WORKING PAPERS

2025/25

23 September 2025

Endogenous Discounting and Economic Dynamics

Krill Borissov, Stefano Bosi, Thai Ha-Huy, Van-Quy Nguyen and Mikhail Pakhnin

CEPS
Centre for Economics at Paris-Saclay

école normale supérieure paris—saclay
université evry val-d'essonne
université PARIS-SACLAY
université PARIS-SACLAY
Maison des Sciences de l'Homme PARIS-SACLAY

Endogenous Discounting and Economic Dynamics*

Kirill Borissov, Stefano Bosi, Thai Ha-Huy,

Van-Quy Nguyen, Mikhail Pakhnin

September 22, 2025

Abstract

We study a discrete-time optimal growth model with endogenous discounting. The discount factor may depend on both consumption and the capital stock, and intertemporal utility is modeled as a discounted sum of instantaneous utilities, with the sum of discount factors equal to one. We show that this specification preserves the invariance of optimal paths and steady states to affine transformations of the instantaneous utility function, providing a general and flexible framework for analyzing economic dynamics under endogenous time preference. We prove that optimal capital paths are monotonic, and steady states depend on initial conditions. We also show the robustness of poverty traps under endogenous discounting: in several examples, for a set of parameters with positive measure, the optimal path converges to a positive steady state only if the initial capital stock exceeds a critical level and otherwise converges to the origin.

Keywords: Economic growth; Time preference; Endogenous discounting; Poverty traps

JEL Classification: C61, D15, D90, O41

*The paper has greatly benefited from comments of participants of the CEPET 2025 Workshop (Udine, Italy), 16th Viennese Conference on Optimal Control and Dynamic Games (Vienna, Austria, 2025), and Workshop on Recent Advances in Optimization, Utility and Fluctuations (Marseille, France, 2025). Mikhail Pakhnin was supported for this work by the Project PID2023-152029OB-I00 financed by MICIU/AEI/10.13039/501100011033/FEDER, EU.

1. Introduction

Time preference is widely recognized as a key driver of economic growth. In classical optimal capital accumulation models, intertemporal trade-offs between current and future consumption are governed by a constant discount factor reflecting the decision maker's degree of patience. However, the relationship between patience and growth is reciprocal: consumption flows and wealth can influence discount factors, suggesting that time preference is endogenous. A substantial literature has examined the dynamic properties of economies with endogenous discounting, in which the discount factor depends either on consumption or on the capital stock (see, among others, Uzawa, 1968; Drugeon, 2000; Erol et al., 2011; Bouché, 2017).

An important implication of endogenous discounting is that the *level* of the instantaneous utility function enters directly into the characterization of optimal paths (see, e.g., Schumacher, 2011, for a discussion). This feature generates several anomalies. For instance, adding a constant to the utility function, which in many other settings leaves consumption decisions unaffected, alters optimal paths and, when discounting depends on capital, can even change steady states. Moreover, the sign of the utility acquires an unexpected behavioral interpretation: when utility is negative, the decision maker may optimally choose to become less patient.

In this paper, we propose a general framework for endogenous discounting that (i) allows the discount factor to depend simultaneously on consumption and the capital stock, and (ii) eliminates the anomalies described above. In our setting, optimal paths and steady states are invariant to affine transformations of the instantaneous utility function, enabling robust conclusions about the dynamics of optimal paths. In particular, we show that any optimal capital path is monotonic, and the steady state to which it converges depends on the initial capital stock, providing a theoretical basis for studying poverty traps driven exclusively by endogenous time preference.

Specifically, we consider a discrete-time optimal growth model where the preferences over feasible capital paths $\{x_t\}_{t=0}^{\infty}$ are represented by a *recursive* intertemporal utility function of the form

$$W(x_t, x_{t+1}, \dots) = \big[1-\delta(x_t, x_{t+1})\big]\, u\big(f(x_t)-x_{t+1}\big)+\delta(x_t, x_{t+1})W(x_{t+1}, x_{t+2}, \dots)\,. \quad (1)$$

Here, $f(x)$ is a standard neoclassical production function, $u(c)$ is an instantaneous utility function of consumption, and consumption is determined as $c_t = f(x_t)-x_{t+1}$.

Intertemporal utility function (1) deserves several comments. First, the discount factor between any two periods t and $t+1$, $\delta(x_t, x_{t+1})$, is not constant but depends on the capital path. Our general specification encompasses many interesting and relevant cases. In particular, when $\delta(x_t, x_{t+1}) = \delta^b(f(x_t) - x_{t+1})$, discounting depends on current *consumption*. When defined as $\delta(x_t, x_{t+1}) = \delta^p(x_t)$, discounting depends on the *current* capital stock, while $\delta(x_t, x_{t+1}) = \delta^f(x_{t+1})$ implies that discounting depends on the *future* capital stock.

Second, we assume that $\delta(x_t, x_{t+1})$ is non-decreasing in its first argument and non-increasing in its second argument; that is, the discount factor is higher when current consumption or capital is higher relative to future capital. This assumption reflects the common intuition that the rich are more patient than the poor, which is supported by numerous empirical and experimental studies.[1] A similar intuition emerges from theoretical models of investment in patience (e.g., Becker and Mulligan, 1997; Stern, 2006), which suggest that the rich devote more resources to overcoming their own weaknesses and, as a result, are rationally less impatient. Also, it is consistent with the observation that wealth is positively correlated with lower mortality rates, thereby increasing the subjective discount factor through the probability of surviving.

Third, we define recursive intertemporal utility as a *convex combination* of the instantaneous utility from current consumption, $u(f(x_t) - x_{t+1})$, and the utility derived from the future consumption path, with the sum of the weights equal to 1 — even when the recursion extends to infinity. Several recent studies (e.g., Wakai, 2008; Chambers and Echenique, 2018; Drugeon et al., 2019; Drugeon and Ha-Huy, 2022), axiomatizing intertemporal choice, put forward the argument that when a decision maker chooses among multiple discount factors, any relevant criterion for evaluating intertemporal utility streams should leave the sum of discount factors equal 1.

Note that when the discount factor δ is constant, criterion (1) reduces to the in-

[1] Lawrance (1991) used U. S. panel data to show that subjective discount rates are 3–5 p.p. higher for households with low permanent incomes than for those with high permanent incomes. Samwick (1998) estimated the distribution of discount rates from U. S. wealth data and found that discount rates decline with income. Harrison et al. (2002) reported results from a field experiment in Denmark confirming that higher-income individuals have significantly lower discount rates. Dohmen et al. (2010), in a laboratory experiment in Germany, found that, controlling for other important characteristics, the higher the income, the lower the experimental measure of impatience.

tertemporal utility function of the standard Ramsey model:

$$W(x_0, x_1, \ldots) \;=\; (1-\delta)\sum_{t=0}^{\infty} \delta^t u(f(x_t) - x_{t+1})\,.$$

In this case, the term $(1-\delta)$ can be ignored, and the sum of discount factors has no effect on the optimal path. By contrast, under endogenous discounting, as we show below, a system of weights summing to 1 plays a crucial role, ensuring that optimal paths remain unchanged when a constant is added to u. This is fundamentally different to existing endogenous discounting models, which do not employ axiomatically relevant normalized discount factors and in which both the level and the sign of u have a significant impact on the optimal path.

For the optimal growth model with the objective function given by (1), we prove the existence of an optimal path starting from any initial capital stock. We show that every optimal path satisfies the Euler equations, which are invariant to affine transformations of the instantaneous utility function.

We characterize the value function and show that it is the unique solution to the Bellman functional equation, finite, continuous, and differentiable almost everywhere. Moreover, the (at most countable) set of initial capital stocks from which the optimal path is *not unique* coincides exactly with the set of points where the value function is not differentiable. We further note that, once initiated, all optimal paths are unique: the continuation of every optimal path (its tail starting from period 1) is uniquely determined.

We prove that the optimal policy correspondence is monotone, which implies that optimal paths are monotonic. Unless starting from a steady state, an optimal path is either strictly increasing and converges to a positive steady state or strictly decreasing and converges to either the origin or a positive steady state.

The monotonicity of optimal paths allows us to study the possibility of poverty traps in economies with endogenous discounting. A poverty trap corresponds to the existence of a critical capital level such that, if the initial capital stock is below this threshold, the optimal path converges to the origin, while if the initial capital stock is above the threshold, the optimal path converges to a positive steady state.

We show that when discounting depends only on future capital, poverty traps cannot arise: either every optimal path converges to the origin, or every optimal path converges to a unique positive steady state. By contrast, when discounting depends on current capital or consumption, we demonstrate that in economies with isoelastic utility and Cobb–Douglas technology, a critical level exists for a positive measure set of parameter values. Thus, our results suggest that poverty traps are far from rare under endogenous discounting.

This paper contributes to several strands of economic literature. First, it advances the analysis of poverty traps. The literature identifies several mechanisms behind poverty traps in growth models, broadly categorized as *technology-related* or *preference-related*. A classical production-side explanation involves non-convex technology (e.g., Dechert and Nishimura, 1983; Akao et al., 2025). In particular, an S-shaped (convex-concave) production function naturally generates multiple steady states and path-dependent dynamics. At small capital stocks, marginal productivity of capital is too low to sustain accumulation, leading to convergence to the origin or a low-level steady state. However, beyond a certain threshold, increasing returns in the convex region foster growth until diminishing returns in the concave region dominate, stabilizing the economy at a high-level steady state.

Among consumption-side explanations, Galor and Weil (2000) (see also Galor, 2005) demonstrate that non-convexities in preferences (the presence of a subsistence consumption level) combined with endogenous fertility (quality-quantity trade-off in fertility decisions) generate multiple growth paths. This unified growth theory accounts for both the Malthusian trap in early stages of development and the transition to sustained economic growth in modern economies.

In this paper, we study an entirely different mechanism and relate the existence of poverty traps to the endogeneity of time preference. We highlight the role of discounting and show that multiple equilibria, including a poverty trap, arise in a one-sector optimal growth model under fairly standard assumptions: a strictly concave production function and a strictly concave objective function.

Second, this paper contributes to the analysis of optimal growth under endogenous time preference. The literature generally distinguishes two main approaches, depending on the source of endogeneity: the discount factor depends either on consumption or on the capital stock.

The setting in which discounting depends on consumption captures the idea of complementarity between successive consumption flows. Uzawa (1968); Epstein and Hynes (1983); Obstfeld (1990) assume that the discount factor decreases with consumption, meaning that individuals become more impatient as they consume more — a property known as *increasing marginal impatience*.[2] A general result under increasing marginal impatience is the existence of a unique steady state that is saddle-path stable, implying that the dynamics of the model with endogenous discounting resemble those of a standard Ramsey model with constant discounting.

In contrast, Das (2003) and Chakrabarty (2012) argue that *decreasing marginal impatience*, whereby individuals with higher consumption levels are more patient, is intuitively more appealing. They show that decreasing marginal impatience leads to multiple steady states without necessarily precluding stability, contrary to earlier beliefs. Notably, when discounting depends on consumption, regardless of the direction of marginal impatience, the instantaneous utility function appears in the optimal paths of consumption and capital.

An alternative approach considers discounting that depends on the capital stock, reflecting the role of wealth in shaping patience. In this case, it is common to assume that the discount factor increases with capital: individuals become more patient as they grow richer. Schumacher (2009); Erol et al. (2011); Strulik (2012); Bouché (2017) study endogenous time preference depending on capital under different assumptions about technology.[3] They find that multiple steady states generically exist, with the lowest one typically interpreted as a steady state of stagnation.

Importantly, in this class of models, both optimal paths and steady states are directly affected by the level and, in particular, the sign of the instantaneous utility function. This feature gives rise to various peculiarities — for example, a negative instantaneous utility function would imply that a higher capital stock reduces total welfare, a result that lacks economic interpretation (see also Schumacher, 2011).

Our paper differs substantially from previous contributions. We consider a general form of endogenous discounting that captures the dependence of the discount factor on both consumption and the capital stock, allowing us to analyze both approaches within a single framework. Moreover, we normalize the sum of discount factors to one, which renders optimal paths and steady states independent of the level of the instantaneous utility function, making our analysis more flexible and robust.

[2] See also Epstein (1987) and Drugeon (1996, 2000).

[3] See also Borissov (2013) for an equilibrium model where agents' discount factors are increasing functions of their relative wealth, and Camacho et al. (2013) for a strategic growth model where agents receive a share of total income proportional to their wealth.

The paper is organized as follows. Section 2 introduces the model, focusing on the existence of an optimal path as well as the properties of the value function and the optimal policy correspondence. Section 3 characterizes optimal paths and establishes their monotonicity. Section 4 studies poverty traps and provides examples of the existence of a critical level of capital for specific discount functions. Section 5 concludes. All proofs are gathered in the Appendix.

2. An economy with endogenous discounting

In this section, we introduce the central object of our study, an optimal growth model with endogenous discounting, and provide some preliminary results. We describe the primitives of the model and discuss the assumptions on preferences, technology and the discount function. We prove the existence of an optimal path and characterize the properties of the value function and the optimal policy correspondence.

2.1 Fundamentals

Throughout the paper, we study the following optimization problem: given $x_0 > 0$,

$$\max_{\{c_t, x_{t+1}\}_{t=0}^{\infty}} \left[\big[1-\delta(x_0,x_1)\big]u(c_0) + \sum_{t=1}^{\infty}\left(\big[1-\delta(x_t,x_{t+1})\big]u(c_t)\prod_{s=0}^{t-1}\delta(x_s,x_{s+1}) \right) \right], \tag{2}$$
$$\text{s. t.} \quad c_t + x_{t+1} \le f(x_t), \quad c_t \ge 0, \quad x_t \ge 0, \quad t \ge 0,$$

where $f(x)$ is the neoclassical production function in intensive form, $u(c)$ is the instantaneous utility function, and $\delta(x,y)$ is a general form of the discount function.

Let us discuss the structure of the problem and the assumptions on the primitives. As in the classic one-sector optimal growth model, we assume that capital fully depreciates each period, and the production function f is strictly increasing and strictly concave:

(P1) *$f : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, with $f(0) = 0$, $f'(x) > 0$, $f''(x) < 0$, and $\lim_{x\to+\infty} f'(x) < 1$.*

The instantaneous utility function of consumption in each period, u, is strictly concave, satisfies the Inada condition, and is bounded from below:

(U1) *$u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, with $u'(c) > 0$, $u''(c) < 0$, and $u'(0) = +\infty$.*

(U2) $\lim_{c \to 0} u(c) > -\infty$.

While condition (U1) is standard in optimal growth models, condition (U2) is specific to endogenous discounting models. Two points are important here. Claim A.1 in Appendix A shows that boundedness from below of the instantaneous utility function is a necessary condition for the objective function in (2) to satisfy the *Pareto property*. Intuitively, if $u(c)$ is not bounded from below, the objective function in (2) may assign a higher value to a consumption path with lower consumption values in each period. To avoid this unreasonable situation, we assume that $u(0)$ is finite.

Moreover, Claim A.2 in Appendix A shows that the specific structure of the objective function, namely, its form as a convex combination of the instantaneous utility and the future intertemporal utility (that is, the sum of discount factors equals 1), makes the solution to problem (2) independent of $u(0)$. Thus, unlike existing endogenous discounting models, we do not require instantaneous utility to be restricted to positive values.[4]

Finally, the discount function $\delta(x, y)$ depends on two variables, current capital x and future capital y. This general form encompasses the three most relevant specifications in the literature. When the discount function is defined as $\delta(x, y) = \delta^b(f(x) - y)$, discounting depends only on consumption. Similarly, if $\delta(x, y) = \delta^p(x)$ or $\delta(x, y) = \delta^f(y)$, endogenous discounting depends only on current or future capital, respectively. We have these natural specifications in mind while retaining the most general functional form of the discount factor.

We assume that the discount function is strictly positive for positive consumption levels, non-decreases with current capital, non-increases with future capital and its cross-derivative is non-negative:[5]

(D1) $\delta : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, 1)$ *is such that* $0 < \delta(x, y) < 1$ *for all* $0 < y < f(x)$.

(D2) $\delta(x, y)$ *is twice continuously differentiable, with* $\delta_1(x, y) \geq 0$ *and* $\delta_2(x, y) \leq 0$.

(D3) $\delta_{12}(x, y) \geq 0$.

Condition (D1) naturally restricts the discount factor to be positive unless the cap-

[4] Endogenous discounting models often assume $u(0) \geq 0$, both in continuous time (Drugeon, 1996; Das, 2003; Schumacher, 2009; Bouché, 2017), and in discrete time (Erol et al., 2011; Chakrabarty, 2012). However, in these models, the specific choice of a positive instantaneous utility function affects the results, whereas in our case, redefining utility as $u(c) - u(0)$ makes no difference.

[5] We denote the partial derivative with respect to the first (second) variable by the subscript 1 (2).

ital stock is zero. Condition (D2) reflects the usual intuition that patience increases with income and wealth: the discount factor is higher when current consumption or capital is greater relative to future capital. This assumption is in line with empirical evidence and is widely used in the literature (see the discussion in the Introduction).

Condition (D3) is also not restrictive. It is trivially satisfied when the discount factor depends only on current or future capital, since the cross derivatives are zero. When δ depends only on consumption, it is easy to verify that $\delta_{12}(x, y) \geq 0$ is equivalent to the concavity of $\delta^b\left(f(x) - y\right)$. Note that (D1)–(D3) always hold when δ is constant.

Furthermore, we require additional conditions relating the instantaneous utility function to the discount function. Define the function ψ as:

$$\psi(x, y) \equiv \left[1 - \delta(x, y)\right]\left[u\big(f(x) - y\big) - u(0)\right].$$

We assume that $\psi(x, y)$ is strictly supermodular:[6]

(UD1) $\psi_{12}(x, y) > 0$ *for all* $0 < y < f(x)$.

(UD2) *The function* $\psi(x, 0)$ *strictly increases with* x.

Claim A.3 in Appendix A shows that these conditions are neither overly restrictive nor counterintuitive. Condition (UD1) holds automatically if δ depends only on current or future capital, since the cross derivatives are zero. If $\delta(x, y) = \delta^b(c)$ with $c = f(x) - y$, (UD1) is equivalent to the concavity of $\left[1 - \delta^b(c)\right]\left[u(c) - u(0)\right]$, which ensures that the objective function in (2) is strictly concave in consumption in each period. When $u(c)$ is bounded, condition (UD2) is equivalent to the Pareto property.

2.2 Existence of an optimal path

In what follows, we assume that (P1), (U1)–(U2), (D1)–(D3), (UD1)–(UD2) hold. Given the initial condition $x_0 > 0$, a capital path $\mathbf{x} = \{x_t\}_{t=0}^{\infty}$ is said to be *feasible* from x_0 if $0 \leq x_{t+1} \leq f(x_t)$ for any t. Let $\Pi(x_0)$ be the set of all capital paths feasible from x_0. Due to increasing monotonicity of $f(x)$, if $x_0 < x_0'$, then $\Pi(x_0) \subset \Pi(x_0')$. A consumption path $\mathbf{c} = \{c_t\}_{t=0}^{\infty}$ is feasible from x_0 if there exists $\mathbf{x} \in \Pi(x_0)$ such that $0 \leq c_t \leq f(x_t) - x_{t+1}$ for any t.

[6] The strict supermodularity means that for $x < x'$ and $y < y'$ such that $0 < y < f(x)$ and $0 < y' < f(x')$, we have $\psi(x, y) + \psi(x', y') > \psi(x, y') + \psi(x', y)$. A sufficient condition for supermodularity is that the cross derivative is positive. See Amir (1996, 2005) for more details.

By (U1) and (D1), constraints in problem (2) are binding at the optimum. Introduce the function W defined on the set of all feasible sequences as

$$W(\mathbf{x}) = \big[1-\delta(x_0,x_1)\big]u(f(x_0)-x_1)+\sum_{t=1}^{\infty}\left(\big[1-\delta(x_t,x_{t+1})\big]u(f(x_t)-x_{t+1})\prod_{s=0}^{t-1}\delta(x_s,x_{s+1})\right).$$

Then the problem (2) is equivalent to the following optimization problem:

$$\max_{\mathbf{x}\in\Pi(x_0)} W(\mathbf{x}).$$

An *optimal path* from x_0 is any capital path $\mathbf{x}$ that solves the above problem.

Let x^M be the solution to $f(x) = x$. For any $\mathbf{x} \in \Pi(x_0)$, we have $x_t \leq \max\{x_0, x^M\}$. A direct application of Tychonov's Theorem (see, among others, Stokey et al., 1989; Le Van and Dana, 2003) implies that $\Pi(x_0)$ is compact in the product topology defined on the space of sequences $\mathbf{x}$, and that $W(\mathbf{x})$ is well-defined and continuous over $\Pi(x_0)$ with respect to the product topology. These observations ensure the existence of an optimal path. The positivity of the optimal consumption and capital paths follows from (D1) and the Inada condition. The following proposition summarizes the above discussion.

Proposition 2.1. (i) *There exists an optimal path* $\mathbf{x}$ *from any* $x_0 > 0$. *The associated optimal consumption path* $\mathbf{c}$ *is given by* $c_t = f(x_t) - x_{t+1}$, $\forall t$.

(ii) *If* $\mathbf{x}$ *is an optimal path and* $\mathbf{c}$ *is the associated optimal consumption path, then* $c_t > 0$ *and* $x_t > 0$, $\forall t$.

Proposition 2.1 allows us to study the optimal path via the corresponding Euler equations.

2.3 Value function and optimal policy correspondence

For $x_0 > 0$, the value function V is defined by

$$V(x_0) \equiv \max_{\mathbf{x}\in\Pi(x_0)} W(\mathbf{x}).$$

The value function is finite because of (U2), (D1) and the existence of a maximum sustainable capital stock. Under these conditions, the value function V is continuous and is the unique solution to the Bellman functional equation. Applying the argument from Proposition 3.4.1 of Le Van and Dana (2003), we arrive at the following proposition.

Proposition 2.2. (i) *The value function V is continuous and strictly increasing, satisfying the Bellman equation: for all $x_0 > 0$,*

$$V(x_0) \;=\; \max_{0\le y\le f(x_0)} \Big\{[1-\delta(x_0,y)]u\left(f(x_0)-y\right)+\delta(x_0,y)V(y)\Big\}.$$

(ii) *The value function V is the unique solution to the Bellman equation which is continuous, strictly increasing, and satisfies the transversality condition:*

$$\lim_{T\to\infty}\prod_{s=0}^{T-1}\delta(x_s,x_{s+1})V(x_T) \;=\; 0\,, \qquad \forall x_0>0, \quad \forall \mathbf{x}\in\Pi(x_0)\,.$$

(iii) *A sequence $\mathbf{x}\in\Pi(x_0)$ is an optimal path if and only if for all t it satisfies*

$$V(x_t) \;=\; \big[1-\delta(x_t,x_{t+1})\big]u(f(x_t)-x_{t+1})+\delta(x_t,x_{t+1})V(x_{t+1})\,.$$

Proposition 2.2 implies that every optimal path is *time-consistent*. However, an optimal path from x_0 is not necessarily unique. The optimal policy $\varphi:\mathbb{R}_+\to\mathbb{R}_+$, defined as

$$\varphi(x_0) \;=\; \operatorname*{argmax}_{0\le y\le f(x_0)} \Big\{[1-\delta(x_0,y)]u(f(x_0)-y)+\delta(x_0,y)V(y)\Big\},$$

is, in general, a correspondence rather than a function: for a given x_0, $\varphi(x_0)$ may be a set rather than a singleton.

The non-emptiness and closedness of the optimal policy correspondence, as well as its equivalence to the optimal path, follow directly from the continuity of the value function via the Maximum Theorem. Furthermore, φ is *monotone*, which is key to studying the monotonicity of optimal paths. The following proposition summarizes the main properties of φ.

Proposition 2.3. (i) *For any $x_0>0$, if $x_1\in\varphi(x_0)$, then $0<x_1<f(x_0)$.*

(ii) *φ is closed and upper hemicontinuous.*

(iii) *A sequence $\mathbf{x}\in\Pi(x_0)$ is the optimal path if and only if $x_{t+1}\in\varphi(x_t)$, $\forall t$.*

(iv) *If $y\in\varphi(x)$ and $y'\in\varphi(x')$ with $x<x'$, then $y\le y'$.*

By (UD1), the function $-\delta_1(x_0,x_1)u\big(f(x_0)-x_1\big)+\big[1-\delta(x_0,x_1)\big]u'\big(f(x_0)-x_1\big)f'(x_0)$ increases with x_1. Taking this into account and applying the argument from the proof of Theorem 6 in Dechert and Nishimura (1983), we obtain that the left derivative of V is given by the minimal value in the set $\varphi(x_0)$, while the right derivative is given by the maximal one. The following proposition characterizes the left and right derivatives of the value function.

PROPOSITION **2.4.** *The value function V has left and right derivatives at any $x_0 > 0$.*

(i) *The left derivative of V is defined as:*

$$V'_-(x_0) = -\delta_1(x_0, x_1)u\big(f(x_0) - x_1\big) + \big[1 - \delta(x_0, x_1)\big]u'\big(f(x_0) - x_1\big)f'(x_0),$$

where $x_1 = \min\{x : x \in \varphi(x_0)\}$.

(ii) *The right derivative of V is defined as:*

$$V'_+(x_0) = -\delta_1(x_0, x_1)u\big(f(x_0) - x_1\big) + \big[1 - \delta(x_0, x_1)\big]u'\big(f(x_0) - x_1\big)f'(x_0),$$

where $x_1 = \max\{x : x \in \varphi(x_0)\}$.

Proposition 2.4 provides the relationship between the differentiability of the value function and the uniqueness of the optimal path. Given x_0, $\varphi(x_0)$ is single-valued if and only if $V'_-(x_0) = V'_+(x_0)$, so that V is differentiable at x_0.

3. MONOTONICITY OF OPTIMAL PATHS

In this section, we characterize optimal paths in an economy with endogenous discounting. We show that there are only three possibilities: (*i*) the optimal path is constant over time; (*ii*) it is strictly increasing and converges to a positive steady state; (*iii*) it is strictly decreasing and converges to either the origin or a positive steady state. We provide specific conditions in terms of the primitives of the model under which it is optimal for the economy to accumulate or reduce its capital stock.

We begin by deriving the Euler equations. By Proposition 2.1, $0 < x_{t+1} < f(x_t)$ for any $t \geq 0$. Equating $\partial W(\mathbf{x})/\partial x_{t+1}$ to zero, we obtain the following proposition.

PROPOSITION **3.1.** *Every optimal path $\mathbf{x} \in \Pi(x_0)$ satisfies the Euler equations: for* *all* $t \geq 0$,

$$\begin{aligned}
&\big[1-\delta(x_t,x_{t+1})\big]u'(f(x_t)-x_{t+1})+\delta_2(x_t,x_{t+1})u(f(x_t)-x_{t+1})\\
&\ = \ \delta(x_t,x_{t+1})\big[1-\delta(x_{t+1},x_{t+2})\big]u'(f(x_{t+1})-x_{t+2})f'(x_{t+1})\\
&-\delta(x_t,x_{t+1})\delta_1(x_{t+1},x_{t+2})u(f(x_{t+1})-x_{t+2})\\
&+\delta_2(x_t,x_{t+1})V(x_{t+1})+\delta(x_t,x_{t+1})\delta_1(x_{t+1},x_{t+2})V(x_{t+2}).
\end{aligned} \tag{3}$$

Although (3) involves the instantaneous utility function u and the value function V, it remains unchanged if a constant is added to u. Thus, unlike other endogenous discounting models, regardless of whether the discount factor depends on consumption (as in, e.g., Das, 2003; Chakrabarty, 2012) or on the capital stock (as in, e.g., Erol et al., 2011), in our case the value of $u(0)$ does not affect the optimal path.

It can be checked that the right-hand side of (3) is strictly increasing in x_{t+2}. Therefore, along each optimal path, starting from period 1, the subsequent optimal path is unique. Moreover, by Proposition 2.4, the value function is differentiable at these capital levels. Using the same arguments as in Dechert and Nishimura (1983), we conclude that the value function is differentiable almost everywhere. Hence the set of initial capital stocks from which the optimal path is *not unique* is at most countable. The following proposition summarizes the above discussion.

Proposition **3.2.** (i) *For every optimal path* $\mathbf{x} = \{x_t\}_{t=0}^{\infty}$*, the value function* V *is differentiable at* x_t *for* $t \geq 1$*, and there exists a unique optimal path from* x_t*.*

(ii) V *is differentiable at* $x_0 > 0$ *if and only if there exists a unique optimal path from* x_0*.*

Consider now the long-run outcomes of the economy. We call $x^* \geq 0$ a *steady state* if the sequence $\{x_t\}_{t=0}^{\infty}$, where $x_t = x^*$ for all t, is an optimal path. Note that the origin, $x^* = 0$, is always a (trivial) steady state. Proposition 3.1 allows us to characterize non-trivial steady states. The following proposition provides a necessary condition for the existence of positive steady states. Moreover, it maintains that when the economy starts from a steady state, it never jumps out of it, and when the economy does not start from a steady state, it never reaches one in a finite number of periods.

Proposition **3.3.** *Let* $x^* > 0$ *be a positive steady state.*

(i) *We have* $\delta(x^*, x^*)f'(x^*) = 1$*, and* $\varphi(x^*) = \{x^*\}$*.*

(ii) *For every optimal path* $\mathbf{x} = \{x_t\}_{t=0}^{\infty}$, *if* $x_0 \neq x^*$, *then* $x_t \neq x^*$ *for any* $t \geq 0$.

The following theorem, building on the monotonicity of the optimal policy correspondence from Proposition 2.3, proves the convergence result: if an optimal path does not start from a steady state, it converges monotonically to some steady state.

THEOREM **3.1.** *Every optimal path is either constant or strictly monotonic and converges to a steady state.*

Suppose that the economy does not start from a steady state. Given an initial capital stock x_0, how do we determine whether it is optimal for the economy to accumulate capital during the transition or decrease the existing stock? The following proposition provides simple sufficient conditions for identification of increasing and decreasing optimal paths.

PROPOSITION **3.4.** (i) *If* $\liminf_{x\to 0} \delta(x,x)f'(x) > 1$, *then every optimal path from* $x_0 > 0$ *converges to a positive steady state. Moreover, for all sufficiently small* x_0, *the optimal paths are strictly increasing.*

(ii) *If there is* $\hat{x} > 0$ *such that* $\delta(x,x)f'(x) < 1$ *for all* $x > \hat{x}$, *then every optimal path from* $x_0 > \hat{x}$ *is strictly decreasing.*

Intuitively, in the optimal growth model with endogenous discounting, at each level of capital there are two competing effects: the return on savings and the discount factor. A higher marginal productivity of capital, $f'(x)$, increases the incentive to save, while a lower discount factor, $\delta(x,y)$, reduces it. The interaction between these two effects determines the monotonicity of optimal paths.

Part (i) of Proposition 3.4 implies that if, for small values of the capital stock, productivity exceeds the efficient rate of time preference, then saving is optimal and the optimal paths are increasing. Part (ii) of Proposition 3.4 states that once the economy has accumulated sufficient capital and productivity falls below the rate of time preference, saving is no longer profitable and the optimal paths are decreasing.

In the general case, when Proposition 3.4 does not apply, the following theorem presents two mutually exclusive integral conditions.

THEOREM **3.2.** *Suppose that* $x_0 > 0$ *is not a steady state. Let* x^* *either be* 0 *or satisfy* $\delta(x^*,x^*)f'(x^*) = 1$.

(i) *Suppose that for all $x^* > x_0$ we have*

$$\int_{x_0}^{x^*} \big[\delta(y,y)f'(y) - 1\big] u'\big(f(y) - y\big) dy \ < \ 0 \,. \tag{4}$$

Then every optimal path from x_0 is strictly decreasing.

(ii) *Suppose that for all $x^* < x_0$ we have*

$$\int_{x^*}^{x_0} \big[\delta(y,y)f'(y) - 1\big] u'\big(f(y) - y\big) dy \ > \ 0 \,. \tag{5}$$

Then every optimal path from x_0 is strictly increasing.

Theorem 3.2 deserves several comments. First, let us provide the intuition behind conditions (4) and (5). Suppose a decision maker endowed with a capital stock y, where $0 < y < f(y)$, faces the following choice: remain at y or increase the capital stock by a small amount. The constant path $(y, y, \ldots)$ yields intertemporal utility $u\big(f(y) - y\big)$. If the decision maker invests $\epsilon > 0$, the resulting capital path becomes $(y, y + \epsilon, y + \epsilon, \ldots)$. The difference in intertemporal utility is given by

$$\begin{aligned}
\Delta W(\epsilon) \ &\equiv \ W(y, y+\epsilon, y+\epsilon, \ldots) - W(y, y, y, \ldots) \\
&= \ \big[1 - \delta(y, y+\epsilon)\big] u\big(f(y) - y - \epsilon\big) + \delta(y, y+\epsilon) u\big(f(y+\epsilon) - y - \epsilon\big) \\
&\quad - \big[1 - \delta(y,y)\big] u\big(f(y) - y\big) - \delta(y,y) u\big(f(y) - y\big) \,.
\end{aligned}$$

Taking the limit as ϵ tends to zero and simplifying, we get

$$\lim_{\epsilon \to 0} \frac{\Delta W(\epsilon)}{\epsilon} \ = \ \big[\delta(y,y)f'(y) - 1\big] u'\big(f(y) - y\big) \,.$$

Thus, $\big[\delta(y,y)f'(y) - 1\big] u'\big(f(y) - y\big)$ is the marginal difference between increasing capital in the future and maintaining the current capital stock. It can be interpreted as a form of net gain measured in terms of utility. If an increase in capital creates a negative net gain, the decision maker prefers to reduce investment, and the optimal path is decreasing. Conversely, if a decrease in capital produces a negative net gain, the decision maker prefers to increase investment, and the optimal path is increasing.

Second, Theorem 3.2 and Proposition 3.4 provide useful tools to analyze whether the economy starting from some x_0 optimally increases or decreases capital stock. As an illustration, consider an example where the function $\delta(x,x)f'(x)$ is single-peaked and the equation $\delta(x,x)f'(x) = 1$ has two positive solutions, $x_s < x^s$.[7] Then $\delta(x,x)f'(x) > 1$ in the interval (x_s, x^s), and $\delta(x,x)f'(x) < 1$ for $x < x_s$ and $x > x^s$.

[7] It is possible to have a continuum of solutions: for example, when $\delta(x,x) = 1/f'(x)$ in some interval, as in the example of Erol et al. (2011). However, this situation only occurs for a set of fundamentals of measure zero.

Suppose that $x_0 < x_s$. Then condition (4) trivially holds for $x^* = x_s$. If additionally condition (4) holds for $x^* = x^s$, then every optimal path starting from $x_0 \in (0, x_s)$ is strictly decreasing and converges to the origin. Similarly, suppose that $x_s < x_0 < x^s$. Then condition (5) trivially holds for $x^* = x_s$. If additionally condition (5) holds for $x^* = 0$, then every optimal path starting from $x_0 \in (x_s, x^s)$ is strictly increasing and converges to x^s. Also, for $x_0 > x^s$, part *(ii)* of Proposition 3.4 implies that every optimal path starting from $x_0 \in (x^s, +\infty)$ is strictly decreasing and converges to x^s.

Third, it follows from Theorem 3.2 that, under certain conditions, the endogenous discounting model parallels well-known findings from the optimal growth model with a constant discount factor.

COROLLARY 3.1. *Let* $\overline{\delta} = \sup_{x \geq 0} \delta(x, x)$ *and* $\underline{\delta} = \inf_{x \geq 0} \delta(x, x)$.

(i) *If* $\overline{\delta} f'(0) < 1$, *then every optimal path converges to the origin.*

(ii) *If* $\underline{\delta} f'(0) > 1$, *then every optimal path converges to a positive steady state.*

(iii) *If* $\delta(x, y) = \delta^f(y)$, *then either every optimal path converges to the origin, or every optimal path converges to a unique positive steady state.*

Part *(i)* of Corollary 3.1 implies that when productivity is always lower than the rate of time preference, it is never optimal to save. Since $\delta(y, y) f'(y) < 1$ for all y, condition (4) holds for all x_0. Conversely, inequality in part *(ii)* of Corollary 3.1 ensures that it is always optimal to save. Then $\delta(y, y) f'(y) > 1$ for all y, and condition (5) holds for all x_0.

Part *(iii)* of Corollary 3.1 maintains that when discounting depends only on future capital, the model with endogenous discounting reproduces the dynamic properties of the standard Ramsey model. This result sharply contrasts with Erol et al. (2011), who study discount factor as a function of future capital (in our terms) and provide a numerical example in which two optimal steady states with local convergence exist.

The crucial difference lies in the assumption about the relationship between patience and wealth. Erol et al. (2011) assume that the discount factor *increases* with future capital, so that the decision maker becomes more patient when expecting higher future wealth. By contrast, our condition (D2) implies that the higher the future capital, the lower the discount factor, which aligns better with the usual intuition.

4. Poverty traps

Finally, we apply our results to study poverty traps under endogenous discounting. In this section, we define a critical level of capital and provide analytical condition for the existence of a poverty trap in a given economy. We also construct several examples to illustrate the existence of a critical level for specific discount functions.

4.1 Critical level: theory

A *poverty trap*, sometimes called a development trap, is a situation in which, for some low initial capital stocks, the optimal capital path is decreasing and converges to the origin, while for other initial capital stocks, it converges to a positive steady state. The threshold that separates two different long-run outcomes in the economy is known as the *critical level.* Formally, we call $x^C > 0$ a *critical level* of capital if, for all $x_0 < x^C$, any optimal path from x_0 converges to the origin, while for all $x_0 > x^C$, any optimal path from x_0 remains bounded away from zero.

The monotonicity of the optimal policy correspondence φ guarantees that, if a critical level x^C exists, it is unique. The following proposition characterizes the existence of the critical level based on the monotonicity of optimal paths.

Proposition **4.1.** *There exists a critical level $x^C > 0$ if and only if there exist an optimal path converging to the origin and an optimal path converging to a positive steady state.*

Intuitively, Proposition 4.1 states that for the existence of a critical level we need two capital stocks, $x_0' > x_0 > 0$, such that the optimal path from x_0 is decreasing and converges to the origin, while the optimal path from x_0' is increasing.

It is tempting to apply Theorem 3.2 to determine whether an economy with endogenous discounting has a critical level. At first glance, it might seem that condition (4) guarantees an optimal path converging to the origin, while condition (5) ensures an optimal path converging to a positive steady state. One might then conclude that if both conditions hold, a critical level exists. However, the problem is more subtle than this intuition suggests.

Consider again the example where the equation $\delta(x,x)f'(x) = 1$ has exactly two solutions, $x_s < x^s$. Both values x_s and x^s are candidates for non-trivial steady states. As we have seen, an optimal path from $x_0 < x_s$ is decreasing and converges to the origin if condition (4) holds for $x^* = x^s$, that is,

$$\int_{x_0}^{x^s} \big[\delta(y,y)f'(y) - 1\big]u'\big(f(y) - y\big)dy \;<\; 0\,.$$

Since $\delta(y,y)f'(y) < 1$ for all $y < x_s$, we have

$$\int_{0}^{x^s} \big[\delta(y,y)f'(y) - 1\big]u'\big(f(y) - y\big)dy \;<\; 0\,.$$

However, since $\delta(y,y)f'(y) > 1$ for (x_s, x^s), it follows that for all $x_0 > 0$,

$$\int_{0}^{x_0} \big[\delta(y,y)f'(y) - 1\big]u'\big(f(y) - y\big)dy \;<\; 0\,,$$

which is exactly the opposite of condition (5) for $x^* = 0$. Thus, it is impossible to deduce whether there exists an increasing optimal path. In other words, the conditions in both parts of Theorem 3.2 cannot hold simultaneously.

Because of this difficulty, a general condition for the existence of a critical level in an economy with endogenous discounting is not straightforward. However, in what follows we provide several examples in which critical levels exist. Moreover, our examples are derived for a set of parameters with positive measure, suggesting that critical levels are not rare in this class of economies.

4.2 Example: Discount function of consumption

Consider the case where discounting depends on consumption, $\delta(x,y) = \delta^b(c)$, with $c = f(x) - y$, and suppose that the discount factor is explicitly defined as:

$$\delta^b(c) \;\equiv\; \frac{c^\theta}{b + c^\theta}\,,$$

where $b > 0$ is a constant whose value will be specified later.

Consider an economy with isoelastic utility $u(c) = c^\rho$ and Cobb–Douglas technology $f(x) = Ax^\alpha$, with $\alpha, \rho \in (0,1)$. We assume that the parameters α, ρ and θ are such that $(1-\alpha)/\alpha < \theta < \min\{\rho, 1-\rho\}$. Observe that $u(0) = 0$. Although the Cobb–Douglas production function has infinite marginal productivity at zero capital, which creates a very strong incentive to accumulate capital when it is scarce, we show that a critical level x^C nevertheless exists.

It is easy to verify that $\delta^b(c)$ is increasing and concave, so that conditions (D1)–(D3) are satisfied. Furthermore, Appendix E shows that the function $\big[1 - \delta^b(c)\big]u(c)$ is increasing and concave in c, implying that conditions (UD1)–(UD2) hold.

We have

$$\delta^b\big(f(x)-x\big)f'(x) \;=\; \frac{\alpha A x^{\alpha-1}\big(Ax^{\alpha}-x\big)^{\theta}}{b+\big(Ax^{\alpha}-x\big)^{\theta}} \;=\; \frac{\alpha A x^{\alpha(1+\theta)-1}\big(A-x^{1-\alpha}\big)^{\theta}}{b+\big(Ax^{\alpha}-x\big)^{\theta}}\,.$$

Consider the function $\zeta(x) = \big(\alpha A x^{\alpha-1}-1\big)\big(Ax^{\alpha}-x\big)^{\theta}$. Then the equation for positive steady states, $\delta^b\left(f(x)-x\right)f'(x)=1$, is equivalent to $\zeta(x)=b$.

Let $x^G=(\alpha A)^{1/(1-\alpha)}$ be the solution to $f'(x)=1$. Since $\alpha(1+\theta)>1$, we have $\zeta(0)=\zeta(x^G)=0$. The following claim (see Appendix E) shows that ζ is single-peaked in the interval $\big[0,x^G\big]$. Thus, for an appropriate choice of the parameter b, the equation $\delta^b\big(f(x)-x\big)f'(x)=1$ has two solutions, meaning that there are exactly two candidates for the positive steady state.

Claim **4.1.** *There exists $\hat{x}\in\big(0,x^G\big)$ such that ζ is increasing in $[0,\hat{x})$ and decreasing in $\big(\hat{x},x^G\big]$.*

Denote the maximum of function ζ in $[0,x^G]$ by $\zeta^*=\zeta(\hat{x})$. If $b>\zeta^*$, then $\delta^b\left(f(x)-x\right)f'(x)<1$ in the interval $\big(0,x^G\big)$. By Proposition 3.4, every optimal path is decreasing and, thus, converges to zero.

If $0<b<\zeta^*$, the equation $\delta^b\left(f(x)-x\right)f'(x)=1$ has exactly two solutions in $\big(0,x^G\big)$. Let $x^s(b)$ be the largest solution and $x_s(b)$ the smallest. Clearly, $x_s(b)<\hat{x}<x^s(b)$ and the two solutions converge to $\hat{x}$ when b tends to ζ^*. The single-peakedness property also implies that, as $x_s(b)$ increases with b, $x^s(b)$ decreases.

It is easy to verify that $\delta^b\big(f(y)-y\big)f'(y)<1$ if $y<x_s(b)$, and $\delta^b\big(f(y)-y\big)f'(y)>1$ if $x_s(b)<y<x^s(b)$. Therefore, for any $b\in(0,\zeta^*)$,

$$\int_0^{x_s(b)}\big[\delta^b(f(y)-y)f'(y)-1\big]u'\big(f(y)-y\big)dy \;<\; 0\,.$$

Let

$$\begin{aligned}\phi(b) \;&\equiv\; \int_0^{x^s(b)}\big[\delta^b(f(y)-y)f'(y)-1\big]u'\big(f(y)-y\big)dy\\ &=\int_0^{x_s(b)}\big[\delta^b(f(y)-y)f'(y)-1\big]u'\big(f(y)-y\big)dy\\ &\quad+\int_{x_s(b)}^{x^s(b)}\big[\delta^b(f(y)-y)f'(y)-1\big]u'\big(f(y)-y\big)dy\,.\end{aligned}$$

Since both $\delta^b(f(y)-y)$ and $x^s(b)$ decrease with b, while $x_s(b)$ increases with b, the function ϕ decreases with b.

When b goes to ζ^*, both $x_s(b)$ and $x^s(b)$ converge to $\hat{x}$. Then, $\lim_{b\to\zeta^*}\phi(b) < 0$. Moreover, since $\lim_{b\to 0} x^s(b) = x^G$,

$$\lim_{b\to 0}\phi(b) \;=\; \int_0^{x^G} \big[f'(y)-1\big]u'\big(f(y)-y\big)dy \;>\; 0\,.$$

Thus, there exists $b^* \in (0,\zeta^*)$ such that $\phi(b^*) = 0$. Fix $r > 0$ such that $0 < b^* - r < b^* + r < x^G$. The monotonicity of function ϕ ensures that $\phi(b)$ is negative in (b^*, b^*+r) and positive in (b^*-r, b^*).

If $b^* < b < b^* + r$, there exists $\underline{x}(b) > 0$ such that

$$\int_{\underline{x}(b)}^{x^s(b)} \big[\delta^b(f(y)-y)f'(y)-1\big]u'\big(f(y)-y\big)dy \;<\; 0\,.$$

Clearly, we can choose $\underline{x}(b)$ to be smaller than $x_s(b)$. Since $\delta^b\,(f(y)-y)\,f'(y) < 1$ in the interval $(0, x_s(b))$, we have

$$\int_{\underline{x}(b)}^{x_s(b)} \big[\delta^b(f(y)-y)f'(y)-1\big]u'\big(f(y)-y\big)dy \;<\; 0\,.$$

By part (i) of Theorem 3.2, every optimal path starting from $x_0 < \underline{x}(b)$ converges to the origin. Using the same argument, by part (ii) of Theorem 3.2, if $b^* - r < b < b^*$, there exists $\overline{x}(b) > 0$ such that every optimal path starting from $x_0 > \overline{x}(b)$ converges to a positive steady state.

It remains to show that there is some $b \in (b^*-r, b^*+r)$ such that both values $\underline{x}(b)$ and $\overline{x}(b)$ exist. The following claim (see Appendix E) establishes this result.

Claim **4.2.** *There exists an open interval $I \subset (b^*-r, b^*+r)$ such that for every $b \in I$, the economy with the discount function $\delta^b(c)$ has a critical level.*

Note that in fact we have proved that critical levels exist for a set of parameters with positive measure.

4.3 Example: Discount function of current capital

Consider now the case where discounting depends only on current capital: $\delta(x,y) = \delta^p(x)$, and suppose that the discount factor is given by

$$\delta^p(x) \;\equiv\; \frac{x^\theta}{p+x^\theta}\,,$$

where $p > 0$ is a constant whose value will be specified later.

We consider the same economy as in the example above: $u(c) = c^\rho$ and $f(x) = Ax^\alpha$, with $\alpha, \rho \in (0, 1)$. However, we now assume that $1 - \alpha < \theta < \alpha\rho$. As before, we have $f'(0) = \infty$, and yet a poverty trap exists.

It is easy to verify that $\delta^p(x)$ is concave and $\big[1 - \delta^p(x)\big] u\big(f(x)\big)$ is strictly increasing, so that condition (UD2) holds. We have

$$\delta^p(x) f'(x) \;=\; \frac{\alpha A x^{\alpha+\theta-1}}{p + x^\theta}\,.$$

Equation $\delta^p(x) f'(x) = 1$ is equivalent to $\chi(x) = p$, where $\chi(x) \equiv x^\theta \big[\alpha A x^{\alpha-1} - 1\big]$. Since $\alpha + \theta > 1$, $\chi(0) = \chi(x^G) = 0$, where x^G is the solution to $f'(x) = 1$. Moreover, the function χ is single-peaked in $[0, x^G]$. Indeed,

$$\chi'(x) \;=\; x^{\theta-1}\left[A(\alpha+\theta-1)x^{\alpha-1} - \theta\right]\,,$$

and hence, $\chi'(x) = 0$ if and only if $x = [A(\alpha+\theta-1)/\theta]^{1/(1-\alpha)} < (\alpha A)^{1/(1-\alpha)} = x^G$.

Following the same arguments used in the previous example from Section 4.2, it is straightforward to show the existence of an interval I such that for all $p \in I$, the corresponding economy with the discount function $\delta^p(x)$ has a critical level.

5. Conclusion

In this paper, we propose a general approach to studying an optimal growth model with endogenous discounting. We allow the discount factor to depend on both consumption and the capital stock, and assume that the sum of discount factors in the intertemporal utility equals one. While this property has no effect in models with a constant discount factor, under endogenous time preference it ensures that optimal paths and steady states are independent of the level and sign of instantaneous utility.

We characterize the value function and the optimal policy correspondence, with a particular focus on the dynamics of optimal paths. We show that, unless starting from a steady state, every optimal path is either strictly increasing or strictly decreasing. We also examine the existence of poverty traps driven solely by endogenous discounting. Our results suggest that when the discount factor depends only on consumption or on current capital, a critical level of capital exists. In this case, two otherwise identical economies may diverge: one starting below the critical level optimally depletes its capital stock and remains stuck in a poverty trap, while the other, starting above the critical level, accumulates capital and grows.

Several open questions remain for future research. Of particular interest are general conditions for the existence of a critical level under arbitrary discount functions, which proved analytically challenging. Another promising direction is to study competitive equilibria with capital externalities in discounting that decentralize the optimal path. We believe that our approach to endogenous discounting provides a useful foundation for a wide range of applications.

Appendix

A. Pareto property and the value of $u(0)$

It is natural to require that the objective function $W(\mathbf{x})$ satisfies the *Pareto property*: for any feasible capital path $\{x_t\}_{t=0}^{\infty}$, if $x'_0 > x_0$, then

$$W(x'_0, x_1, x_2, \ldots) \; > \; W(x_0, x_1, x_2, \ldots)\,.$$

This condition is equivalent to the requirement that for $c'_0 > c_0$, consumption path $(c'_0, c_1, c_2, \ldots)$ dominates $(c_0, c_1, c_2, \ldots)$ in terms of intertemporal utility. In classic optimal growth models with a constant discount factor, this property is guaranteed by the fact that $u(c)$ is increasing. However, when the discount factor is endogenous, the situation changes. The following claim ensures that a necessary condition for the Pareto property is that $u(c)$ is bounded from below.

Claim **A.1.** *If* $\lim_{c\to 0} u(c) = -\infty$*, then* $W(\mathbf{x})$ *does not satisfy the Pareto property.*

Proof. Since

$$W(x, x, \ldots) \; = \; \big[1 - \delta(x, x)\big] u(f(x) - x) + \delta(x, x) W(x, x, \ldots)\,,$$

we have $W(x, x, \ldots) = u(f(x) - x)$. Fix a triplet $(x, \hat{x}, x')$, such that $\hat{x} > x$, $\delta(\hat{x}, x') > \delta(x, x')$ and

$$u\big(f(x') - x'\big) \; < \; \frac{\big[1 - \delta(x, x')\big] u(f(x) - x') - \big[1 - \delta(\hat{x}, x')\big] u(f(\hat{x}) - x')}{\delta(\hat{x}, x') - \delta(x, x')}\,.$$

Then $W(\hat{x}, x', x', \ldots) < W(x, x', x', \ldots)$, violating the Pareto property. QED

Thus, unbounded from below instantaneous utility functions are not appropriate for studying endogenous discounting. Suppose now that $u(c)$ is bounded. The following claim shows that the value of $u(0)$ has no particular role.

Claim **A.2.** *Optimal path does not depend on $u(0)$.*

Proof. Since the objective function $W(\mathbf{x})$ satisfies (1) for all t, we have

$$\begin{aligned} W(x_t, x_{t+1}, \ldots) - u(0) \;&=\; \big[1 - \delta(x_t, x_{t+1})\big]\big[u\big(f(x_t) - x_{t+1}\big) - u(0)\big] \\ &+ \delta(x_t, x_{t+1})\big[W(x_{t+1}, x_{t+2}, \ldots) - u(0)\big]\,. \end{aligned} \tag{A.1}$$

Iterating over time, we obtain

$$\begin{aligned} W(x_0, x_1, \ldots) \;&= u(0) + \big[1 - \delta(x_0, x_1)\big]\big[u(f(x_0) - x_1) - u(0)\big] \\ &+ \sum_{t=1}^{\infty}\left(\big[1 - \delta(x_t, x_{t+1})\big]\big[u(f(x_t) - x_{t+1}) - u(0)\big]\prod_{s=0}^{t-1}\delta(x_s, x_{s+1})\right). \end{aligned}$$

Thus, the optimization problem (2) is invariant under changes in $u(0)$. Without loss of generality, we can always consider $u(c) - u(0)$ as the instantaneous utility. QED

By contrast, optimal paths in models that do not normalize the sum of discount factors to 1 depend on $u(0)$. Indeed, if intertemporal utility is just a sum of utilities discounted by an endogenous discount factor, we have

$$\begin{aligned} W(x_0, x_1, x_2, \ldots) \;&=\; u(c_0) + \sum_{t=1}^{\infty} u(c_t)\prod_{s=0}^{t-1}\delta(x_s, x_{s+1}) \\ &=\; \big[u(c_0) - u(0)\big] + \sum_{t=1}^{\infty}\big[u(c_t) - u(0)\big]\prod_{s=0}^{t-1}\delta(x_s, x_{s+1}) + u(0) + u(0)\sum_{t=1}^{\infty}\prod_{s=0}^{t-1}\delta(x_s, x_{s+1})\,. \end{aligned}$$

Unlike the standard Ramsey model or our setting, where only $u(c) - u(0)$ matters, here the optimal path starting from any $x_0 > 0$ alters once $u(0)$ changes. To illustrate this point, suppose that discounting depends on future capital, $\delta(x, y) = \delta^f(y)$, and consider the necessary condition for a steady state (cf. Eq. 9 in Erol et al., 2011):

$$u\big(f(x) - x\big) \;=\; \frac{\big[1 - \delta^f(x)\big]\big[1 - \delta^f(x) f'(x)\big]}{\delta^{f\prime}(x)}\; u'\big(f(x) - x\big)\,.$$

The left-hand side depends only on $u(c)$, while the right-hand side does not. Thus, adding a constant to $u(c)$ affects the solution to the above equation and changes the steady state capital stock, which is not the case in our paper.

Finally, we show that the Pareto property imposes another condition relating $u(c)$ to $\delta(x, y)$. Recall that $\psi(x, y) = \big[1 - \delta(x, y)\big]\big[u\big(f(x) - y\big) - u(0)\big]$. The following claim studies the properties of $\psi(x, y)$ and proves that once $u(0)$ is finite, the Pareto property is equivalent to the strictly increasing monotonicity of the function $\psi(x, 0)$.

CLAIM **A.3.** (i) *If either $\delta(x,y)=\delta^p(x)$ or $\delta(x,y)=\delta^f(y)$, then $\delta_{12}(x,y)=0$ and $\psi_{12}(x,y)>0$ for all $0<y<f(x)$.*

(ii) *If $\delta(x,y)=\delta^b\big(f(x)-y\big)$, then the supermodularity of ψ is equivalent to the strict concavity of $\big[1-\delta^b(c)\big]\big[u(c)-u(0)\big]$ with respect to c.*

(iii) *When $u(c)$ is bounded, the Pareto property is equivalent to condition (UD2).*

Proof. For any $0<y<f(x)$, the cross derivative of ψ is given by:

$$\begin{aligned}\psi_{12}(x,y) \;&=\; -\delta_{12}(x,y)\big[u\big(f(x)-y\big)-u(0)\big]+\delta_1(x,y)u'\big(f(x)-y\big)\\ &\quad -\delta_2(x,y)u'\big(f(x)-y\big)f'(x)-\big[1-\delta(x,y)\big]u''\big(f(x)-y\big)f'(x)\,.\end{aligned}$$

PART (i). When the discount factor depends on current or future capital, the cross derivative $\delta_{12}(x,y)=0$. Since $\delta_2(x,y)$ and $u''\big(f(x)-y\big)$ are non-positive, we have

$$\begin{aligned}\psi_{12}(x,y) \;&=\; \delta_1(x,y)u'\big(f(x)-y\big)-\delta_2(x,y)u'\big(f(x)-y\big)f'(x)\\ &\quad -\big[1-\delta(x,y)\big]u''\big(f(x)-y\big)f'(x) \;>\; 0\,.\end{aligned}$$

PART (ii). Let $\hat{u}(c)=u(c)-u(0)$. Obviously, $\hat{u}(c)\geq 0$ for $c\geq 0$. When discounting depends on consumption, $\delta(x,y)=\delta^b\big(f(x)-y\big)$, setting $c=f(x)-y$, we have

$$\begin{aligned}\psi_{12}(x,y) \;&=\; -\delta_{12}(x,y)\hat{u}\big(f(x)-y\big)+\delta_1(x,y)\hat{u}'\big(f(x)-y\big)\\ &\quad -\delta_2(x,y)\hat{u}'\big(f(x)-y\big)f'(x)-\big[1-\delta(x,y)\big]\hat{u}''\big(f(x)-y\big)f'(x)\\ &=\; \delta^{b\prime\prime}\big(f(x)-y\big)f'(x)\hat{u}\big(f(x)-y\big)+\delta^{b\prime}\big(f(x)-y\big)f'(x)\hat{u}'\big(f(x)-y\big)\\ &\quad +\delta^{b\prime}\big(f(x)-y\big)\hat{u}'\big(f(x)-y\big)f'(x)-\big[1-\delta^b\big(f(x)-y\big)\big]\,\hat{u}''\big(f(x)-y\big)f'(x)\\ &=\; \big[\delta^{b\prime\prime}(c)\hat{u}(c)+2\delta^{b\prime}(c)\hat{u}'(c)-\big[1-\delta^b(c)\big]\hat{u}''(c)\big]f'(x)\,.\end{aligned}$$

Thus, $\psi_{12}(x,y)>0$ if and only if

$$\Big(\big[1-\delta^b(c)\big]\hat{u}(c)\Big)'' \;=\; -\delta^{b\prime\prime}(c)\hat{u}(c)-2\delta^{b\prime}(c)\hat{u}'(c)+\big[1-\delta^b(c)\big]\hat{u}''(c) \;<\; 0\,.$$

PART (iii). To prove necessity, observe that if the Pareto property is satisfied, then $W(x_0,0,0,\ldots)=u(0)+\big[1-\delta(x_0,0)\big]\big[u\big(f(x_0)\big)-u(0)\big]$ strictly increases with x_0.

To prove sufficiency, we show that under condition (UD2), for any feasible path $\{x_t\}_{t=0}^{\infty}$, the function $W(x_0,x_1,\ldots)$ strictly increases with x_0. By (A.1), we have

$$W(x_0,x_1,\ldots)-u(0) \;=\; \big[1-\delta(x_0,x_1)\big]\big[u\big(f(x_0)-x_1\big)-u(0)\big]+\delta(x_0,x_1)\big[W(x_1,x_2,\ldots)-u(0)\big]\,.$$

Function $\delta(x_0, x_1)$ does not decrease with x_0. Furthermore, since $\psi_{12}(x_0, x_1) > 0$, for any $0 < x_1 < f(x_0)$, we have $\psi_1(x_0, x_1) > \psi_1(x_0, 0) > 0$. Therefore, $\psi(x_0, x_1)$ increases with x_0, implying that the Pareto property holds. QED

B. Proof of Proposition 2.3

Part (i) comes directly from Proposition 2.1. The Maximum Theorem implies part (ii). Part (iii) follows from Proposition 2.2.

To prove part (iv), as in the proof of Claim A.3, let $\hat{u}\big(f(x)-y\big) = u\big(f(x)-y\big) - u(0)$, and $\hat{V}(x) = V(x) - V(0)$. Observe that $\hat{u}\big(f(x)-y\big) \geq 0$ and $\hat{V}(x) \geq 0$.

It is clear that for $0 \leq y \leq f(x)$, since $V(0) = u(0)$, we have

$$\big[1-\delta(x,y)\big]\hat{u}\big(f(x)-y\big)+\delta(x,y)\hat{V}(y) \;=\; \big[1-\delta(x,y)\big]u\big(f(x)-y\big)+\delta(x,y)V(y)-u(0)\,.$$

We show that, for any $x < x'$ and $y \in \varphi(x)$, $y' \in \varphi(x')$, we have $y \leq y'$. Assume the contrary, that is $y > y'$. We have

$$\begin{aligned}
\big[1-\delta(x,y)\big]\hat{u}\big(f(x)-y\big)+\delta(x,y)\hat{V}(y) \;&\geq\; \big[1-\delta(x,y')\big]\hat{u}\big(f(x)-y'\big)+\delta(x,y')\hat{V}(y'),\\
\big[1-\delta(x',y')\big]\hat{u}\big(f(x')-y'\big)+\delta(x',y')\hat{V}(y') \;&\geq\; \big[1-\delta(x',y)\big]\hat{u}\big(f(x')-y\big)+\delta(x',y)\hat{V}(y).
\end{aligned}$$

Summing these two inequalities, we get

$$\psi(x,y)+\psi(x',y')+\delta(x,y)\hat{V}(y)+\delta(x',y')\hat{V}(y') \geq \psi(x',y)+\psi(x,y')+\delta(x,y')\hat{V}(y')+\delta(x',y)\hat{V}(y)$$

and, by the supermodularity property of ψ,

$$\psi(x,y)+\psi(x',y') \;<\; \psi(x',y)+\psi(x,y')\,.$$

Thus, with $x < x'$ and $y > y'$,

$$\delta(x,y)\hat{V}(y)+\delta(x',y')\hat{V}(y') \;>\; \delta(x,y')\hat{V}(y')+\delta(x',y)\hat{V}(y)\,,$$

which implies that

$$\hat{V}(y')\big[\delta(x',y')-\delta(x,y')\big] \;>\hat{V}(y)\big[\delta(x',y)-\delta(x,y)\big]$$

or, equivalently,

$$\hat{V}(y')\int_x^{x'}\delta_1(z,y')dz \;>\; \hat{V}(y)\int_x^{x'}\delta_1(z,y)dz\,.$$

However, since the cross derivative of δ is non-negative, we have $\delta_1(z,y) \geq \delta_1(z,y')$ for any $z \in (x,x')$, leading to a contradiction.

C. Proof of the results from Section 3

C.1 Proof of Proposition 3.2

To prove part (i), note that the Euler equation (3) can be written as

$$\begin{aligned}
&\big[1-\delta(x_t,x_{t+1})\big]u'(f(x_t)-x_{t+1})+\delta_2(x_t,x_{t+1})\big[u(f(x_t)-x_{t+1})-u(0)\big]\\
&=\ \delta(x_t,x_{t+1})\psi_1(x_{t+1},x_{t+2})+\delta_2(x_t,x_{t+1})\big[V(x_{t+1})-V(0)\big]\\
&+\delta(x_t,x_{t+1})\delta_1(x_{t+1},x_{t+2})\big[V(x_{t+2})-V(0)\big]\,.
\end{aligned}$$

It is straightforward to verify that each derivative with respect to x_{t+2} in the right-hand side is positive. Therefore, $\varphi(x_{t+1})$ is a singleton. This guarantees that the sequence $(x_1, x_2, \ldots)$ is the unique optimal path starting from x_1.

Part (ii) is a direct consequence of part (i) and Proposition 2.4.

C.2 Proof of Proposition 3.3

Part (i). Replace x_t and x_{t+1} by x^* in the Euler equations and observe that if x^* is a steady state, then $V(x^*) = u\big(f(x^*)-x^*\big)$. According to Proposition 3.2, since $(x^*, x^*, \ldots)$ is an optimal path, $\varphi(x^*) = \{x^*\}$.

Part (ii). Assume the contrary. Then, there exists some $t \geq 0$ such that $x_t \neq x^*$, and $x_{t+1} = x^*$. The sequence $(x_t, x^*, x^*, \ldots)$ is an optimal path starting from x_t. We replace x_{t+1} by x^* in the Euler equation:

$$\begin{aligned}
&\big[1-\delta(x_t,x^*)\big]u'\big(f(x_t)-x^*\big)+\delta_2(x_t,x^*)u\big(f(x_t)-x^*\big)\\
&=\delta(x_t,x^*)\big[1-\delta(x^*,x^*)\big]u'\big(f(x^*)-x^*\big)f'(x^*)\\
&-\delta(x_t,x^*)\delta_1(x^*,x^*)u\big(f(x^*)-x^*\big)\\
&+\delta_2(x_t,x^*)V(x^*)+\delta(x_t,x^*)\delta_1(x^*,x^*)V(x^*).
\end{aligned}$$

Since $V(x^*) = u\big(f(x^*)-x^*\big)$, the Euler equation is equivalent to

$$\begin{aligned}
&\big[1-\delta(x_t,x^*)\big]u'\big(f(x_t)-x^*\big)+\delta_2(x_t,x^*)u\big(f(x_t)-x^*\big)\\
&=\delta(x_t,x^*)\big[1-\delta(x^*,x^*)\big]u'\big(f(x^*)-x^*\big)f'(x^*)+\delta_2(x_t,x^*)V(x^*).
\end{aligned}$$

In the case $x_t < x^*$,

$$V(x^*)\ =\ u\big(f(x^*)-x^*\big)\ >\ u\big(f(x_t)-x^*\big)\,.$$

Since $\delta(x^*, x^*) f'(x^*) = 1$ and $\delta_2(x_t, x^*) \leq 0$, we have

$$\begin{aligned}
&\big[1 - \delta(x_t, x^*)\big] u'\big(f(x_t) - x^*\big) \\
&= \delta(x_t, x^*)\big[1 - \delta(x^*, x^*)\big] u'\big(f(x^*) - x^*\big) f'(x^*) + \delta_2(x_t, x^*)\big[V(x^*) - u\big(f(x_t) - x^*\big)\big] \\
&\leq \delta(x_t, x^*)\big[1 - \delta(x^*, x^*)\big] u'\big(f(x^*) - x^*\big) f'(x^*) \\
&\leq \delta(x^*, x^*)\big[1 - \delta(x^*, x^*)\big] u'\big(f(x^*) - x^*\big) f'(x^*) \\
&= \big[1 - \delta(x^*, x^*)\big] u'\big(f(x^*) - x^*\big) \;<\; \big[1 - \delta(x_t, x^*)\big] u'\big(f(x_t) - x^*\big)\,,
\end{aligned}$$

a contradiction. In the case $x_t > x^*$, applying the same arguments but with reversed inequalities also leads to a contradiction.

C.3 Proof of Theorem 3.1

Consider an optimal path $\{x_t\}_{t=0}^{\infty}$. If x_0 is a steady state, then by Propositions 3.2–3.3, $\{x_t\}_{t=0}^{\infty}$ is constant.

Suppose now that x_0 is not a steady state. By Proposition 3.3, there is no t such that x_t is a steady state. Moreover, if there is some t such that $x_{t+1} = x_t$, then x_t is a steady state, since $x_{t+1} \in \varphi(x_t)$: a contradiction.

Thus, $x_{t+1} \neq x_t$ for every t. If $x_0 < x_1$, since $x_1 \in \varphi(x_0)$ and $x_2 \in \varphi(x_1)$, Proposition 2.3 implies that $x_1 < x_2$. By induction, we get $x_t < x_{t+1}$ for any t. If $x_0 > x_1$, using the same arguments, we find that the path $\{x_t\}_{t=0}^{\infty}$ is strictly decreasing. The monotonicity of $\{x_t\}_{t=0}^{\infty}$ entails that this sequence has a limit, and the hemicontinuity of the optimal policy correspondence guarantees that this limit is a steady state.

C.4 Proof of Proposition 3.4

Part (i). Assume the contrary. Then there exists an optimal path $\{x_t\}_{t=0}^{\infty}$ from some non-steady state $x_0 > 0$ that converges to the origin.

Since $\liminf_{x \to 0} \delta(x, x) f'(x) > 1$, there exists $z > 0$ such that $\delta(x, x) f'(x) > 1$ for any $x < z$. The convergence of $\{x_t\}_{t=0}^{\infty}$ to the origin implies the existence of a period T such that $x_t < z$ for every $t \geq T$.

Recall that

$$V(x_T) = [1 - \delta(x_T, x_{T+1})]\, u\big(f(x_T) - x_{T+1}\big) + \delta(x_T, x_{T+1})\big[1 - \delta(x_{T+1}, x_{T+2})\big] u\big(f(x_{T+1}) - x_{T+2}\big) + \dots.$$

Since $(x_T, x_T, \ldots)$ is feasible,

$$V(x_T) \;\geq\; u\big(f(x_T) - x_T\big)\,.$$

Let us now prove the following inequality for any $t \geq T$:

$$u\big(f(x_t) - x_t\big) \geq [1 - \delta(x_t, x_{t+1})]\, u\big(f(x_t) - x_{t+1}\big) + \delta(x_t, x_{t+1}) u\big(f(x_{t+1}) - x_{t+1}\big).$$

Indeed, consider the following function:

$$\eta(y) \;=\; [1 - \delta(x_t, y)]\, u\big(f(x_t) - y\big) + \delta(x_t, y) u\big(f(y) - y\big)\,. \tag{C.1}$$

Taking the derivative with respect to y, and recalling that δ non-increases with the second argument, we obtain for any $y \in (x_{t+1}, x_t)$:

$$\begin{aligned}
\eta'(y) = &- [1 - \delta(x_t, y)]\, u'\big(f(x_t) - y\big) - \delta_2(x_t, y) u\big(f(x_t) - y\big) \\
&+ \delta_2(x_t, y) u\big(f(y) - y\big) + \delta(x_t, y) u'\big(f(y) - y\big)\, [f'(y) - 1] \\
&= -\delta_2(x_t, y)\, \big[u\big(f(x_t) - y\big) - u\big(f(y) - y\big)\big] \\
&- [1 - \delta(x_t, y)]\, u'\big(f(y) - y\big) + \delta(y, y) u'\big(f(y) - y\big)\, [f'(y) - 1] \\
&\geq - [1 - \delta(y, y)]\, u'\big(f(y) - y\big) + \delta(y, y) u'\big(f(y) - y\big)\, [f'(y) - 1] \\
&= \big[\delta(y, y) f'(y) - 1\big] u'\big(f(y) - y\big) \;>\; 0\,.
\end{aligned}$$

The function η is strictly increasing in the interval (x_{t+1}, x_t), which implies that $\eta(x_t) > \eta(x_{t+1})$. Thus, the inequality holds and

$$\begin{aligned}
u\big(f(x_T) - x_T\big) \;>\; &[1 - \delta(x_T, x_{T+1})]\, u\big(f(x_T) - x_{T+1}\big) + \delta(x_T, x_{T+1}) u\big(f(x_{T+1}) - x_{T+1}\big) \\
&> [1 - \delta(x_T, x_{T+1})]\, u\big(f(x_T) - x_{T+1}\big) \\
&+ \delta(x_T, x_{T+1})\, [1 - \delta(x_{T+1}, x_{T+2})]\, u\big(f(x_{T+2}) - x_{T+2}\big) \\
&> \;\ldots\; > \; V(x_T)\,,
\end{aligned}$$

a contradiction.

Since no optimal path converges to the origin, for all x_0 sufficiently close to 0, any optimal path from x_0 is increasing and converges to a positive steady state.

Part (ii). Clearly, we need to consider only the case $x_0 \leq x^M$. In this case, the sequence $(x_0, x_0, \ldots)$ is feasible and $V(x_0) \geq u\big(f(x_0) - x_0\big)$. We apply the same arguments as in the proof of part (i), but with reversed inequalities. If the optimal capital path $\{x_t\}_{t=0}^{\infty}$ is increasing, the function η given by (C.1) is strictly decreasing in the interval (x_t, x_{t+1}). Indeed, for any $y \in (x_t, x_{t+1})$,

$$\begin{aligned}
\eta'(y) &= -[1-\delta(x_t,y)]\, u'\big(f(x_t)-y\big) - \delta_2(x_t,y)u\big(f(x_t)-y\big) \\
&\quad + \delta_2(x_t,y)u\big(f(y)-y\big) + \delta(x_t,y)u'\big(f(y)-y\big)\,[f'(y)-1] \\
&= -\delta_2(x_t,y)\,\big[u\big(f(x_t)-y\big) - u\big(f(y)-y\big)\big] \\
&\quad - [1-\delta(x_t,y)]\, u'\big(f(y)-y\big) + \delta(y,y)u'\big(f(y)-y\big)\,[f'(y)-1] \\
&\le -[1-\delta(y,y)]\, u'\big(f(y)-y\big) + \delta(y,y)u'\big(f(y)-y\big)\,[f'(y)-1] \\
&= \big[\delta(y,y)f'(y)-1\big]u'\big(f(y)-y\big) \;<\; 0\,.
\end{aligned}$$

The function η is strictly decreasing in the interval (x_t, x_{t+1}), which implies $\eta(x_t) > \eta(x_{t+1})$. Hence,

$$\begin{aligned}
u\big(f(x_0)-x_0\big) &> [1-\delta(x_0,x_1)]\, u\big(f(x_0)-x_1\big) + \delta(x_0,x_1)u\big(f(x_1)-x_1\big) \\
&> [1-\delta(x_0,x_1)]\, u\big(f(x_0)-x_1\big) + \delta(x_0,x_1)\,[1-\delta(x_1,x_2)]\, u\big(f(x_2)-x_2\big) \\
&> \;\ldots\; > \; V(x_0)\,,
\end{aligned}$$

a contradiction.

C.5 Proof of Theorem 3.2

Part (i). Let $\{x_t\}_{t=0}^{\infty}$ be the optimal path starting from x_0 and assume that it is increasing and it converges to a positive steady state, say x^*.

We prove that for any t,

$$V(x_{t+1})-V(x_t) \ge \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_{t+1})-x_{t+1}\big)-\big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_t)-x_{t+1}\big).$$

Indeed, the inequality is equivalent to

$$\begin{aligned}
&[1-\delta(x_t,x_{t+1})]V(x_{t+1}) + \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_t)-x_{t+1}\big)+\delta(x_t,x_{t+1})V(x_{t+1}) \\
&\ge V(x_t) + \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_{t+1})-x_{t+1}\big).
\end{aligned}$$

Since $\big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_t)-x_{t+1}\big)+\delta(x_t,x_{t+1})V(x_{t+1}) = V(x_t)$, we need to show

$$\big[1-\delta(x_t,x_{t+1})\big]V(x_{t+1}) \;\ge\; \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_{t+1})-x_{t+1}\big)\,.$$

This trivially holds because $V(x_{t+1}) \geq u\big(f(x_{t+1}) - x_{t+1}\big)$. Therefore, we obtain

$$\begin{aligned}
V(x_{t+1}) - V(x_t) \ &\geq \ \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_{t+1})-x_{t+1}\big) - \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_t)-x_{t+1}\big) \\
&\geq \big[1-\delta(x_t,x_{t+1})\big]\int_{x_t}^{x_{t+1}} u'\big(f(y)-x_{t+1}\big)f'(y)dy \\
&\geq \big[1-\delta(x_t,x_{t+1})\big]\int_{x_t}^{x_{t+1}} u'\big(f(y)-y\big)f'(y)dy \\
&\geq \int_{x_t}^{x_{t+1}} \big[1-\delta(y,y)\big]u'\big(f(y)-y\big)f'(y)dy.
\end{aligned}$$

We have also

$$\begin{aligned}
V(x^*) - V(x_0) &= \sum_{t=0}^{\infty}\big[V(x_{t+1}) - V(x_t)\big] \\
&\geq \sum_{t=0}^{\infty}\int_{x_t}^{x_{t+1}} \big[1-\delta(y,y)\big]u'\big(f(y)-y\big)f'(y)dy \\
&= \int_{x_0}^{x^*} \big[1-\delta(y,y)\big]u'\big(f(y)-y\big)f'(y)dy.
\end{aligned}$$

Since x^* is a steady state, $V(x^*) = u\big(f(x^*) - x^*\big)$ and

$$\begin{aligned}
&\int_{x_0}^{x^*} \big[1-\delta(y,y)\big]u'\big(f(y)-y\big)f'(y)dy \\
&\leq \ V(x^*) - V(x_0) \ \leq \ u\big(f(x^*)-x^*\big) - u\big(f(x_0)-x_0\big) \\
&< u\big(f(x^*)-x^*\big) - u\big(f(x_0)-x_0\big) + \int_{x_0}^{x^*}\big[1-\delta(y,y)f'(y)\big]u'\big(f(y)-y\big)dy \\
&= \int_{x_0}^{x^*} u'\big(f(y)-y\big)\big[f'(y)-1\big]dy + \int_{x_0}^{x^*}\big[1-\delta(y,y)f'(y)\big]u'\big(f(y)-y\big)dy \\
&= \int_{x_0}^{x^*} \big[1-\delta(y,y)\big]u'\big(f(y)-y\big)f'(y)dy,
\end{aligned}$$

a contradiction.

Part (ii). Let $\{x_t\}_{t=0}^{\infty}$ be the optimal path from x_0 which is decreasing and converges to some steady state x^*. We know that either $x^* = 0$ or x^* is solution to $\delta(x,x)f'(x) = 1$.

We prove that for any t,

$$V(x_t) - V(x_{t+1}) \ \leq \ -\big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_{t+1})-x_{t+1}\big) + \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_t)-x_{t+1}\big).$$

Indeed, this inequality is equivalent to

$$\begin{aligned}
&[1-\delta(x_t,x_{t+1})V(x_{t+1}] + \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_t)-x_{t+1}\big) + \delta(x_t,x_{t+1})V(x_{t+1}) \\
&\geq \ V(x_t) + \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_{t+1})-x_{t+1}\big).
\end{aligned}$$

Since $\big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_t)-x_{t+1}\big)+\delta(x_t,x_{t+1})V(x_{t+1}) = V(x_t)$, it remains to prove that

$$[1-\delta(x_t,x_{t+1})]V(x_{t+1}) \geq \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_{t+1})-x_{t+1}\big).$$

This holds because $V(x_{t+1}) \geq u\big(f(x_{t+1})-x_{t+1}\big)$. Thus, we obtain

$$\begin{aligned}
V(x_t)-V(x_{t+1}) &\leq \big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_t)-x_{t+1}\big)-\big[1-\delta(x_t,x_{t+1})\big]u\big(f(x_{t+1})-x_{t+1}\big) \\
&\leq \big[1-\delta(x_t,x_{t+1})\big]\int_{x_{t+1}}^{x_t} u'\big(f(y)-x_{t+1}\big)f'(y)dy \\
&\leq \big[1-\delta(x_t,x_{t+1})\big]\int_{x_{t+1}}^{x_t} u'\big(f(y)-y\big)f'(y)dy \\
&\leq \int_{x_{t+1}}^{x_t}\big[1-\delta(y,y)\big]u'\big(f(y)-y\big)f'(y)dy.
\end{aligned}$$

We also have

$$\begin{aligned}
V(x_0)-V(x^*) &= \sum_{t=0}^{\infty}\big[V(x_t)-V(x_{t+1})\big] \\
&\leq \sum_{t=0}^{\infty}\int_{x_{t+1}}^{x_t}\big[1-\delta(y,y)\big]u'\big(f(y)-y\big)f'(y)dy \\
&= \int_{x^*}^{x_0}\big[1-\delta(y,y)\big]u'\big(f(y)-y\big)f'(y)dy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{x^*}^{x_0}\big[1-\delta(y,y)\big]u'\big(f(y)-y\big)f'(y)dy \\
&\geq\; V(x_0)-V(x^*) \;\geq\; u\big(f(x_0)-x_0\big)-u\big(f(x^*)-x^*\big) \\
&> u\big(f(x_0)-x_0\big)-u\big(f(x^*)-x^*\big)-\int_{x^*}^{x_0}\big[\delta(y,y)f'(y)-1\big]u'\big(f(y)-y\big)dy \\
&= \int_{x^*}^{x_0}u'\big(f(y)-y\big)\big[f'(y)-1\big]dy-\int_{x^*}^{x_0}\big[\delta(y,y)f'(y)-1\big]u'\big(f(y)-y\big)dy \\
&= \int_{x^*}^{x_0}\big[1-\delta(y,y)\big]u'\big(f(y)-y\big)f'(y)dy,
\end{aligned}$$

a contradiction.

C.6 Proof of Corollary 3.1

Parts (i) and (ii) follow directly from Theorem 3.2. To prove part (iii), observe that, since δ^f decreases with x, the function $\delta^f(x)f'(x)$ strictly decreases. If $\delta^f(0)f'(0) \leq 1$, then, by part (i), every optimal path is decreasing and converges to the origin.

If $\delta^f(0)f'(0) > 1$, denote by x^s the unique solution to the equation $\delta^f(x)f'(x) = 1$. We have $\delta^f(x)f'(x) > 1$ for $x < x^s$ and $\delta^f(x)f'(x) < 1$ for $x > x^s$. By part (i) of this Corollary and part (ii) of Proposition 3.4, every optimal path $\{x_t\}_{t=0}^{\infty}$ increases if $x_0 < x^s$ and decreases if $x_0 > x^s$. In both cases, the path converges to x^s.

D. Proof of Proposition 4.1

In view of monotonicity of optimal paths, necessity is obvious. To prove sufficiency, assume the existence of an optimal path $\{\underline{x}_t\}_{t=0}^{\infty}$ that converges to zero, and an optimal path $\{\overline{x}_t\}_{t=0}^{\infty}$ that converges to a positive steady state. Let x^C be the infimum of the set of capital stocks x_0 that satisfy the condition that there exists an optimal path starting from x_0, bounded away from zero.

The existence of a path converging to zero implies $x^C > 0$. Indeed, if $x^C = 0$, by the definition of x^C, any optimal path starting from $x_0 > 0$ should be bounded away from zero, which is a contradiction.

Since we have chosen x^C in this way, if $x_0 < x^C$, every optimal path starting from x_0 will converge to the origin. On the other hand, if $x_0 > x^C$, Proposition 2.3 guarantees that any optimal path starting from x_0 will be bounded away from zero.

E. Proofs for the Example from Section 4.2

Consider the function $\big[1-\delta^b(c)\big]u(c)$. Calculus gives:

$$\frac{d}{dc}\big[1-\delta^b(c)\big]u(c) \;=\; \frac{d}{dc}\left(\frac{bc^\rho}{b+c^\theta}\right) \;=\; \frac{b^2\rho c^{\rho-1} + bc^{\theta+\rho-1}(\rho-\theta)}{\left(b+c^\theta\right)^2} \;>\; 0\,.$$

Since $\rho < 1$ and $\rho + \theta < 1$, the numerator decreases in c. Hence, the function $\big[1-\delta^b(c)\big]u(c)$ is increasing and concave in c.

Proof of Claim 4.1. Recall that $\zeta(0) = \zeta(x^G) = 0$. Let $\hat{x} \in \operatorname{argmax}_{0\le x\le x^G}\zeta(x)$. For $0 \le x \le x^G$, we have $f'(x) > 1$ and $f(x) > x$, so $0 < \hat{x} < x^G$. We prove that $\hat{x}$ is the unique solution to $\zeta'(x) = 0$ in $\big(0, x^G\big)$.

Indeed, $\zeta(x) = g(x)^\theta g'(x)$ with $g(x) = f(x) - x$. Then

$$\zeta'(x) \;=\; \theta g(x)^{\theta-1}\big[g'(x)\big]^2 + g(x)^\theta g''(x) \;=\; g(x)^{\theta-1}\left(\theta\big[g'(x)\big]^2 + g(x)g''(x)\right).$$

The equation $\zeta'(x) = 0$ is equivalent to

$$\xi(x) \equiv \theta\big(f'(x) - 1\big)^2 + \big(f(x) - x\big)f''(x) = 0.$$

Let us prove that $\xi(x) = 0$ has a unique solution. Focus first on its derivative:

$$\begin{aligned}
\xi'(x) &= 2\theta\big(f'(x) - 1\big)f''(x) + \big(f'(x) - 1\big)f''(x) + \big(f(x) - x\big)f'''(x) \\
&= (2\theta + 1)\big(\alpha A x^{\alpha-1} - 1\big)\alpha(\alpha - 1)Ax^{\alpha-2} + \big(Ax^\alpha - x\big)\alpha(\alpha - 1)(\alpha - 2)Ax^{\alpha-3} \\
&= \alpha(\alpha - 1)Ax^{\alpha-3}\big[(2\theta + 1)\big(\alpha A x^\alpha - x\big) + (\alpha - 2)\big(Ax^\alpha - x\big)\big] \\
&= -2\alpha(1 - \alpha)Ax^{\alpha-2}\left[(\alpha\theta + \alpha - 1)Ax^{\alpha-1} - \left(\theta + \frac{\alpha - 1}{2}\right)\right].
\end{aligned}$$

Clearly, the equation

$$(\alpha\theta + \alpha - 1)Ax^{\alpha-1} - \left(\theta + \frac{\alpha - 1}{2}\right) = 0$$

has a unique positive solution:

$$\tilde{x} = \left(\frac{\alpha\theta + \alpha - 1}{\theta + \frac{\alpha-1}{2}}A\right)^{\frac{1}{1-\alpha}} < (\alpha A)^{\frac{1}{1-\alpha}} = x^G.$$

Note that $\xi'(x) < 0$ if $x \in (0, \tilde{x})$ and $\xi'(x) > 0$ if $x \in \big(\tilde{x}, x^G\big)$. Moreover,

$$\xi(\tilde{x}) < \xi(x^G) < 0.$$

Because of this inequality, $\xi(x) = 0$ has no solution in the interval $(\tilde{x}, x^G)$.

Combining $\lim_{x\to 0} \xi(x) = +\infty$ with the monotonicity of ξ in the interval $(0, \tilde{x})$, the equation $\xi(x) = 0$ has a unique solution in $(0, \tilde{x})$. Thus the equation $\zeta'(x) = 0$ has unique solution in the interval $\big(0, x^G\big)$, say $\hat{x}$. The uniqueness of this solution ensures that the function ζ is increasing in $(0, \hat{x})$ and decreasing in $\big(\hat{x}, x^G\big)$. QED

Proof of Claim 4.2. To avoid confusion, denote by $\mathcal{E}(b)$ the economy corresponding to the discount function δ^b. The corresponding intertemporal utility function is denoted by $W^b(\mathbf{x})$.

Fix any $x_0 \in (0, x^G)$ such that $x_0 < x_s(b)$ for any $b \in (b^* - r, b^* + r)$, and N large enough such that $Nr > 1$. For each $n \geq N$, consider two sequences of intervals $\underline{I}_n \equiv \big(b^* + \frac{1}{n+1}, b^* + \frac{1}{n}\big)$ and $\overline{I}_n \equiv \big(b^* - \frac{1}{n}, b^* - \frac{1}{n+1}\big)$. By the choice of N, both $\underline{I}_n$ and $\overline{I}_n$ are subsets of $(b^* - r, b^* + r)$.

We prove the existence of n such that, either for any $b \in \underline{I}_n$ or for any $b \in \overline{I}_n$, the economy $\mathcal{E}(b)$ has a critical level.

Assume the contrary. Proposition 4.1 implies that, for each n, there exists $\underline{b}_n \in \underline{I}_n$ and $\overline{b}_n \in \overline{I}_n$ such that, starting from x_0, any optimal path $\{x_t(\underline{b}_n)\}_{t=0}^{\infty}$ of the economy $\mathcal{E}(\underline{b}_n)$ converges to the origin, and any optimal path $\{x_t(\overline{b}_n)\}_{t=0}^{\infty}$ of the economy $\mathcal{E}(\overline{b}_n)$ converges to a positive steady state. Since x_0 is smaller than any possible steady state of $\mathcal{E}(\underline{b}_n)$ and $\mathcal{E}(\overline{b}_n)$, in the first case, the optimal path strictly decreases, while in the second one, strictly increases. Clearly, both $\underline{b}_n$ and $\overline{b}_n$ converge to b^* when n goes to infinity.

The compactness of the set $\Pi(x_0)$ with respect to the product topology entails the existence of a subsequence $\{\underline{b}_{n_k}\}_{k=0}^{\infty}$ such that the sequence of optimal paths $\big\{\{x_t(\underline{b}_{n_k})\}_{t=0}^{\infty}\big\}_{k=0}^{\infty}$ converges to a sequence $\{\underline{x}_t\}_{t=0}^{\infty}$ under this topology. By the same argument, there exists a subsequence $\{\overline{b}_{n_l}\}_{l=0}^{\infty}$ such that $\big\{\{x_t(\overline{b}_{n_l})\}_{t=0}^{\infty}\big\}_{l=0}^{\infty}$ converges to a sequence $\{\overline{x}_t\}_{t=0}^{\infty}$ under the product topology. Clearly, $\{\underline{x}_t\}_{t=0}^{\infty}$ is non-increasing and $\{\overline{x}_t\}_{t=0}^{\infty}$ is non-decreasing.

Take now any feasible path $\mathbf{x} \in \Pi(x_0)$ and notice that

$$\begin{aligned} W^{\underline{b}_{n_k}}(x_0, x_1(\underline{b}_{n_k}), x_2(\underline{b}_{n_k}), \ldots) &\geq W^{\underline{b}_{n_k}}(x_0, x_1, x_2, \ldots)\,, \\ W^{\overline{b}_{n_l}}(x_0, x_1(\overline{b}_{n_l}), x_2(\overline{b}_{n_l}), \ldots) &\geq W^{\overline{b}_{n_l}}(x_0, x_1, x_2, \ldots)\,. \end{aligned}$$

Let k and l go to infinity, to obtain

$$\begin{aligned} W^{b^*}(x_0, \underline{x}_1, \underline{x}_2, \ldots) &\geq W^{b^*}(x_0, x_1, x_2, \ldots)\,, \\ W^{b^*}(x_0, \overline{x}_1, \overline{x}_2, \ldots) &\geq W^{b^*}(x_0, x_1, x_2, \ldots)\,. \end{aligned}$$

Since $\{x_t\}_{t=0}^{\infty}$ is arbitrary, both sequences $\{\underline{x}_t\}_{t=0}^{\infty}$ and $\{\overline{x}_t\}_{t=0}^{\infty}$ are optimal paths of the economy $\mathcal{E}(b^*)$ starting from x_0. According to Theorem 3.1, since we have chosen $x_0 < x_s(b^*)$, these sequences are strictly monotonic. The first sequence is strictly decreasing. It converges to the origin, since x_0 is smaller than any candidate for a positive steady state of the economy $\mathcal{E}(b^*)$. The second sequence is strictly increasing. Therefore, x_0 is the critical level of the economy $\mathcal{E}(b^*)$. Recalling that we have chosen x_0 arbitrarily less than $x_s(b^* - r)$, we reach a contradiction: each $x_0 < x_s(b^* - r)$ is a critical level of $\mathcal{E}(b^*)$.

This contradiction comes from the hypothesis that, for any n, there exists $\underline{b}_n \in \underline{I}_n$ and $\overline{b}_n \in \overline{I}_n$ such that the corresponding economies have no critical level. Hence, there exists some n such that, either for $b \in \underline{I}_n$ or for $b \in \overline{I}_n$, the economy with the discount function δ^b has critical level. QED

References

Akao K. I., Kamihigashi T., Nishimura K. (2025). Critical Capital Stock in a Continuous-Time Growth Model with a Convex-Concave Production Function. *Journal of Mathematical Economics* 103146.

Amir R. (1996). Sensitivity Analysis of Multi-Sector Optimal Economic Dynamics. *Journal of Mathematical Economics* 25, 123-141.

Amir R. (2005). Supermodularity and Complementarity in Economics: An Elementary Survey. *Southern Economic Journal* 71, 636-660.

Becker G. S., Mulligan C. B. (1997). The Endogenous Determination of Time Preference. *Quarterly Journal of Economics* 112, 729-758.

Borissov K. (2013). Growth and Distribution in a Model with Endogenous Time Preferences and Borrowing Constraints. *Mathematical Social Sciences* 66, 117-128.

Bouché S. (2017). Learning by Doing, Endogenous Discounting and Economic Development. *Journal of Mathematical Economics* 73, 34-43.

Camacho C., Saglam C., Turan A. (2013). Strategic Interaction and Dynamics under Endogenous Time Preference. *Journal of Mathematical Economics* 49, 291-301.

Chakrabarty D. (2012). Poverty Traps and Growth in a Model of Endogenous Time Preference. *The B. E. Journal of Macroeconomics* 12, 1-33.

Chambers C., Echenique F. (2018). On Multiple Discount Rates. *Econometrica* 86, 1325-1346.

Das M. (2003). Optimal Growth with Decreasing Marginal Impatience. *Journal of Economic Dynamics and Control* 27, 1881-1898.

Dechert D., Nishimura K. (1983). A Complete Characterization of Optimal Growth Paths in an Aggregated Model with a Non-Concave Production Function. *Journal of Economic Theory* 31, 332-354.

Dohmen T., Falk A., Huffman D., Sunde U. (2010). Are Risk Aversion and Impatience Related to Cognitive Ability? *American Economic Review* 100, 1238-1260.

Drugeon J. P. (1996). Impatience and Long-run Growth. *Journal of Economic Dynamics and Control* 20, 281-313.

Drugeon J. P. (2000). On the Roles of Impatience in Homothetic Growth Paths. *Economic Theory* 15, 139-161.

Drugeon J. P., Ha-Huy T. (2022). A Not so Myopic Axiomatization of Discounting. *Economic Theory* 73, 349-376.

Drugeon J. P., Ha-Huy T., Nguyen T. D. H. (2019). On Maximin Dynamic Programming and the Rate of Discount. *Economic Theory* 67, 703-729.

Epstein L. G., Hynes J. A. (1983). The Rate of Time Preference and Dynamic Economic Analysis. *Journal of Political Economy* 91, 611-635.

Epstein L. G. (1987). A Simple Dynamic General Equilibrium Model. *Journal of Economic Theory* 41, 68-95.

Erol S., Le Van C., Saglam C. (2011). Existence, Optimality and Dynamics of Equilibria with Endogenous Time Preference. *Journal of Mathematical Economics* 47, 170-179.

Galor O., Weil D. N. (2000). Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond. *American Economic Review* 90, 806-828.

Galor O. (2005). From Stagnation to Growth: Unified Growth Theory. *Handbook of Economic Growth* 1A, 171-293.

Harrison G. W., Lau M. I., Williams M. B. (2002). Estimating Individual Discount Rates in Denmark: A Field Experiment. *American Economic Review* 92, 1606-1617.

Lawrance E. C. (1991). Poverty and the Rate of Time Preference: Evidence from Panel Data. *Journal of Political Economy* 99, 54-77.

Le Van C., Dana R. A. (2003). *Dynamic Programming in Economics*. Kluwer Academic Publishers.

Obstfeld M. (1990). Intertemporal Dependence, Impatience, and Dynamics. *Journal of Monetary Economics* 26, 45-75.

Samwick A. A. (1998). Discount Rate Heterogeneity and Social Security Reform. *Journal of Development Economics* 57, 117-146.

Schumacher I. (2009). Endogenous Discounting via Wealth, Twin-peaks and the Role of Technology. *Economics Letters* 103, 78-80.

Schumacher I. (2011). Endogenous Discounting and the Domain of the Felicity Function. *Economic Modelling* 28, 574-581.

Stern M. L. (2006). Endogenous Time Preference and Optimal Growth. *Economic Theory* 29, 49-70.

Stokey N. L., Lucas R. E., with Prescott E. C. (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.

Strulik H. (2012). Patience and Prosperity. *Journal of Economic Theory* 147, 336-352.

Uzawa H. (1968). Time Preference, the Consumption Function, and Optimum Asset Holdings. In: J. Wolfe (Ed.), *Value, Capital and Growth*. Chicago University Press.

Wakai K. (2008). A Model of Utility Smoothing. *Econometrica* 76, 137-153.