a non-increasing optimal path starting from $x_0$.

Part 3. Assume the contrary. Then, there exists a decreasing sequence $(b_n)_{n=0}^{\infty}$ converging to zero such that, for any $n$, every optimal path of the economy $\mathcal{E}(b_n)$ is non-increasing.

We observe that for $b = b_n$ for some $n \geq 0$,

$$V^b(y) < \frac{u\big(f(y)\big)}{1 - \delta^b\big(f(y)\big)}, \tag{7}$$

for any $y \in \big(0, x^G\big)$. Indeed, an optimal path $\big(y_t\big)_{t=0}^{\infty}$ starting from $y$ is non-increasing. Thus, $y_t \leq y$, $c_t = f(y_t) - y_{t+1} < f(y)$ for any $t$ and

$$V(y) < u\big(f(y)\big) + \delta^b\big(f(y)\big)u\big(f(y)\big) + \delta^b\big(f(y)\big)^2 u\big(f(y)\big) + \ldots = \frac{u\big(f(y)\big)}{1 - \delta^b\big(f(y)\big)}.$$

We will prove the existence of $B > 0$ such that

$$\int_0^{x^s(b)} \delta^{b\prime}(c_y) f'(y) \left[\frac{u\big(f(y)\big)}{1 - \delta^b\big(f(y)\big)} - \frac{u\big(f(y) - y\big)}{1 - \delta^b\big(f(y) - y\big)}\right] dy < B, \tag{8}$$

for $b = b_n$ for some $n$. We have

$$\begin{aligned}
&\frac{u\big(f(y)\big)}{u\big(f(y) - y\big)} \frac{1 - \delta^b\big(f(y) - y\big)}{1 - \delta^b\big(f(y)\big)} \\
&= \frac{u\big(f(y)\big)}{u\big(f(y) - y\big)} \frac{b + f(y)}{b + f(y) - y} < \frac{u\big(f(y)\big)}{u\big(f(y) - y\big)} \frac{f(y)}{f(y) - y} \\
&= \frac{1}{\left[1 - \frac{y}{f(y)}\right]^{\rho+1}} < \frac{1}{\left[1 - \frac{x^G}{f(x^G)}\right]^{\rho+1}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^{x^s(b)} \delta^{b\prime}(c_y) f'(y) \left[\frac{u\big(f(y)\big)}{1 - \delta^b\big(f(y)\big)} - \frac{u\big(f(y) - y\big)}{1 - \delta^b\big(f(y) - y\big)}\right] dy \\
< \;& \hat{B} \int_0^{x^s(b)} \delta^{b\prime}(c_y) f'(y) \frac{u\big(f(y) - y\big)}{1 - \delta^b\big(f(y) - y\big)} dy,
\end{aligned} \tag{9}$$

where

$$\hat{B} \equiv \frac{1}{\left[1 - \frac{x^G}{f(x^G)}\right]^{\rho+1}} - 1.$$

Moreover,

$$
\begin{aligned}
&\hat{B}\int_0^{x^s(b)} \delta^{b\prime}(c_y) f'(y) \frac{u\big(f(y)-y\big)}{1-\delta^b\big(f(y)-y\big)} dy \\
&= \hat{B}\int_0^{x^s(b)} \frac{b}{(b+c_y)^2} f'(y) u\big(f(y)-y\big) \frac{b+c_y}{b} dy \\
&= \hat{B}\int_0^{x^s(b)} \frac{f'(y)u\big(f(y)-y\big)}{b+c_y} dy \le \hat{B}\int_0^{x^G} \frac{f'(y)u\big(f(y)-y\big)}{b+c_y} dy \\
&\le \hat{B}\int_0^{x^G} \frac{f'(y)u\big(f(y)-y\big)}{f(y)-y} dy = \frac{\hat{B}}{\rho}\int_0^{x^G} f'(y)u'\big(f(y)-y\big) dy \\
&= \frac{\hat{B}}{\rho}\int_0^{x^G} \left[f'(y)-1\right] u'\big(f(y)-y\big) dy + \frac{\hat{B}}{\rho}\int_0^{x^G} u'\big(f(y)-y\big) dy \\
&= \frac{\hat{B}}{\rho} u\big(f\left(x^G\right)-x^G\big) + \frac{\hat{B}}{\rho}\int_0^{x^G} u'(c_y) dy.
\end{aligned}
$$

Fix $\hat{y} \in \left(0, x^G\right)$ such that, if $y < \hat{y}$, we have $f(y) - y > y$, with the consequence that $u'(c_y) < u'(y)$.

We observe that

$$
\begin{aligned}
\int_0^{x^G} u'(c_y) dy &= \int_0^{\hat{y}} u'(c_y) dy + \int_{\hat{y}}^{x^G} u'(c_y) dy \\
&< \int_0^{\hat{y}} u'(y) dy + u'(c_{\hat{y}}) \int_{\hat{y}}^{x^G} dy = u(\hat{y}) + u'(c_{\hat{y}})\left(x^G - \hat{y}\right).
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
\hat{B}\int_0^{x^s(b)} \delta^{b\prime}(c_y) f'(y) \frac{u\big(f(y)-y\big)}{1-\delta^b\big(f(y)-y\big)} dy &< \frac{\hat{B}}{\rho} u\big(f\left(x^G\right)-x^G\big) + \frac{\hat{B}}{\rho}\int_0^{x^G} u'(c_y) dy \\
&< \frac{\hat{B}}{\rho} u\big(f\left(x^G\right)-x^G\big) + \frac{\hat{B}}{\rho}\left[u(\hat{y}) + u'(c_{\hat{y}})\left(x^G-\hat{y}\right)\right] \\
&\equiv B.
\end{aligned}
$$

According to (7), (8) and (9), if $b = b_n$ for some $n$, we have

$$
\int_0^{x^s(b)} \delta^{b\prime}(c_y) f'(y) \left[V^b(y) - \frac{u\big(f(y)-y\big)}{1-\delta^b\big(f(y)-y\big)}\right] dy < B. \tag{10}
$$

Now, let's prove that

$$
\lim_{b\to 0}\int_0^{x_s(b)} \left[\delta^b(c_y) f'(y) - 1\right] \left[\frac{u'(c_y)}{1-\delta^b(c_y)} + \frac{u(c_y)\delta^{b\prime}(c_y)}{\left[1-\delta^b(c_y)\right]^2}\right] dy = 0. \tag{11}
$$

Recall that, for any $b$, when $y < x_s(b)$, $\delta^b(c_y) f'(y) < 1$. Moreover, $\lim_{b\to 0} x_s(b) = 0$.

Fix $\varepsilon \in (0,1)$. For $b$ sufficiently small, we have $x_s(b) < \hat{y}$ and

$$\delta^b(c_y) < \frac{1}{f'(y)} < \frac{1}{f'\big(x_s(b)\big)} < \varepsilon$$

with $y < x_s(b)$ . Recall also that $c_y > y$ for $y < x_s(b_n)$, since $x_s(b) < \hat{y}$.

For $b$ small enough,

$$\int_0^{x_s(b)} \frac{u'(c_y)}{1-\delta^b(c_y)} dy < \int_0^{x_s(b)} \frac{u'(c_y)}{1-\varepsilon} dy = \frac{1}{1-\varepsilon} \int_0^{x_s(b)} u'(c_y) dy$$
$$< \frac{1}{1-\varepsilon} \int_0^{x_s(b)} u'(y) dy = \frac{u\big(x_s(b)\big)}{1-\varepsilon}.$$

Hence,

$$\lim_{b \to 0} \int_0^{x_s(b)} \frac{u'(c_y)}{1-\delta^b(c_y)} dy = 0. \tag{12}$$

In addition, for small value of $b$,

$$\int_0^{x_s(b)} \frac{u(c_y)\delta^{b\prime}(c_y)}{\big[1-\delta^b(c_y)\big]^2} dy$$
$$< \frac{1}{1-\varepsilon} \int_0^{x_s(b)} \frac{u(c_y)\delta^{b\prime}(c_y)}{1-\delta^b(c_y)} dy = \frac{1}{1-\varepsilon} \int_0^{x_s(b)} \frac{u(c_y)\frac{b}{(b+c_y)^2}}{\frac{b}{b+c_y}} dy$$
$$= \frac{1}{1-\varepsilon} \int_0^{x_s(b)} \frac{u(c_y)}{b+c_y} dy \leq \frac{1}{1-\varepsilon} \int_0^{x_s(b)} \frac{u(c_y)}{c_y} dy$$
$$= \frac{1}{(1-\varepsilon)\rho} \int_0^{x_s(b)} u'(c_y) dy < \frac{1}{(1-\varepsilon)\rho} \int_0^{x_s(b)} u'(y) dy < \frac{u\big(x_s(b)\big)}{(1-\varepsilon)\rho}.$$

Hence,

$$\lim_{b \to 0} \int_0^{x_s(b)} \frac{u(c_y)\delta^{b\prime}(c_y)}{\big[1-\delta^b(c_y)\big]^2} dy = 0. \tag{13}$$

Combining (12) and (13) with $\delta^b(c_y)f'(y) < 1$ for $y < x_s(b)$, we obtain (11).

Now, let's prove that

$$\lim_{b \to 0} \int_{x_s(b)}^{x^s(b)} \big[\delta^b(c_y)f'(y) - 1\big] \left[ \frac{u'(c_y)}{1-\delta^b(c_y)} + \frac{u(c_y)\delta^{b\prime}(c_y)}{\big[1-\delta^b(c_y)\big]^2} \right] dy = \infty. \tag{14}$$

Indeed, fixing $0 < x_* < x^* < x^G$ and $\hat{b}$ small enough such that, for $b < \hat{b}$, we have

$$0 < x_s(b) < x_* < x^* < x^s(b) < x^G,$$

we obtain

$$\int_{x_s(b)}^{x^s(b)} \left[\delta^b(c_y)f'(y) - 1\right] \left[\frac{u'(c_y)}{1-\delta^b(c_y)} + \frac{u(c_y)\delta^{b\prime}(c_y)}{\left[1-\delta^b(c_y)\right]^2}\right] dy$$
$$> \int_{x_*}^{x^*} \left[\delta^b(c_y)f'(y) - 1\right] \left[\frac{u'(c_y)}{1-\delta^b(c_y)} + \frac{u(c_y)\delta^{b\prime}(c_y)}{\left[1-\delta^b(c_y)\right]^2}\right] dy$$
$$> \int_{x_*}^{x^*} \left[\delta^{\hat{b}}(c_y)f'(y) - 1\right] \left[\frac{u'(c_y)}{1-\delta^b(c_y)} + \frac{u(c_y)\delta^{b\prime}(c_y)}{\left[1-\delta^b(c_y)\right]^2}\right] dy.$$

For every $y \in (x_*, x^*)$, $\lim_{b\to 0} \delta^b(c_y) = 1$. Therefore,

$$\lim_{b\to 0} \int_{x_*}^{x^*} \left[\delta^{\hat{b}}(c_y)f'(y) - 1\right] \left[\frac{u'(c_y)}{1-\delta^b(c_y)} + \frac{u(c_y)\delta^{b\prime}(c_y)}{\left[1-\delta^b(c_y)\right]^2}\right] dy = \infty.$$

and we get (14).

Using (10), (11) and (14), we find $\lim_{n\to\infty} \phi(b_n) = \infty$.

Hence, for $n$ sufficiently large, $\phi(b_n) > 0$. By Lemma 2.4, there exists $\overline{x}(b_n) < x^s(b_n)$ such that any optimal path starting from $x_0 > \overline{x}(b_n)$ converges to $x^s(b_n)$: a contradiction with the hypothesis that every optimal path of $\mathcal{E}(b_n)$ is non-increasing. Thus, Part 3 is proven. QED

Let us return to the proof of Proposition 3.1.

First, we will prove the existence of $b$ such that $\mathcal{E}(b)$ has a poverty trap. Assume the contrary. Then, for any fixed $b$, either every optimal path converges to the origin, or every optimal path converges to a positive steady state.

Although we cannot calculate explicit critical values of the fundamentals, we can prove the existence of values that generate a poverty trap.

In this respect, we introduce an auxiliary function:

$$\phi(b) \equiv \int_0^{x^s(b)} \left[\delta^b(c_y)f'(y) - 1\right] \left[\frac{u'(c_y)}{1-\delta^b(c_y)} + \frac{u(c_y)\delta^{b\prime}(c_y)}{\left[1-\delta^b(c_y)\right]^2}\right] dy$$
$$- \int_0^{x^s(b)} \delta^{b\prime}(c_y)f'(y)\left[V^b(y) - \frac{u(c_y)}{1-\delta^b(c_y)}\right] dy. \quad (15)$$

When $b = b^M$, both $x_s(b)$ and $x^s(b)$ both equal to $\hat{x}$. In this case, $\delta^b(c_y)f'(y) \leq 1$ for any $y$. Then, $\phi\left(b^M\right) < 0$. By Lemma 2.4, for $b$ close enough to $b^M$, there exists $\underline{x}(b) > 0$ such that every optimal path starting from $x_0 < \underline{x}(b)$ converges to the origin. Since $\mathcal{E}(b)$ has no poverty trap, every optimal path of these economies converges to the origin for any value of $x_0$.

Let's summarize the results we have obtained so far, assuming no poverty trap for any $b \in \left(0, b^M\right)$. First, there exists $b$ such that every optimal path in the economy $\mathcal{E}(b)$ converges to a positive steady state. Second, $\phi\left(b^M\right) < 0$ and, for any $b$ close enough to $b^M$, every optimal path of the economy $\mathcal{E}(b)$ converges to the origin.

Let us take a closer look at the second result: if we fix $b$ close enough to $b^M$, then, for any $b' \in \left(b, b^M\right)$, every optimal path of the economy $\mathcal{E}(b')$ converges to the origin.

Let $b^*$ the infimum of values $b$ such that, for any $b < b' < b^M$, any optimal path of the economy $\mathcal{E}(b')$ converges to the origin. As a consequence of point (3) in Lemma 7.1, $b^* > 0$.

Fix $r > 0$ such that $0 < b^* - r < b^* + r < b^M$ and $x_0 < x_s\left(b^* - r\right)$. By the definition of $b^*$, there exists a sequence $\left(\bar{b}_n\right)_{n=0}^{\infty}$ converging to $b^*$ with $\bar{b}_n \in (b^* - r, b^*)$ for any $n$ such that, starting from $x_0$, any optimal path $\left(x_t(\bar{b}_n)\right)_{t=0}^{\infty}$ of the economy $\mathcal{E}(\bar{b}_n)$ converges to a positive steady state. There exists also a sequence $\left(\underline{b}_n\right)_{n=0}^{\infty}$ with $\underline{b}_n \in (b^*, b^* + r)$ converging to $b^*$ such that, starting from $x_0$, any optimal path $\left(x_t(\underline{b}_n)\right)_{t=0}^{\infty}$ of the economy $\mathcal{E}(\underline{b}_n)$ converges to the origin.

Since $x_0$ is smaller than any possible steady state of $\mathcal{E}(\bar{b}_n)$ and $\mathcal{E}(\underline{b}_n)$, the optimal path is strictly increasing in the former case, and strictly decreasing in the latter. By Lemma 7.1, starting from $x_0$, the economy $\mathcal{E}(b^*)$ has a non-increasing optimal path $\left(\underline{x}_t\right)_{t=0}^{\infty}$, and a non-decreasing optimal path $\left(\overline{x}_t\right)_{t=0}^{\infty}$, both starting from $x_0$.

According to Proposition 2.5, since $x_0$ is smaller than any positive candidate for a steady state of the economy $\mathcal{E}(b^*)$, these sequences are strictly monotonic. The first sequence is strictly decreasing and converges to the origin. The second sequence is strictly increasing. Therefore, $x_0$ is the poverty trap of the economy $\mathcal{E}(b^*)$. Recalling that we have chosen $x_0$ arbitrarily less than $x_s(b^* - r)$, we reach a contradiction: each $x_0 < x_s(b^* - r)$ is a poverty trap of $\mathcal{E}(b^*)$.

This contradiction comes from the hypothesis that, for any $b \in \left(0, b^M\right)$, the economy $\mathcal{E}(b)$ has no poverty trap. Therefore, the exists some $b$ such that the economy $\mathcal{E}(b)$ exhibits a poverty trap. In other words, $\mathcal{C} \neq \emptyset$.

We prove the following claim: there exists $b \in \mathcal{C}$ and $\varepsilon > 0$ such that for any $b' \in (b - \varepsilon, b)$, the economy $\mathcal{E}(b')$ has a poverty trap.

Assume that the claim is false and choose $b \in \mathcal{C}$ and $x_0 > 0$ that is smaller than both the poverty trap of $\mathcal{E}(b)$ and $x_s(b)$. Observe that there exists $\varepsilon > 0$ small enough such that for $b' \in (b - \varepsilon, b)$, every optimal path of $\mathcal{E}(b')$ starting from $x_0$ is non-increasing.

Indeed, the contrary implies the existence of an increasing sequence $\left(b_n\right)_{n=0}^{\infty}$ that converges to $b$, and strictly increasing sequences $\left(x_t(n)\right)_{t=0}^{\infty}$ that are optimal path of the economy $\mathcal{E}(b_n)$ starting from $x_0$. By Lemma 7.1, this entails the existence of an optimal path of the economy $\mathcal{E}(b)$ starting from $x_0$ that is non-decreasing, a contradiction with the choice of $x_0$.

Hence, there exists $\varepsilon > 0$ small enough such that for $b' \in (b-\varepsilon, b)$, every optimal path of $\mathcal{E}(b')$ starting from $x_0$ is non-increasing. Under the hypothesis that the claim is false, there exists a sequence $(b_n)_{n=0}^{\infty}$ converging to $b$, with $b_n \in (b-\varepsilon, b)$ for any $n$, such that $\mathcal{E}(b_n)$ has no poverty trap and each optimal path of these economies starting from $x_0$ is non-increasing. Thus, each optimal path of these economies is decreasing and converges to the origin. According to Lemma 7.1, for any $x_0 > 0$, the economy $\mathcal{E}(b)$ has an optimal path starting from $x_0$ that is non-increasing.

We will prove that every optimal path of $\mathcal{E}(b)$ is non-increasing.[8] Assume the contrary: there exists $\tilde{x}_0 > 0$ and an optimal path starting from $\tilde{x}_0$ that is strictly increasing and converges to $x^*$. By the monotonicity of the optimal policy correspondence, there exists a strictly increasing optimal path that converges to $x^*$ from $x'_0 \in (\tilde{x}_0, x^*)$. Moreover, $\tilde{x}_0$ is not a steady state and $\tilde{x}_0 < x^s(b)$, the largest candidate for a steady state. The level $x^*$ is either $x_s(b)$ or $x^s(b)$.

Consider a non-increasing optimal path starting from $\tilde{x}_0$. Since $\tilde{x}_0$ is not a steady state, by Proposition 2.5, this sequence is strictly decreasing. If this sequence converges to $x_s(b)$, any member belongs to the interval $(x_s(b), x^s(b))$ with $\delta\left(f(x)-x\right)f'(x) > 1$, in contradiction to Lemma 2.2.

Hence, the sequence $(\tilde{x}_t)_{t=0}^{\infty}$ converges to the origin. In other words, $(0, \tilde{x}_0)$ is a poverty trap of $\mathcal{E}(b)$. Since the argument leading to this conclusion can be applied for any $x'_0 \in (\tilde{x}_0, x^*)$, we arrive at a contradiction.

Therefore, under the assumption that the claim is false, we conclude that, for any $b \in \mathcal{C}$, every optimal path of economy $\mathcal{E}(b)$ is non-increasing. Lemma 7.1 ensures that $b^m \equiv \inf \mathcal{C} > 0$.

Let $(\underline{b}_n)_{n=0}^{\infty}$ be a sequence of parameters in $\mathcal{C}$ converging to $b^m$. We do not need the strict monotonicity of the sequence: if $b^m \in \mathcal{C}$, we can simply fix $\underline{b}_n \equiv b^m$ for any $n$. Since every optimal path of $\mathcal{E}(\underline{b}_n)$ is non-increasing, by Lemma 7.1, for every $x_0 > 0$,

[8] The poverty trap is $x^s(b)$: optimal paths starting from $x_0 < x^s(b)$ converge to the origin, while those starting from $x_0 > x^s(b)$ converge to $x^s(b)$.

there exists a non-increasing optimal path of $\mathcal{E}(b^m)$ starting from $x_0$.

Let $b_*$ the infimum of values $b$ with the following property: for each $x_0 > 0$ and any $b'$ such that $b \leq b' \leq b^m$, there exists an optimal path of $\mathcal{E}(b')$ starting from $x_0$ that is non-increasing.

We observe that $b_* > 0$. This is clear if $b_* = b^m$. Focus instead on the case $b_* < b^m$. Then, for any $b \in (b_*, b^m)$ and any $0 < x_0 < x_s(b)$, there exists an optimal path of $\mathcal{E}(b)$ starting from $x_0$ that is non-increasing. The choice of $x_0$ ensures that this path converges to the origin. By the definition of $b^m$, the economy $\mathcal{E}(b)$ has no poverty trap. Hence, every optimal path of $\mathcal{E}(b)$ is decreasing and converges to the origin. Since this is true for any $b \in (b_*, b^m)$, by Lemma 7.1, we conclude that $b_* > 0$.

By the choice of $b_*$ and Lemma 7.1, for any $x_0 < x_s(b_*)$, there exists a non-increasing optimal path starting from $x_0$. The choice of $x_0$ guarantees that this path converges to the origin.

Choose $r \in (0, b_*)$ and $0 < x_0 < x_s(b_* - r)$. By the definition of $b_*$, there exists a strictly increasing sequence $\left(\bar{b}_n\right)_{n=0}^{\infty}$ with $\bar{b}_n \in (b_* - r, b_*)$ for any $n$ and $\lim_{n\to\infty} \bar{b}_n = b^*$, such that the economy $\mathcal{E}(b_n)$ has no poverty trap. Since $x_0 < x_s(b_n)$, starting from $x_0$, any optimal path $\left(x_t(\bar{b}_n)\right)_{t=0}^{\infty}$ of the economy $\mathcal{E}(\bar{b}_n)$ is strictly increasing. Hence, there exists a non-decreasing optimal path of $\mathcal{E}(b_*)$ starting from $x_0$. In other words, starting from every $x_0 < x_s(b_* - r)$, there exists a non-decreasing optimal path. Recall that there exists also a decreasing optimal path converging to the origin. This implies that $x_0$ is a poverty trap. Since the choice of $x_0 < x_s(b_* - r)$ was arbitrary, we arrive at a contradiction.

The claim is true and there exist $b \in \mathcal{C}$ and $\varepsilon > 0$ such that $(b - \varepsilon, b) \subset \mathcal{C}$. Simply choose $\underline{b} \equiv b - \varepsilon$ and $\bar{b} \equiv b$. QED

## 7.13 Proof of Proposition 4.1

The proof is standard. We follow exactly the same arguments used in the proofs of Propositions 2, 3, 4 and Theorems 1, 2 in Le Van and Morhaim (2002). QED

## 7.14 Proof of Proposition 4.2

The proofs of Parts (1) and (2) are also standards. For part (3), we use the supermodularity of the function $u\big(f(x) - y\big)$. We observe that

$$\frac{\partial^2}{\partial x \partial y} u\big(f(x) - y\big) = -u''\big(f(x) - y\big) f'(x) > 0.$$

Assume that $y' < y$. We have

$$\begin{aligned} u\big(f(x) - y\big) + \delta(y)V(y) &\geq u\big(f(x) - y'\big) + \delta(y')V(y'), \\ u\big(f(x') - y'\big) + \delta(y')V(y') &\geq u\big(f(x') - y\big) + \delta(y)V(y). \end{aligned}$$

Summing these inequalities, we get

$$u\big(f(x) - y\big) + u\big(f(x') - y'\big) \geq u\big(f(x) - y'\big) + u\big(f(x') - y\big),$$

a contradiction with the supermodularity property of $u\big(f(x) - y\big)$. QED

## 7.15 Proof of Proposition 4.3

Part (1). We remind that $0 < x_{t+1} < f(x_t)$ for any $t \geq 0$. Taking the derivative of the intertemporal utility function with respect to $x_{t+1}$, we obtain

$$\begin{aligned} & u'\big(f(x_t) - x_{t+1}\big) \\ & = \delta'(x_{t+1}) u\big(f(x_{t+1}) - x_{t+2}\big) + \delta(x_{t+1}) u'\big(f(x_{t+1}) - x_{t+2}\big) f'(x_{t+1}) \\ & + \delta'(x_{t+1})\delta(x_{t+2})V(x_{t+2}) = \delta(x_{t+1}) u'\big(f(x_{t+1}) - x_{t+2}\big) f'(x_{t+1}) + \delta'(x_{t+1})V(x_{t+1}). \end{aligned}$$

Part (2). We note that the right-hand side of Euler equation (3) is strictly increasing with respect to $x_{t+2}$. Hence, for $t \geq 1$, $\varphi(x_t)$ is a singleton. QED

## 7.16 Proof of Proposition 4.4

Replacing $x_t$, $x_{t+1}$ and $x_{t+2}$ by $x^*$ in the Euler equation (3), we obtain Part (1). To prove part (2), let $x_{t+1} = x^*$. Replacing $x_{t+1}, x_{t+2}$ by $x^*$ in the Euler equation gives

$$u'\big(f(x_t) - x^*\big) = \delta(x^*) u'\big(f(x^*) - x^*\big) f'(x^*) + \delta'(x^*)V(x^*).$$

This implies $x_t = x^*$. Hence, if $x_t \neq x^*$, the next state $x_{t+1} \neq x^*$.

The monotonicity property is a direct consequence of Proposition 4.2. QED

## 7.17 Proof of Proposition 4.5

Before proving the proposition, we introduce an auxiliary function $\psi(y)$:

$$\psi(y) \equiv \big[\delta(y)f'(y) - 1\big]u'\big(f(y) - y\big) + \frac{\delta'(y)u\big(f(y) - y\big)}{1 - \delta(y)}.$$

Part (1). Clearly, $\psi(y) > 0$ if and only if

$$\delta(y)f'(y) > 1 - \frac{\delta'(y)u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]u'\big(f(y) - y\big)}.$$

We have $\psi(y) > 0$ for $0 < y < \hat{x}$. Assume the existence of an optimal path $(x_t)_{t=0}^{\infty}$ starting from $x_0 > 0$ and converging to zero. Monotonicity ensures that this sequence is strictly decreasing. There exists some $T$ such that $x_t < \hat{x}$ for any $t \geq T$. We will prove that, if $t \geq T$,

$$\frac{u\big(f(x_t) - x_t\big)}{1 - \delta(x_t)} > u\big(f(x_t) - x_{t+1}\big) + \delta(x_{t+1})\frac{u\big(f(x_{t+1}) - x_{t+1}\big)}{1 - \delta(x_{t+1})}.$$

Define

$$\zeta(y) \equiv u\big(f(x_t) - y\big) + \delta(y)\frac{u\big(f(y) - y\big)}{1 - \delta(y)}.$$

$\zeta$ is strictly increasing in $(x_{t+1}, x_t)$. Indeed,

$$\begin{aligned}
\zeta'(y) &= -u'\big(f(x_t) - y\big) + \delta'(y)\frac{u\big(f(y) - y\big)}{1 - \delta(y)} \\
&\quad + \delta(y)\frac{\big[1 - \delta(y)\big]u'\big(f(y) - y\big)\left[f'(y) - 1\right] + \delta'(y)u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]^2} \\
&> -u'\big(f(y) - y\big) + \delta'(y)\frac{u\big(f(y) - y\big)}{1 - \delta(y)} \\
&\quad + \delta(y)\frac{\big[1 - \delta(y)\big]u'\big(f(y) - y\big)\left[f'(y) - 1\right] + \delta'(y)u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]^2} \\
&= \frac{\psi(y)}{1 - \delta(y)} > 0.
\end{aligned}$$

Hence, $\zeta$ is strictly increasing in $(x_{t+1}, x_t)$ and we have $\zeta(x_t) > \zeta(x_{t+1})$, which implies

$$\begin{aligned}
\frac{u\big(f(x_T) - x_T\big)}{1 - \delta\big(f(x_T) - x_T\big)} &> u\big(f(x_T) - x_{T+1}\big) + \delta(x_{T+1})\frac{u\big(f(x_{T+1}) - x_{T+1}\big)}{1 - \delta(x_{T+1})} \\
&> u\big(f(x_T) - x_{T+1}\big) + \delta(x_{T+1})u\big(f(x_{T+1}) - x_{T+2}\big) \\
&\quad + \delta(x_{T+1})\delta(x_{T+2})\frac{u\big(f(x_{T+2}) - x_{T+2}\big)}{1 - \delta(x_{T+2})} > \ldots \geq V(x_T),
\end{aligned}$$

a contradiction.

Part (2). Using similar arguments, we can prove that, if the optimal path $(x_t)_{t=0}^{\infty}$ is increasing, the function $\zeta$ is decreasing in $(x_t, x_{t+1})$, which leads to a contradiction.

QED

## 7.18 Proof of Corollary 4.1

Apply the proof of Proposition 4.5. QED

## 7.19 Proof of Proposition 4.6

Part (1). Assume that there exists an increasing optimal path $(x_t)_{t=0}^{\infty}$ starting from $x_0$ with limit $x^*$. For any $t \geq 0$, we have

$$\begin{aligned} V(x_{t+1}) - V(x_t) &\geq u\big(f(x_{t+1}) - x_{t+1}\big) + \delta(x_{t+1})V(x_{t+1}) - u\big(f(x_t) - x_{t+1}\big) - \delta(x_{t+1})V(x_{t+1}) \\ &= u\big(f(x_{t+1}) - x_{t+1}\big) - u\big(f(x_t) - x_{t+1}\big) = \int_{x_t}^{x_{t+1}} u'\big(f(y) - x_{t+1}\big)f'(y)dy \\ &\geq \int_{x_t}^{x_{t+1}} u'\big(f(y) - y\big)f'(y)dy. \end{aligned}$$

This implies

$$\begin{aligned} V(x^*) - V(x_0) &= \sum_{t=0}^{\infty}\big[V(x_{t+1}) - V(x_t)\big] \geq \sum_{t=0}^{\infty}\int_{x_t}^{x_{t+1}} u'\big(f(y) - y\big)f'(y)dy \\ &= \int_{x_0}^{x^*} u'\big(f(y) - y\big)f'(y)dy. \end{aligned}$$

On the other hand,

$$\begin{aligned} V(x^*) - V(x_0) &\leq \frac{u\big(f(x^*) - x^*\big)}{1 - \delta(x^*)} - \frac{u\big(f(x_0) - x_0\big)}{1 - \delta(x_0)} \\ &= \int_{x_0}^{x^*} \frac{\big[1 - \delta(y)\big]u'\big(f(y) - y\big)\left[f'(y) - 1\right] + \delta'(y)u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]^2}dy. \end{aligned}$$

This implies

$$\int_{x_0}^{x^*} \frac{\big[1 - \delta(y)\big]u'\big(f(y) - y\big)\left[f'(y) - 1\right] + \delta'(y)u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]^2}dy \geq \int_{x_0}^{x^*} u'\big(f(y)-y\big)f'(y)dy.$$

Taking the left-hand side minus the right-hand side, we get

$$\int_{x_0}^{x^*} \left(\big[\delta(y)f'(y) - 1\big]\frac{u'\big(f(y) - y\big)}{1 - \delta(y)} + \frac{\delta'(y)u\big(f(y) - y\big)}{\big[1 - \delta(y)\big]^2}\right) dy \geq 0,$$

a contradiction.

Part (2). Using similar arguments, we can prove that assuming a decreasing optimal path starting from $x_0$ leads to a contradiction. QED

## 7.20 Proof of Corollary 4.2

Part (1). Choose $\overline{x} < x^s$ close enough to $x^s$ such that

$$\int_0^{\overline{x}} \left( \big[\delta(y)f'(y) - 1\big] \frac{u'\big(f(y)-y\big)}{1-\delta(y)} + \frac{\delta'(y)u\big(f(y)-y\big)}{\big[1-\delta(y)\big]^2} \right) dy > 0,$$

and apply Proposition 4.6.

Part (2). Choose $\underline{x} < x_s$ close enough to 0 such that

$$\int_{\underline{x}}^{x^s} \left( \big[\delta(y)f'(y) - 1\big] \frac{u'\big(f(y)-y\big)}{1-\delta(y)} + \frac{\delta'(y)u\big(f(y)-y\big)}{\big[1-\delta(y)\big]^2} \right) dy < 0,$$

and apply Proposition 4.6. QED

## 7.21 Proof of Proposition 5.1

We consider the function $\zeta(x) \equiv (1+\rho\alpha)Ax^\alpha - (1+\rho)x$. Clearly, (4) is equivalent to $\zeta(x) = \rho b$. Let

$$\hat{x} \equiv \left( \frac{1+\rho\alpha}{1+\rho} A \right)^{\frac{1}{1-\alpha}}.$$

The function $\zeta$ is strictly concave, with $\zeta(0) = \zeta(\hat{x}) = 0$. Define

$$b^M \equiv \frac{1}{\rho} \max_{x\in[0,\hat{x}]} \zeta(x).$$

If $b > b^M$, then $\zeta(x) < \rho b$ for any $x \geq 0$. If $0 < b < b^M$, then the equation $\zeta(x) = \rho b$ has exactly two positive solutions: $x_s(b) < x^s(b)$. The first one is strictly increasing with respect to $b$, while the second one is strictly decreasing. We obtain $\lim_{b\to 0} x_s(b) = 0$ and $\lim_{b\to 0} x^s(b) = \hat{x}$.

Let us denote the economy associated to the discount function $\delta^b$ by $\mathcal{E}(b)$, and the corresponding intertemporal utility function by $W^b(x_0, x_1, x_2, \ldots)$.

Reminding that, if $b = b^M$, for any $y \in [0, \hat{x}]$, we have

$$\delta^b(y)f'(y) \leq 1 - \frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{\big[1-\delta^b(y)\big]u'\big(f(y)-y\big)},$$

which is equivalent to

$$\left[\delta^b(y)f'(y)-1\right]\frac{u'\big(f(y)-y\big)}{1-\delta^b(y)}+\frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{\left[1-\delta^b(y)\right]^2}\leq 0.$$

Consider

$$\Phi(b)\equiv\int_0^{x^s(b)}\left(\left[\delta^b(y)f'(y)-1\right]\frac{u'\big(f(y)-y\big)}{1-\delta^b(y)}+\frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{\left[1-\delta^b(y)\right]^2}\right)dy.$$

Clearly, $\Phi\left(b^M\right)<0$.

At this stage of the proof, we need to introduce a preparatory Lemma.

Lemma **7.2.** *1. Fix $x_0>0$. Let $b>0$ and a sequence of parameters $(b_n)_{n=0}^{\infty}$ that converges to $b$. For each $n$, let $\big(x_t(n)\big)_{t=0}^{\infty}$ be an optimal path of the economy $\mathcal{E}(b_n)$ starting from $x_0$. If, for any $n$, $\big(x_t(n)\big)_{t=0}^{\infty}$ is non-decreasing, then the economy $\mathcal{E}(b)$ has a non-decreasing optimal path starting from $x_0$.*

*2. Fix $x_0>0$. Let $b>0$ and a sequence of parameters $(b_n)_{n=0}^{\infty}$ that converges to $b$. For each $n$, let $\big(x_t(n)\big)_{t=0}^{\infty}$ be an optimal path of the economy $\mathcal{E}(b_n)$ starting from $x_0$. If, for any $n$, $\big(x_t(n)\big)_{t=0}^{\infty}$ is non-increasing, then the economy $\mathcal{E}(b)$ has a non-increasing optimal path starting from $x_0$.*

*3. For $b>0$ close enough to zero, we have $\Phi(b)>0$.*

**Proof of Lemma 7.2**

Part (1) and (2): the proof follows exactly the same arguments as in the proof of Lemma 7.1.

We consider Part (3). We will prove that, for $b$ close enough to zero, $\Phi(b)>0$.

Fix $\varepsilon\in(0,1)$ and $\hat{y}$ such that, if $y<\hat{y}$, we have $f(y)-y>y$. For $b$ small enough, $x_s(b)<\hat{y}$ and $\delta^b(y)<1/f'(y)<\varepsilon$ for any $y<x_s(b)$.

We have for small $b$,

$$\begin{aligned}\int_0^{x_s(b)}\frac{u'\big(f(y)-y\big)}{1-\delta^b(y)}dy&<\int_0^{x_s(b)}\frac{u'\big(f(y)-y\big)}{1-\varepsilon}dy\\&=\frac{1}{1-\varepsilon}\int_0^{x_s(b)}u'\big(f(y)-y\big)dy\\&<\frac{1}{1-\varepsilon}\int_0^{x_s(b)}u'(y)dy=\frac{u\big(x_s(b)\big)}{1-\varepsilon}.\end{aligned}$$

and, hence,

$$\lim_{b\to 0}\int_0^{x_s(b)} \frac{u'\big(f(y)-y\big)}{1-\delta^b(y)}dy = 0. \tag{16}$$

Moreover,

$$\begin{aligned}
&\int_0^{x_s(b)} \frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{\left[1-\delta^b(y)\right]^2}dy \\
&< \frac{1}{1-\varepsilon}\int_0^{x_s(b)} \frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{1-\delta^b(y)}dy = \frac{1}{1-\varepsilon}\int_0^{x_s(b)} \frac{\frac{b}{(b+y)^2}u\big(f(y)-y\big)}{\frac{b}{b+y}}dy \\
&= \frac{1}{1-\varepsilon}\int_0^{x_s(b)} \frac{u\big(f(y)-y\big)}{b+y}dy \le \frac{1}{1-\varepsilon}\int_0^{x_s(b)} \frac{u\big(f(y)\big)}{y}dy \\
&= \frac{A^\rho}{1-\varepsilon}\int_0^{x_s(b)} y^{\rho\alpha-1}dy = \frac{A^\rho}{(1-\varepsilon)\rho\alpha}\int_0^{x_s(b)} \left(y^{\rho\alpha}\right)' dy = \frac{A^\rho\left[x_s(b)\right]^{\rho\alpha}}{(1-\varepsilon)\rho\alpha}.
\end{aligned}$$

Thus,

$$\lim_{b\to 0}\int_0^{x_s(b)} \frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{\left[1-\delta^b(y)\right]^2}dy = 0. \tag{17}$$

The limits in (16) and (17), jointly with $\delta^b(y)f'(y) < 1$ for $y < x_s(b)$, imply

$$\lim_{b\to 0}\int_0^{x_s(b)} \left(\left[\delta^b(y)f'(y)-1\right]\frac{u'\big(f(y)-y\big)}{1-\delta^b(y)} + \frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{\left[1-\delta^b(y)\right]^2}\right)dy = 0. \tag{18}$$

We can now show that

$$\lim_{b\to 0}\int_{x_s(b)}^{x^s(b)} \left(\left[\delta^b(y)f'(y)-1\right]\frac{u'\big(f(y)-y\big)}{1-\delta^b(y)} + \frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{\left[1-\delta^b(y)\right]^2}\right)dy = \infty. \tag{19}$$

Fix $0 < x_* < x^* < x^G$ (solution to $f'(x) = 1$) and $\hat{b}$ small enough such that, for $b < \hat{b}$,

$$0 < x_s(b) < x_* < x^* < x^s(b) < \hat{x},$$

and $\delta^{\hat{b}}(y)f'(y) > 1$ for any $x_* \le y \le x^*$. We obtain

$$\begin{aligned}
&\int_{x_s(b)}^{x^s(b)} \left(\left[\delta^b(y)f'(y)-1\right]\frac{u'\big(f(y)-y\big)}{1-\delta^b(y)} + \frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{\left[1-\delta^b(y)\right]^2}\right)dy \\
&> \int_{x_*}^{x^*} \left(\left[\delta^b(y)f'(y)-1\right]\frac{u'\big(f(y)-y\big)}{1-\delta^b(y)} + \frac{\delta^{b\prime}(y)u\big(f(y)-y\big)}{\left[1-\delta^b(y)\right]^2}\right)dy \\
&> \int_{x_*}^{x^*} \left[\delta^{\hat{b}}(y)f'(y)-1\right]\frac{u'\big(f(y)-y\big)}{1-\delta^b(y)}dy.
\end{aligned}$$

For any $y \in (x_*, x^*)$, $\lim_{b\to 0} \delta^b(y) = 1$. Hence,

$$\lim_{b\to 0} \int_{x_*}^{x^*} \left[\delta^{\hat{b}}(y) f'(y) - 1\right] \frac{u'\big(f(y) - y\big)}{1 - \delta^b(y)} dy = \infty.$$

From (18) and (19), we have $\lim_{b\to 0} \Phi(b) = \infty$. Thus, there exists $b$ small enough such that $\Phi(b) > 0$. QED

We can now return to the proof of Proposition 5.1.

First, we show the existence of $b$ such that the economy $\mathcal{E}(b)$ has a poverty trap.

Assume the contrary. Remind that, if $b$ is close enough to $b^M$, we have $\Phi(b) < 0$. By Proposition 4.6, there are optimal paths converging to the origin. Then, any optimal path of $\mathcal{E}(b)$ converges to the origin, otherwise a poverty trap exists.

Conversely, if $b$ is close enough to zero, we find $\Phi(b) > 0$. By Proposition 4.6, there are optimal paths converging to a positive steady state. Then, every optimal path of $\mathcal{E}(b)$ converges to a positive steady state.

Let $b^*$ be the infimum of values $b$ with the following property: for any $b < b' < b^M$, every optimal path of the economy $\mathcal{E}(b')$ converges to the origin. By Lemma 7.2, $b^* > 0$.

From now on, we can apply the arguments in Section 3, taking care to replace Lemma 7.1 by Lemma 7.2, and obtain a contradiction. These arguments also imply the existence of $\underline{b} < \overline{b}$ such that, for any $b \in (\underline{b}, \overline{b})$, the economy $\mathcal{E}(b)$ experiences a poverty trap. QED

# References


Akao, K.-I., T. Kamihigashi and K. Nishimura (2025): Critical capital stock in a continuous time growth model with a convex-concave production function. *Journal of Mathematical Economics* 119, https://doi.org/10.1016/j.jmateco.2025.103146.

Amir, R. (1996): Sensitivity analysis of multisector optimal of economic dynamics. *Journal of Mathematical Economics* 25, 123-141.

Amir, R. (2005): Supermodularity and complementarity in economics: An elementary survey. *Southern Economic Journal* 71, 636-660.

Borissov, K., S. Bosi, T. Ha-Huy, V.-Q. Nguyen and M. Pakhnin (2025): Endogeneous discounting and economic dynamics. *Mimeo.*

Cass, D. (1965): Optimum growth in an aggregative model of capital accumulation. *Review of Economic Studies* 32, 233-340.

Chambers, C.P. and F. Echenique (2018): On multiple discount rates. *Econometrica* 86, 1325-1346.

Clark, C.W. (1971): Economically optimal policies for the utilization of biologically renewable resource. *Mathematical Biosciences* 12, 245-260.

Dechert, W.D. and K. Nishimura (1983): A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. *Journal of Economic Theory* 31, 332-354.

Erol, S., C. Le Van and C. Saglam (2011): Existence, optimality and dynamics of equilibria with endogenous time preference. *Journal of Mathematical Economics* 47, 170-179.

Epstein, L.G. (1983): Stationary cardinal utility and optimal growth under uncertainty. *Journal of Economic Theory* 31, 133-152.

Epstein, L.G. (1987): A simple dynamic general equilibrium model. *Journal of Economic Theory* 41, 68-95.

Fisher, I. (1930): *The Theory of Interest*. New York, Macmillan.

Hung, N.M., C. Le Van and P. Michel (2009): Non-convex aggregate technology and optimal economic growth. *Economic Theory* 40, 457-471.

Koopmans, T.C. (1965): On the concept of optimal economic growth. In: Johansen, J., Ed., *The Econometric Approach to Development Planning*. Amsterdam, North Holland, 225-287.

Le Van, C. and R.-A. Dana (2003): *Dynamic Programming in Economics*. Dordrecht, Kluwer Academic Publishers.

Le Van, C. and L. Morhaim (2002): Optimal growth models with bounded or unbounded returns: A unifying approach. *Journal of Economic Theory* 105, 158-187.

Majumdar, M. and T. Mitra (1982): Intertemporal allocation with a non-convex technology: The aggregative framework. *Journal of Economic Theory* 27, 101-136.

Phelps, P. (1961): The golden rule of accumulation: a fable for growthmen. *American Economic Review* 51, 638-643.

Ramsey, F.P. (1928): A mathematical theory of saving. *Economic Journal* 38, 543-559.

Skiba, A.K. (1978): Optimal growth with a convex-concave production function. *Econometrica* 46, 527-539.

Stokey, N.L., R.E. Lucas, with E.C. Prescott (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press.

Tao, T. (2011): *An Introduction to Measure Theory*. American Mathematical Society, Graduate Studies in Mathematics 126.

Uzawa, H. (1968): Time preference, the consumption function, and optimum asset holdings. In: Wolfe, J. Ed., *Value, Capital and Growth: Papers in Honour of Sir John Hicks*. Chicago, Aldine Publishing Company, 485-505.

Vasilev, A. (2022): A real-business-cycle model with endogenous discounting and a government sector. *Notas Economicas, Faculty of Economics, University of Coimbra* 54, 73-86.