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Abstract

This paper introduces consumption externalities in a one-sector Ramsey economy featuring heterogeneous households and borrowing constraints. Externalities are taken into account by writing that the felicity functions depend upon the consumption of all the households in the economy. Focusing on the class of equilibria in which the most patient household owns the whole capital stock, it is proved that there exist non-convergent Ramsey equilibria even though the Maximum Income Monotonicity (MIM) condition holds.

Key words: Consumption externalities; borrowing constraints; heterogeneous households; local bifurcation.

JEL Classification: E22; E23; O40

1. Introduction

1.1 Motivation and Results

The present paper considers the issue of how consumption externalities, that is nonmarket interdependence between households, affect the dynamics exhibited by the one-sector Ramsey model with heterogeneous households and borrowing constraints. It will be assumed that, besides their own consumption, households are influenced by the consumption of others. To be more specific, each household's felicity function will depend at any date on the state of the economy which is determined by the overall consumption distribution.[1]

The standard Ramsey model provides a framework for understanding a competitive market economy. Even though the span of intertemporal trades is restricted, the allocation of resources is determined exclusively by market mechanism. Each agent interacts with society solely via markets. However, social relations, although hardly insignificant for the welfare of individuals and the allocation of resources, are largely beyond the scope of the competitive market. To put it differently, it could be said that the dependence between individual actions actually goes beyond the balance between demand and supply in equilibrium. Such a dependence originates in non-market interactions, the latter being usually termed externalities. Widespread externalities then appear as an appropriate device to account for non-market interactions within competitive market economies.

In focus will be put on a special class of equilibrium, in which the turnpike property holds. The latter, which is actually satisfied by the stationary equilibrium, will be ensured by a myopia argument. The interest of this class of equilibria is that they are easily characterized by making use of the dynamical systems approach, initiated by Becker and Foias (1990) and extensively used since then (see e.g. Becker and Foias (1994), Sorger (1994)). The main result could stated as follows: in the presence of consumption externalities, the Maximum Income Monotonicity condition is not sufficient to ensure the

[1]Other types of households' dependencies could have been considered. Consumption externalities could impact the time preference rather than the felicity function; households could be concerned with the distribution of wealth rather than consumption, along the line of Balasko (2015). But it should be noted that the analysis would have been much more intricate. See Kochov and Song (2023) for a study in a context of infinite repreated games.

convergence of the capital stock towards its steady state value. On the opposite, it could be argued that social interactions not mediated by markets can ensure convergence that would not be obtained in their absence; stabilizing non-market interdependencies, so to speak.

As a matter of fact, widespread externalities are those created by and simultaneously affecting large numbers of individuals. Unlike local externalities, they are related to the entire society, and cannot be removed by negotiations between individuals. Widespread consumption externalities are thus a device to formalize out of markets dependencies among individuals within large societies. McKenzie (1955) was the first to prove explicitly the existence of competitive equilibrium in a finite, convex economy where each consumer's preferences depend on the allocation of resources among other consumers (see also Arrow and Hahn (1971)). The issue has received particular attention in recent years (see, e.g., Geanakoplos and Polemarchakis (2008), Bonisseau and del Mercato (2010), Dufwenberg et al. (2011), del Mercato and Platino (2017), Velez (2017), Nguyen (2021) and del Mercato and Nguyen (2023)).

1.2 Related literature

Dynamic general equilibrium models seek knowledge about the time paths of prices, consumption, and wealth of decentralized market economies. One class of these models considers one-good economies populated by finitely many households, each a distinct individual with different tastes and endowments, in order to examine the interaction between rates of time preference, the completeness of markets, and the technological possibilities for capital accumulation.

In a complete Arrow-Debreu markets economy or, similarly, in a sequential markets economy where individuals are allowed to borrow and lend subject to repaying all loans, a heterogenous distribution of discounting rates leads to the emergence of dominant household: the consumption of relatively more impatient households is driven towards zero as their incomes are entirely devoted to debt service; in the long run only the most patient household has positive wealth, consumes the entire output of the economy, and determines prices. This result, known as Ramsey's conjecture, has been proven by Bewley (1982) and

Coles (1986).[2]

In an incomplete markets economy or, similarly, in a sequential markets economy where individuals cannot discount their future labor incomes, heterogeneous discounting does not imply that the whole consumption ultimately goes to the most patient individual. However, Becker (1980) showed that the stationary equilibrium of such heterogeneous households no-borrowing economy features a dramatically skewed distribution of wealth and consumption: only the most patient household owns capital, impatient households consume at the minimum.

What about non-stationary equilibria? In a comprehensive survey, Becker (2006) points out that Ramsey's conjecture about the eventual capital ownership pattern does not hold in general. More precisely, the only major result that can be proven under standard assumptions is the so-called recurrence property: every household other than the most patient one must attain the zero-capital state infinitely often.[3] Furthermore, it has been shown in Becker and Foias (1987, 1994) and Sorger (1994) that Ramsey equilibria can display non-convergent behavior, even when the turnpike property holds, i.e., even when eventually the most patient household owns the entire capital stock.

Focusing on the economy's primitives, Becker and Foias (1987) came up with the first sufficient condition for the convergence of the capital stock in every Ramsey equilibrium, the Capital Income Monotonicity (CIM) condition. They proved that if the production technology is such that the capital income is monotone increasing in the capital stock, the wealth distribution becomes degenerate in finite time, or in other words, the turnpike property holds. Additionally, all variables converge asymptotically toward their steady state values. In a recent contribution, Becker et al. (2014) established that a weaker condition, the monotonicity of the maximal income that any household can have - the Maximum Income Monotonicity (MIM) condition, is indeed sufficient.[4]

[2]The accuracy of the Ramsey conjecture is obtained provided each household's tastes are represented by a time additive and separable utility function with a fixed rate of time preference. This outcome occurs either asymptotically (at every finite date the households have positive but small and shrinking consumption) or eventually (in finite time). The latter result is due to the assumption that marginal utility is bounded, even at zero consumption.

[3]Indeed, in an example due to M.L. Stern, reported in Becker (2006), the impatient household holds positive capital infinitely often. Becker et al. (2014) provide a reciprocal example in which the most patient household reaches a no capital position infinitely often.

[4]Attempts have been made to seek alternative conditions which guarantee the convergence of the

To finish, it is worthwhile to point out that the properties of the continuous-time formulation of the Ramsey model stand in stark contrast to the ones of the discrete-time version. As a matter of fact, Mitra and Sorger (2013) proved that in the continuous-time Ramsey economy: (i) the unique steady state equilibrium is globally asymptotically stable, and (ii) along every Ramsey equilibrium the most patient household eventually owns the whole stock of capital.

1.3 Organization

This paper is organized as follows. Section 2 is devoted to the model and basic assumptions, with the definition of Ramsey equilibrium with externalities, and a small global analysis. Section 3 establishes the existence of non-convergent equilibria, even though the turnpike property applies and the MIM condition holds. Section 4 concludes. All proofs are gathered in the Appendix.

2. The Ramsey economy with dependent preferences

This section describes the economy under consideration. Except for the assumption that individual tastes are dependent, this is the standard competitive Ramsey model with borrowing constraints comprehensively surveyed by Becker (2006).

2.1 Fundamentals

Time is discrete; periods are indexed by $t \geq 0$. The production sector consists of a set of identical competitive firms, which transform labor and capital into a homogeneous output good. The set of firms has unit measure. Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$. The common technology is described by the linearly homogeneous production function $F : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$. At the beginning of period t, every firm hires L_t unit of labor and K_t unit

equilibrium capital sequence. For instance, Borrissov and Dubey (2015) relax the no borrowing condition by letting the households to be able to borrow against their next period wage income and show that irrespective of production function, the capital stock sequence converges; see also Becker et al. (2015).

of capital in order to produce the amount of output $F(K_t, L_t)$. Let denotes the rental rates for labor and capital in period t by w_t and r_t, respectively. In every period $t \geq 0$, firms solve the static problem:

$$\mathrm{P}^f = \max_{(K_t, L_t)} F(K_t, L_t) - r_t K_t - w_t L_t. \tag{1}$$

In order to state assumptions about the technology, it will be useful to define a reduced production function written only in terms of capital. Define the function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ by $f(K) = F(K, \ell)$, where ℓ is the total labor endowment of the economy; see below. It is assumed that

Assumption 1. Assumption on production function: *The reduced production function f is continuous on $\mathbb{R}_+$ and C^2 on $\mathbb{R}^2_{++}$ with $f(0) = 0$, $f'(K) > 0$, and $f''(K) < 0$ for all $K \in \mathbb{R}_{++}$. In addition, it holds that $\lim_{K \to 0} f'(K) = +\infty$ and $\lim_{K \to \infty} f'(K) < 1$ (Inada conditions).*

Under Assumption 1, whenever $0 < r_t < \infty$, there is a unique positive stock K_t which solves P^f at each t:

$$f'(K_t) = r_t. \tag{2}$$

The corresponding wage, w_t, is positive and given by the zero profit condition:

$$w_t = \frac{1}{L_t} \times \left[f(K_t) - K_t f'(K_t) \right]. \tag{3}$$

As regards the consumption sector, there is a finite number H of households labeled by $h \in \mathcal{H} := \{1, ..., H\}$. The lifetime preferences of households $h \in \mathcal{H}$ are described by an additively separable utility function characterized by $(u^h; \delta_h)$, where u^h is the felicity or one-period utility function and δ_h denotes the constant discount factor. This paper being aimed at illustrating how nonmarket interactions impinge upon Ramsey equilibria, it will be assumed that besides her own consumption, each household cares about the consumption by all the other households in the economy. The consumption of others may matter because individuals are altruistic, envious, non-conformist, or even malevolent. In this setting,

$$u^h : \mathbb{R}^{H-1}_+ \times \mathbb{R} \mapsto \mathbb{R},$$

so that $u^h(c^h, c^{-h})$ represents household h's felicity associated with the consumption c^h and the consumption by the other households $c^{-h} := (c^i)_{i \in \mathcal{H} \setminus h}$.

Assumption 2. Assumption on preferences:

i) *We have* $1 > \delta_1 > \delta_2 \geq \delta_3 \geq \cdots \geq \delta_H > 0$.

ii) *For each* $h \in \mathcal{H}$ *the function* $u^h : \mathbb{R}_+^{H-1} \times \mathbb{R} \mapsto \mathbb{R}_+$ *is continuous and* $\mathcal{C}^2$ *on* $\mathbb{R}_+ \times \mathbb{R}_+^{H-1}$. *In addition, for each* $c^{-h} \in \mathbb{R}_+^{H-1}$, *the function* $u^h(\cdot, c^{-h})$ *is strictly increasing and strictly concave on* $\mathbb{R}_{++}$, *satisfying the Inada condition:*

$$\lim_{c^h \to 0} \frac{\partial u^h}{\partial c^h}(c^h, c^{-h}) = \infty.$$

iii) *For each* $h \in \mathcal{H}$ *the felicity function* u^h *is non-separable in externalities:*

$$\frac{\partial}{\partial c^i}\left(\frac{\partial u^h}{\partial c^h}\right) \neq 0, \ \forall i \neq h.$$

iv)

$$\sup_{c \in \mathbb{R}_+^H, \tilde{c}^{-h} \in \mathbb{R}_+^{H-1}} \frac{\frac{\partial u^h}{\partial c^h}(c^h, c^{-h})}{\frac{u^h}{\partial c^h}(c^h, \tilde{c}^{-h})} < \infty.$$

Assumptions 2(i) and 2(ii) are standard. They assume the existence of a most patient agent, the boundedness from below of the utility functions, their concavity and the Inada property.

Assumption 2(iii) implies that externalities do influence not only the felicity levels but also the marginal rate of substitution. In other words, an agent's evaluation of a trade is allowed to depend on the trades engaged by other members of the economy. More precisely, the marginal rate of substitution between any pair of adjacent dates for household $h \in \mathcal{H}$ depends on the consumption of all other households at those dates:

$$\text{MRS}_{t,t+1}^h = \frac{\frac{\partial u^h}{\partial c^h}\left(c_t^h, c_t^{-h}\right)}{\delta_h \frac{\partial u^h}{\partial c^h}\left(c_{t+1}^h, c_{t+1}^{-h}\right)}. \tag{4}$$

This feature is critical. As a matter of fact, the mere dependence of u^h on c^{-h} does not mean that the economic behavior of a household will depend upon the consumption of the others. If, for instance, each household's felicity function is additive separable in the consumption of the rest of households, the presence of externalities would have welfare effects, but it would not affect the behavior of any household.[5] One expects consumption

[5]In a pure exchange economy Dufwenberg et al. (2011) showed that with additively separable utility functions, equilibrium prices and allocations are those of the economy without externalities.

externalities to affect the outcome of competitive markets if and only if they have an effect on the marginal rates of substitution. Sole non-additively separable externalities introduce intricate interdependencies.

Assumption 2(iv) ensures that the marginal rate of substitution $\mathrm{MRS}^h_{t,t+1}$ is bounded away from zero and infinity.

Household $h \in \mathcal{H}$ is endowed with $k^h \geq 0$ units of capital at time $t = 0$ and $\ell^h > 0$ units of labor at all dates $t \geq 0$.[6] Let x^h_t denote the capital stock held by household h at the beginning of period t. For simplicity, and without loss of generality, it will be assumed that capital fully depreciates within the period.

The characteristic of the competitive environment is that every household behaves as though she were unable to influence the market prices or the actions of other households. Given the prices sequences $\{w_t\}_{t\geq 0}$ and $\{r_t\}_{t\geq 0}$, and the sequence of consumption patterns of other households $\{c^{-h}_t\}_{t\geq 0}$, each household $h \in \mathcal{H}$ solves:

$$\mathrm{P}^h = \max_{\{c^h_t, x^h_{t+1}\}} \sum_{t=0}^{\infty} \delta^t_h u^h(c^h_t, c^{-h}_t), \tag{5}$$

subject to

$$c^h_t + x^h_{t+1} = r_t x^h_t + w_t \ell^h, \tag{6}$$

$$c^h_t \geq 0,\ x^h_{t+1} \geq 0,\ x^h_0 = k^h,\ \{c^{-h}_t\}_{t\geq 0} \text{ given.} \tag{7}$$

where the constraint (7) states that households must have non-negative wealth at each time; they are not allowed to finance present consumption by borrowing against future income.

The no arbitrage condition for $\{c^{h\star}_t, x^{h^*}_{t+1}\}_{t\geq 0}$ to solve P^h are $c^{h\star}_t > 0$ and:

$$\frac{\partial u^h}{\partial c^h_t}\left(c^{h\star}_t, c^{-h\star}_t\right) \geq \delta_h r_{t+1} \frac{\partial u^h}{\partial c^h_{t+1}}, \left(c^{h\star}_{t+1}, c^{-h\star}_{t+1}\right) \text{ with `` = '' if } x^{h\star}_{t+1} > 0. \tag{8}$$

The Euler condition is sufficient beging provided that

$$\lim_{t\to\infty} (\delta_h)^t \frac{\partial u^h}{\partial c^h_t}\left(c^{h\star}_t, c^{-h\star}_t\right) x^{h\star}_{t+1} = 0.$$

A collection $\mathcal{E} = (f, \{u^h, \delta_h, k^h, \ell^h\}_{h\in\mathcal{H}})$ satisfying Assumptions 1-4, $k^h \geq 0$, $\sum_{h\in\mathcal{H}} k^h > 0$, and $\ell^h > 0$ is said to be a *dependent household felicities Ramsey economy*, or an economy for short.

[6]Notice that, because the utility is derived solely from consumption goods, the competitive household will offer its entire endowment of labor services to the market in each time period.

2.2 The equilibrium concept and a global analysis

The assumed competitive organization of markets along the widespread nature of externalities justify using a noncooperative perfect foresight equilibrium. More specifically, a competitive equilibrium in a Ramsey economy with widespread externalties is defined so that:

(i) Agents (households and firms) maximize their goals by perfectly anticipating and taking as given both the sequences of prices and levels of externalities.

(ii) The induced demands and supplies balance at every point of time.

(iii) The resulting levels of externalities coincide at every date with expected levels.

Definition **2.1.** *The sequences of rental rates* $\{r_t^\star, w_t^\star\}_{t\geq 0}$ *and allocations* $\{K_t^\star, L_t^\star, (c_t^{h\star}, x_t^{h\star})_{h\in\mathcal{H}}\}_{t\geq 0}$ *constitute an equilibrium for an economy* $\mathcal{E}$ *provided that:*

1) *for all* $h \in \mathcal{H}$, $\{c_t^{h\star}, x_t^{h\star}\}_{t\geq 0}$ *solves* P^h *given* $\{r_t^\star, w_t^\star\}_{t\geq 0}$ *and* $\{\mathbf{c}^{-h\star}\}$*;*

2) *for each* $t \geq 0$, $(K_t^\star, L_t^\star)$ *solves* P^f *given* $(r_t^\star, w_t^\star)$*;*

3) *the capital market clears:* $K_0^\star = \sum_{h\in\mathcal{H}} k^h$ *and, for all* $t \geq 1$, $K_t^\star = \sum_{h\in\mathcal{H}} x_t^{h\star}$*;*

4) *the labor market clears: for each* $t \geq 0$, $L_t^\star = \sum_{h\in\mathcal{H}} \ell^h := \ell$.

Walras law ensuring balance on the output market, i.e., $\sum_{h\in\mathcal{H}}(c_t^{h\star} + x_{t+1}^{h\star}) = f(K_t^\star)$, *for all* $t \geq 1$, $\sum_{h\in\mathcal{H}}(c_0^{h\star} + k^h) = f(K_0^\star)$.

A Ramsey equilibrium $\{r_t, w_t, K_t, L_t, (c_t^h, x_t^h)_{h\in\mathcal{H}}\}$ is a *stationary* Ramsey equilibrium provided that, for all $t \geq 0$, $r_t = \overline{r}$, $w_t = \overline{w}$, $K_t = \overline{K}$, $L_t = \overline{L}$, $c_t^h = \overline{c}^h$, $x_t^h = \overline{x}^h$.

Let

$$r(x) := f'(x)x, \tag{9}$$

$$w(x) := \frac{f(x) - f'(x)x}{\ell}. \tag{10}$$

Definition **2.2.** *The economy satisfies the Income Monotonicity Condition (IMC) if the function*

$$\mathcal{I}^h(x) := r(x) + w(x)l^h$$

is increasing with respect to x*, for all* h*.*

It is clear that if $r(x)$ is increasing with respect to x, then the IMC condition is satisfied.

A turnpike equilibirum is defined as an equilibrium such that $x_t^{h\star} = 0$ with $h \in \mathcal{H}^\iota$, with prices $r_t^\star = r(x_t^{1\star})$, $w_t^\star = w(x_t^{1\star})$. Precisely,

$$x_0^{1\star} = k^1, \text{ and } x_t^{1\star} > 0, \forall t \geq 1, \tag{11}$$

$$\frac{\partial u^1}{\partial c_t^1}\left(c_t^{1\star}, (w_t^\star \ell^h)_{h\in\mathcal{H}^\iota}\right) = \delta_1 r_{t+1}^\star \frac{\partial u^1}{\partial c_{t+1}^1}\left(c_{t+1}^{1\star}, (w_{t+1}^\star \ell^h)_{h\in\mathcal{H}^\iota}\right), \forall t \geq 0, \tag{12}$$

$$\frac{\partial u^h}{\partial c_t^h}\left(w_t^\star \ell^h, \left(c_t^{1\star}, (w_t^\star \ell^i)_{i\in\mathcal{H}^\iota\setminus\{h\}}\right)\right) \geq \delta_h r_{t+1}^\star \frac{\partial u^h}{\partial c_t^h}\left(w_t^\star \ell^h, \left(c_t^{1\star}, (w_t^\star \ell^i)_{i\in\mathcal{H}^\iota\setminus\{h\}}\right)\right), \forall t \geq 0, \forall h \in \mathcal{H}^\iota, \tag{13}$$

$$\lim_{t\to\infty} (\delta_h)^t \frac{\partial u^h}{\partial c_t^h}\left(c_t^{h\star}, c_t^{-h\star}\right) x_{t+1}^{h\star} = 0, \forall h \in \mathcal{H}. \tag{14}$$

A stationary turnpike equilibrium is a turnpike equilibrium that is stationary.

For instance, we focus on the existence of an equilibrium for this economy. Under Assumptions 1 and 2 that we imposed on the whole article from now on, an equilibrium exists.

Proposition 2.1. *The economy $\mathcal{E}$ admits an equilibrium.*

Let $\overline{K}$ be the solution to $\delta_1 f'(K) = 1$ and $\underline{K}$ be the capital accumulation such that

$$f'(\underline{K}) = \frac{1}{\delta_H} \times \sup_{c\in\mathbb{R}_+^H, \tilde{c}^{-h}\in\mathbb{R}_+^{H-1}} \frac{\frac{\partial u^h}{\partial c^h}(c^h, c^{-h})}{\frac{u^h}{\partial c^h}(c^h, \tilde{c}^{-h})}.$$

By the concavity of the production function f, we have $\underline{K} < \overline{K}$. Proposition 2.2 provides some results about global dynamic of the economy. The capital sequence is bounded from below, there is no agent with a consumption sequence that converges to zero, and the capital stock level of every impatient agent visits infinitely often the zero level. As a direct consequence of this result, a stationary equilibrium must satisfy the turpike property.

Proposition 2.2. *Assume that $\{r_t, w_t, K_t, L_t, (c_t^h, x_t^h)_{h\in\mathcal{H}}\}_{t=0}^\infty$ is an equilibrium.*

i) *We have*

$$\limsup_{t\to\infty} K_t \geq \overline{K}.$$

ii) *For every $t \geq 0$, $K_t \geq \min\{K_0, \underline{K}\}$.*

iii) *For every h,*

$$\limsup_{t\to\infty} c_t^h > 0.$$

iv) *For every $h \geq 2$, there exist an infinite number of time t such that $x_t^h = 0$.*

3. Maximal Income Monotonicity condition and non-convergent Ramsey equilibria

The existence theorem is silent about the qualitative properties of Ramsey equilibria. This section is aimed at shedding some light on these properties. The explanatory inquiry hereafter is interested in wether or not the aggregate capital path converges in an equilibrium configuration. In the setting without external effects, Theorem 4 in Becker et al. (2014) showed that if the maximal income that any household can receive in monotone increasing, then the aggregate capital sequence along *every*, *i.e.*, irrespective of the initial wealth distribution, equilibrium path is convergent, and eventually the turnpike property holds. They called this sufficient condition the *Maximal Income Monotonicity* (MIM) condition.

It will be established that the MIM condition is no longer sufficient to ensure convergence whenever households' preferences are dependent. First, by constructing a *two* households economy admitting an equilibrium which steadily exhibits the recurrence property. Second, by characterizing a special class of equilibria in which the most patient household owns the entire stock of capital of the economy, *i.e.*, called the turnpike property.

3.1 Non-convergence: an example

This section considers an economy $(f,(u^1,\delta_1,k^1,\ell^1),(u^2,\delta_2,k^2,\ell^2))$ satisfying Assumptions 1-2, and $0<\delta_2<\delta_1<1$. It aims to construct an equilibrium exhibiting the recurrence property permanently notwithstanding the fact that the MIM condition on f is met. To properly state the latter, observe that the maximal income a single household h can receive is when that household owns the whole stock of capital, *i.e*, $x^h = K$. In this happens, the said income would be $w(K)\ell^h + r(K)K$, where $w(K)$ and $r(K)$ are the equilibrium wage rate for labour and rental rate for capital, respectively. The MIM condition means that the latter is monotonously increasing in K.

Let $\bar{K}$ be the unique solution to the equation $\delta_1 r(K) = 1$. For each value $K^\star > \bar{K}$, denote by $K_\star$ the capital satisfying $K_\star < \bar{K} < K^\star$ and:

$$r(K^\star)r(K_\star) = \frac{1}{\delta_1^2}.$$

Now it is obvious that if $K^\star$ converges to $\bar{K}$, the same happens for $K_\star$. We will prove the following claim: for $K^\star$ and $K_\star$ sufficiently close to $\bar{K}$, there exist $(\tilde{x}^1, \tilde{x}^2, c, \tilde{c}_0^2, \tilde{c}_1^2) \in \mathbb{R}^5_{++}$ such that:

$$c + K_\star = w(K^\star)\ell^1 + r(K^\star)\tilde{x}^1, \tag{15a}$$

$$\tilde{c}_0^2 = w(K^\star)\ell^2 + r(K^\star)\tilde{x}^2, \tag{15b}$$

$$c + \tilde{x}^1 = w(K_\star)\ell^1 + r(K_\star)K_\star, \tag{15c}$$

$$\tilde{c}_1^2 + \tilde{x}^2 = w(K_\star)\ell^2, \tag{15d}$$

$$\tilde{x}^1 + \tilde{x}^2 = K^\star. \tag{15e}$$

Indeed, equations (15a) and (15c), yields:

$$\tilde{x}^1 = \frac{(w(K_\star) - w(K^\star))\ell^1 + (1 + r(K_\star))K_\star}{1 + r(K^\star)}.$$

We can re-write this equality as:

$$\tilde{x}^1 - K^\star = \frac{w(K_\star)\ell^1 + r(K_\star)K_\star - (w(K^\star)\ell^1 + r(K^\star)K^\star) + K_\star - K^\star}{1 + r(K^\star)}.$$

The MIM condition thus entails $\tilde{x}^1 < K^\star$, which, from (15e), is equivalent to $\tilde{x}^2 > 0$. One verifies that as $K^\star$ converges to $\overline{K}$, $(\tilde{x}^1, c, \tilde{c}_0^2, \tilde{c}_1^2)$ converge correspondingly to

$$\left(\bar{K}, w(\bar{K})\ell^1 + \left(\frac{1}{\delta_1} - 1\right)\bar{K}, w(\bar{K})\ell^2, w(\bar{K})\ell^2\right) \in \mathbb{R}^4_{++}.$$

It follows that, as claimed, for $K^\star$ and $K_\star$ close enough to $\bar{K}$, the following values are strictly positive: $(\tilde{x}^1, \tilde{x}^2, c, \tilde{c}_0^2, \tilde{c}_1^2) \in \mathbb{R}^5_{++}$.

To pursue, notice that $c + \tilde{c}_0^2 + K_\star = f(K^\star)$, and $c + \tilde{c}_1^2 + K^\star = f(K_\star)$. Hence, $\tilde{c}_0^2 > \tilde{c}_0^1$.

Fix $\varepsilon > 0$ small enough so as $\tilde{c}_0^2 - r(K^\star)\varepsilon > \tilde{c}_1^2 + \varepsilon$. Let

$$\begin{aligned} x^1 &= \tilde{x}^1 + \varepsilon, \\ x^2 &= \tilde{x}^2 - \varepsilon, \\ c_0^1 &= c + r(K^\star)\varepsilon, \\ c_1^1 &= c - \varepsilon, \\ c_0^2 &= \tilde{c}_0^2 - r(K^\star)\varepsilon, \\ c_1^2 &= \tilde{c}_1^2 + \varepsilon. \end{aligned}$$

Making use en (15a)-(15e) one obtains:

$$\begin{aligned} c_0^1 + K_\star &= w(K^\star)\ell^1 + r(K^\star)x^1, \\ c_0^2 &= w(K^\star)\ell^2 + r(K^\star)x^2, \\ c_1^1 + x^1 &= w(K_\star)\ell^1 + r(K_\star)K_\star, \\ c_1^2 + x^2 &= w(K_\star)\ell^2, \\ x^1 + x^2 &= K^\star. \end{aligned}$$

Now, assume that felicity functions u^h are given by:

$$u^h(c_t^h, c_t^{-h}) = u(c_t^h)v^h(c_t^{-h}), \;\; h = 1, 2,$$

where u is a concave function, v^h is a continuous, increasing function such that $v^h(c) > 0$ for $c \geq 0$.

Recall that $c_1^1 < c_0^1$, and $c_0^2 > c_1^2$. It is obvious that

$$\begin{aligned} u'(c_0^1) &< \delta_1 r(K_\star)u'(c_1^1), \\ u'(c_1^2) &> \delta_2 r(K^\star)u'(c_0^2). \end{aligned}$$

It is clear that we can choose functions v^1 and v^2 strictly increasing such that

$$\begin{aligned} u'(c_0^1)v^1(c_0^2) &= \delta_1 r(K_\star)u'(c_1^1)v^1(c_1^2), \\ u'(c_1^2)v^2(c_1^1) &= \delta_2 r(K^\star)u'(c_0^2)v^2(c_0^1). \end{aligned}$$

Keeping in mind that $r(K^\star)r(K_\star) = \frac{1}{\delta_1^2} < \frac{1}{\delta_2^2}$, the previous equalities imply:

$$\begin{aligned} u'(c_1^1)v^1(c_1^2) &= \delta_1 r(K^\star)u'(c_0^1)v^1(c_0^2), \\ u'(c_0^2)v^2(c_0^1) &> \delta_2 r(K_\star)u'(c_1^2)v^2(c_1^1). \end{aligned}$$

Now, consider the sequence of rental rates $\{r_t, w_t\}$ such that for any $s \geq 0$,

$$\begin{aligned} r_{2s} &= r(K^\star), \\ r_{2s+1} &= r(K_\star), \\ w_{2s} &= w(K^\star), \\ w_{2s+1} &= w(K_\star). \end{aligned}$$

For $s \geq 0$, let

$$\begin{aligned} x^1_{2s} &= x^1, \\ x^1_{2s+1} &= K_\star, \\ x^2_{2s} &= x^2, \\ x^2_{2s+1} &= 0, \\ c^1_{2s} &= c^1_0, \\ c^1_{2s+1} &= c^1_1, \\ c^2_{2s} &= c^2_0, \\ c^2_{2s+1} &= c^2_1. \end{aligned}$$

Finally, it is easy to verify that the sequence of aggregate capital stocks $\{K^\star, K_\star, K^\star, K_\star, ...\}$; the dominant household's stocks $\{x^1, K_\star, x^1, K_\star, \ldots\}$ and consumption stream $\{c^1_0, c^1_1, c^1_0, c^1_1, \ldots\}$; the impatient household's holdings of capital $\{x^2, 0, x^2, 0, \ldots\}$ and sequence of consumptions, $\{c^2_0, c^2_1, c^2_0, c^2_1, \ldots\}$ verify $x^1_t + x^2_t = K_t$, the budget balance conditions

$$c^h_t + x^h_{t+1} = w^h_\ell + r_t x^h_t, \;\; h = 1, 2.$$

Moreover, they satisfy the no arbitrage and transversality conditions for each household. They consequently constitute a Ramsey equilibrium.

3.2 Non-convergence: the dynamical approach

One of our main purposes is to characterize a special class of equilibria in which the turnpike property holds. The motivation for focusing upon this specific solution is twofold: on the one hand, the turnpike property holds at the stationary equilibrium; on the other, it allows to examine the behavior of the non-stationary equilibria by making use of the dynamical systems approach initiated by Becker and Foias (1990).[7]

3.2.1 The turnpike property: a myopia argument

The equilibrium path has the turnpike property when the capital stocks held by relatively impatient households, *i.e.*, the ones of whom discount factors are below the highest discount factor in the economy, are zero. Starting from an *arbitrary* endowment of capital $\{k^h : k^h \geq 0\}$, the turnpike property should be understood as meaning that every household other that the most patient one *eventually* reach a no capital position and maintain that state thereafter.

In the narrower sense used here, the turnpike property on the capital ownership pattern holds *for all time*. This clearly requires that the capital endowment of relatively impatient households must be zero. The economies considered in the sequel will therefore satisfy the skewed capital endowment condition: The initial distributions of capital are such as $k^1 > 0$ and $k^h = 0$, for all $h \in \mathcal{H}^\iota$.

However, even though impatient households have a zero capital stock ownership position at time 0, they receive a positive wage, and they always have the option of acquiring capital. Thus, in order for the turnpike property to hold *for all time*, the equilibrium path must be constructed in such way that only the most patient household has the willingness to accumulate capital. Lemmas 3.1 and 3.2 state the necessary and sufficient conditions for this to happen.

Clearly, whenever the turnpike property holds, the resulting properties of the model are deduced by examining this special case where the aggregate capital stock and the most patient household's stock are the same. The resulting paths of aggregate capital stocks and consumptions for the most patient household, together with the assignment of wage income to the relatively more impatient households always expresses an equilibrium for some economy. That is, the felicity functions of impatient households and their discount factors can always be chosen to support the specially constructed path as a Ramsey equilibrium.

Following Becker and Tsyganov (2002), the turnpike property obtains whenever relatively

[7] To cite some contributions where the turnpike property obtains, see Becker and Foias (1994), Becker and Tsyganov (2002), and Sorger (1994).

impatient households are sufficiently *myopic* in comparison with the dominant household's time preference. Notice that the myopia argument is particularly suitable here inasmuch as, by assumption, discount factors are exogenous, i.e., not related to the state of the economy.

Formally, fix the felicity functions and consider the sequence $\{c_t^{1\star}, x_t^{1\star}\}_{t\geq 0}$, with $x_0^{1\star} = k^1 > 0$, constructed from the most patient household's no arbitage conditions together with that household's budget balance relations:

$$\frac{\partial u^1}{\partial c_t^1}\left(c_t^{1\star}, \left(w(x_t^{1\star})\ell^h\right)_{h\in\mathcal{H}^\iota}\right) = \delta_1 r\left(x_{t+1}^{1\star}\right)\frac{\partial u^1}{\partial c_{t+1}^1}\left(c_{t+1}^{1\star}, \left(w(x_{t+1}^{1\star})\ell^h\right)_{h\in\mathcal{H}^\iota}\right) \tag{16}$$

$$c_t^{1\star} + x_{t+1}^{1\star} = r\left(x_t^{1\star}\right)x_t^{1\star} + w(x_t^{1\star})\ell^1, \tag{17}$$

hypothesizing that $c_t^{h\star} = w(x_t^{1\star})\ell^h$ for each $h \in \mathcal{H}^\iota$. Now, let

$$\underline{\delta}_h := \inf_{t,t+1}\frac{\frac{\partial u^h}{\partial c_t^h}\left(w(x_t^\star)\ell^h, \left(c_t^{1\star}, (w(x_t^\star)\ell^i)_{h\in\mathcal{H}^\iota\setminus\{h\}}\right)\right)}{r(x_{t+1}^\star)\frac{\partial u^h}{\partial c_{t+1}^1}\left(w(x_{t+1}^\star)\ell^h, \left(c_{t+1}^{1\star}, (w(x_{t+1}^\star)\ell^i)_{h\in\mathcal{H}^\iota\setminus\{h\}}\right)\right)}. \tag{18}$$

Clearly, if $\delta_h \leq \underline{\delta_h}$ the agent's h no arbitrage conditions (13) will remain slack along the constructed path; this means that perfectly foreseeing the sequence of rental prices $\{w_t^\star, r_t^\star\} = \{w(x_t^{1\star}), r(x_t^{1\star})\}$, household h has no incentive to acquire capital. It follows that $\{c_t^{1\star}, x_t^{1\star}\}_{t\geq 0}$, constructed from equations (16)-(17), together with $\{c_t^{h\star}, x_t^{h\star}\}_{t\geq 0} = \{w(x_t^{1\star})\ell^h, 0\}_{t\geq 0}$, for each $h \in \mathcal{H}^\iota$, is a Ramsey equilibrium along which the most patient household owns all the capital. This inspires us to Lemma 3.1.

For each capital level $k^1 > 0$, let $\Pi(k^1)$ be the set of sequences $\{x_t^1\}_{t=0}^\infty$ such that $x_0^1 = k^1$ and

$$0 \leq x_{t+1}^1 \leq r(x_t^1)x_t^1 + w(x_t^1)l^1,$$

for every $t \geq 0$.

Lemma **3.1.** *Fix* $k^1 > 0$.

i) *For each sequence* $\{x_t^1\}_{t=0}^\infty \in \Pi(k^1)$, *let* $c_t^h = w(x_t^1)l^h$, *for* $h \geq 2$. *There exists solution to* $\{\hat{x}_t^1\}_{t=0}^\infty \in \Pi(k^1)$ *to the following problem:*

$$\max \sum_{t=0}^{\infty}(\delta_1)^t u^1(\hat{c}_t^1, c_t^{-h}),$$
$$s.\ c.\ \hat{c}_t^1 + \hat{x}_{t+1}^1 = r(x_t^1)\hat{x}_t^1 + w(x_t^1)l^1,$$
$$\hat{x}_0^1 = k^1, \hat{c}_t^1 \geq 0 \text{ for every } t.$$

ii) *For* $\boldsymbol{x}^1 \in \Pi(k^1)$*, let* $T^1(\boldsymbol{x}^1) = \hat{\boldsymbol{x}}^1$*. There exists a sequence* $\boldsymbol{x}^{1\star} \in \Pi(k^1)$ *such that* $\boldsymbol{x}^{1\star} = T^1(\boldsymbol{x}^{1\star})$*.*

For $\{x_t^{1\star}\}_{t=0}^{\infty}$ that is a fixed point of T^1 in Lemma 3.1, let $\underline{\delta}_h$ be defined as in (18). Observe that $\underline{\delta}_h$ crucially depends upon δ_1 and k^1, and even on the chosen fixed point of T^1. Therefore, and roughly speaking, saying that $\delta_h \leq \underline{\delta}_h$ is tantamount to regarding relative impatient households as strongly myopic in comparison to the dominant one.

Assumption 3. *For each* $h \in \mathcal{H}^\iota$*,* $\delta_h \leq \underline{\delta}_h$*.*

A sequence $\{c_t, x_t\}_{t\geq 0}$ is called a turnpike consumption-capital sequence if the following conditions are satisfied:

$$x_0 > 0, \tag{19}$$

$$\frac{\partial u^1}{\partial c_t}\left(c_t, \left(w(x_t)\ell^h\right)_{h\in\mathcal{H}^\iota}\right) = \delta_1 r\left(x_{t+1}\right)\frac{\partial u^1}{\partial c_{t+1}}\left(c_{t+1}, \left(w(x_{t+1})\ell^h\right)_{h\in\mathcal{H}^\iota}\right), \tag{20}$$

$$c_t + x_{t+1} = r\left(x_t\right)x_t + w(x_t)\ell^1, \tag{21}$$

$$\lim_{t\to\infty}(\delta_1)^t\frac{\partial u^1}{\partial c_t}\left(c_t, \left(w(x_t)\ell^h\right)_{h\in\mathcal{H}^\iota}\right)x_{t+1} = 0. \tag{22}$$

Lemma 3.2 paves the necessary conditions for the existence of a turnpike equilibrium path.

Lemma **3.2.** *The sequence* $\{r_t, w_t, (c_t^h, x_t^h)_{h\in\mathcal{H}^\iota}\}_{t\geq 0}$ *is turnpike equilibrium if* $\{c_t^1, x_t^1\}_{t\geq 0}$ *is a turnpike consumption-capital sequence and*

$$r_t = r(x_t) \text{ and } w_t = w(x_t),$$
$$c_t^1 = c_t \text{ and } x_t^1 = x_t,$$
$$c_t^h = w_t\ell^h \text{ and } x_t^h = 0,\ \forall h \in \mathcal{H}^\iota.$$

The above-mentioned considerations suggest Proposition 3.1.

Proposition **3.1.** i) *There exists unique stationary turnpike equilibrium. This equilibrium satisfies* $k^1 = \overline{K}$*.*

ii) *Fix* $k^1 > 0$*. By adding the myopic assumption, the economy* $\mathcal{E}$ *admits a turnpike equilibrium beginning from* $K_0 = k^1$*.*

3.2.2 A local dynamics analysis

The object of this subsection is to show that there exists dependent felicities Ramsey equilibria satisfying the turnpike property (by construction) that fail to converge even when the MIM condition is satisfied.

When the turnpike property holds, the equilibrium dynamics can be described by a two-dimensional dynamical system. The first equation of that system is the budget balance relation of the dominant household, which can be written:

$$x_{t+1} = g(x_t) - c_t,$$

where $g(x) := r(x)x + w(x)\ell^1$; g denotes the dominant agent's income function. As regards the second equation, Assumptions 1-3 imply the existence of a continuous function $F : \mathbb{R}_{++} \times \mathbb{R}_{++} \mapsto \mathbb{R}_{++}$ such that

$$\frac{\partial u^1}{\partial c_t}\left(c, \left(w(x)\ell^h\right)_{h\in\mathcal{H}^\iota}\right) = \delta r\left(G(x,c)\right)\frac{\partial u^1}{\partial c_{t+1}}\left(F(x,c), w\left(G(x,c)\right)\ell^h\right)_{h\in\mathcal{H}^\iota}\right),$$

where $G(x,c) := g(x) - c$. A Ramsey equilibrium with the turnpike property is then an orbit $\{x_t, c_t\}_{t\geq 0}$, such that $(x_t, c_t) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$, generated by the equations

$$x_{t+1} = G(x_t, c_t), \tag{23}$$

$$c_{t+1} = F(x_t, c_t), \tag{24}$$

from the initial condition (x_0, c_0).[8]

The constant orbit $\{x_t, c_t\}_{t\geq 0} = \{\overline{x}, \overline{c}\}$, where $(\overline{x}, \overline{c}) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$ is the unique fixed point of the system (23)-(24), is a stationary Ramsey equilibrium if $\overline{x} = \overline{K}$. If $x_0 \neq \overline{x}$, the Ramsey equilibrium is non-stationary. If it happens that $\lim_{t\to\infty} x_t = \overline{x}$, the Ramsey equilibrium is *asymptotically* stationary. Should this not be the case, and with a slight abuse of terminology, the Ramsey equilibrium will be referred to as *non-convergent*.

[8]In the current economic model, only the aggregate capital endowment $x_0 = k^1$ is a given data; c_0 must be properly chosen in order for the orbit to represent an equilibrium trajectory, *i.e.*, to satisfy the transversality condition (22).

Let

$$\begin{aligned}
\theta &:= \frac{\partial u^1}{\partial c}\left(\bar{c}, ((w(\bar{x})\ell^h)_{h\in\mathcal{H}^\imath}\right), \\
\eta &:= -\frac{\partial^2 u^1}{(\partial c)^2}\left(\bar{c}, ((w(\bar{x})\ell^h)_{h\in\mathcal{H}^\imath}\right), \\
\varepsilon &:= \sum_{h\in\mathcal{H}^\imath} \frac{\partial^2 u^1}{\partial c \partial c^h}\left(\bar{c}, w(\bar{x})\ell^h\right)\frac{\ell^h}{\ell}.
\end{aligned}$$

The differentiation of the system (23)-(24), evaluated at $(\bar{x}, \bar{c})$, yields:

$$\begin{aligned}
dx_{t+1} &= -dc_t + g'(\bar{x})dx_t, \\
-\eta dc_{t+1} + (\delta r'(\bar{x})\theta + w'(\bar{x})\ell\varepsilon)dx_{t+1} &= -\eta dc_t + w'(\bar{x})\ell\varepsilon dx_t.
\end{aligned}$$

The Jacobian matrix is determined as:

$$J = \frac{1}{\eta}\begin{pmatrix} \eta g'(\bar{x}) & -\eta \\ (1-g'(\bar{x}))\, w'(\bar{x})\varepsilon - g'(\bar{x})\delta r'(\bar{x})\theta & \eta + \delta r'(\bar{x})\theta + w'(\bar{x})\varepsilon \end{pmatrix}.$$

The characteristic polynomial for the Jacobian matrix J is $\lambda^2 - T\lambda + D = 0$, where

$$\begin{aligned}
T &= 1 + g'(\bar{x}) - \frac{1}{\eta}(\delta_1 r'(\bar{x})\theta + w'(\bar{x})\ell\varepsilon), \\
D &= g'(\bar{x}) - \frac{1}{\eta}w'(\bar{x})\ell\varepsilon.
\end{aligned}$$

The eigenvalues of the corresponding jacobian matrix J are real; one of them, at least, with a modulus greater than one. An *hyperbolic* fixed point $(\bar{x}, \bar{c})$ is therefore either locally unstable or a saddle point. In this latter case, the local stable manifold is one-dimensional. An orbit $\{x_t, c_t\}_{t\geq 0}$ lying on that manifold is asymptotically stationary, and thus a Ramsey equilibrium. Furthermore, for any x_0 sufficiently closes to $\bar{x}$, there exits an unique c_0 such that (x_0, c_0) lies on the stable manifold. This leads to the conclusion that the Ramsey equilibrium is locally unique, that is, there are no other equilibria, *i.e.*, no other orbit, in any close neighborhood of the steady state.

PROPOSITION **3.2.** *There exists an economy $\mathcal{E}$ which has a non-convergent Ramsey equilibrium.*

Proposition 3.2 tells us that a Ramsey equilibrium with the turnpike property may not be asymptotically stationary. As is readily seen, this could occur even if the *Income Monotonicity Condition* (IMC) holds true.

Let

$$\underline{\varepsilon} := -\frac{\delta r'(\bar{x})\theta}{2w'(x)\ell}.$$

From Proposition 3.2, if we add additional condition about Income Monotonicity Condition, the economy may exhibit non-convergent equilibrium, under condition $\varepsilon > \underline{\varepsilon}$. Observe that if $r(x)$ is increasing with respect to x, then the condition is satisfied with any $\varepsilon > 0$.

Corollary 3.1. *There exist economies satisfying Assumptions 1, 2, 3 and the Income Monotonicity Condition that admit a non-convergent equilibrium.*

Corollary 3.1 shows that, in the presence of external effects, the IMC is not sufficient to rule out non-convergence. A point of interest is actually the sign of:

$$S := \frac{\partial^2 u^1}{(\partial c^1)^2}\left(c^1, c^{-1}\right) + \sum_{h \in \mathcal{H}} \frac{\partial^2 u^1}{\partial c^1 \partial c^h}\left(c^1, c^{-1}\right)\frac{\ell^h}{\ell}.$$

Observe that at steady state,

$$S = -\eta + \varepsilon.$$

A positive value of S means that the overall effect of changes in the consumption (c^{-1}) of impatient households on the dominant household's marginal felicity (ε at the steady state), prevails over the *negative* effect of a change in the consumption c^1 of the dominant households (η in absolute value at the steady state). In other words, the most patient household's marginal felicity is more sensitive to the consumption of the others as a whole than to his own consumption.

One would then describe the positive external effects as being of the *first-order*. Should the latter phenomenon requested in order for non-convergent equilibria to exist, one would then say that these equilibria rest upon quite strong externalities.[9] As it will be shown below, this is indeed not the case.

[9]In the case of finite pure exchange economies with consumption external effects, Bonisseau and del Mercato (2010) established that the standard assumptions do not suffice to guarantee the generic regularity of the competitive equilibrium. An additional assumption on the *second order* external effects on utility, that ensures that the external effects on one consumer's marginal utilities is dominated by the effect of his own consumption, is required. In the absence of this assumption, they provide an example where equilibria are indeterminate for all initial endowments.

Proposition **3.3.** *Consider the set of economies satisfying the Income Monotonicity Condition, and such that $\varepsilon > \underline{\varepsilon}$. The subset of economies with $\eta > \varepsilon$ exhibiting non-convergent equilibria is non-empty.*

To gain further insights into the issue of how an economy's primitives relate to the existence of a non-convergent equilibrium it is convenient to to rewrite $g'(x)$ and $w'(x)\ell/(1+g'(x))$ in the following way:

$$g'(x) = r(x)\left(1 + \frac{r'(x)x}{r(x)} + \frac{w'(x)x}{w(x)}\frac{w(x)}{r(x)}\frac{\ell^1}{x}\right), \tag{25}$$

$$\frac{w'(x)\ell}{1+g'(x)} = \frac{\frac{w'(x)x}{w(x)}\frac{w(x)}{r(x)}\frac{\ell}{x}}{\frac{1}{r(x)} + \frac{r'(x)x}{r(x)} + 1 + \frac{w'(x)x}{w(x)}\frac{w(x)}{r(x)}\frac{\ell^1}{x}}. \tag{26}$$

Now, let

$$s(x) := \frac{f'(x)x}{f(x)}, \tag{27}$$

$$\sigma(x) := \frac{f'(x)(f(x) - xf'(x))}{xf(x)f''(x)}. \tag{28}$$

denote the share of capital in total income and the elasticity substitution between capital and labor, respectively. It is immediate to see that:

$$\frac{w'(x)x}{w(x)} = \frac{s(x)}{\sigma(x)},$$

$$\frac{r'(x)x}{r(x)} = -\frac{1-s(x)}{\sigma(x)},$$

$$\frac{w(x)}{r(x)} = \frac{1-s(x)}{s(x)}\frac{x}{\ell}.$$

In order to simplify the notation, hereafter the argument of the various functions will be omitted when referring to the steady state, *e.g.*, $s := s(\bar{x})$. Remembering that $\delta_1 r(\bar{x}) = 1$, substitutions and simplifications finally deliver:

$$g' = \frac{1}{\delta_1}\left(1 - \frac{1-s}{\sigma}\lambda\right), \tag{29}$$

$$\frac{w'\ell}{1+g'} = \frac{\frac{1-s}{\sigma}}{1+\delta_1 - \frac{1-s}{\sigma}\lambda}, \tag{30}$$

where $\lambda := 1 - \ell^1/\ell \in (0,1)$ denotes de share of the impatient households in the economy labour force.

Something can now be said about the economies in which non-convergence of the capital stock is eventually reconcilable with the increasing dominant household's incomes (IMC) and in parallel "moderate", *i.e.*, *second-order*, external effects.

It is worthy delineating these economies in terms of share of the impatient households, $\ell - \ell^i$, in the total labor force ℓ. The outcomes of that delineation are summarized in the following proposition.

Proposition **3.4.** *Make Assumptions 1-3. Let $\sigma \geq 0$ and $s \in (0,1)$ denote the steady state values of the factors elasticity of substitution and the capital share, respectively. Let $\lambda := 1 - \ell^1/\ell$ be the share of the impatient households in the labour force. Assume that:*

$$\lambda \in \left(\max\left\{0, \frac{\sigma}{1-s}(1+\delta_1) - 1\right\}, \min\left\{\frac{\sigma}{1-s}, 1\right\}\right), \text{ for } \sigma \in \left(0, 2\frac{1-s}{1+\delta_1}\right).$$

Then there are economies in which the IMC holds, the external effects are second-order in preferences, that exhibit non-convergent equilibria.

4. Conclusion

This paper has introduced consumption externalities in a standard Ramsey model with heterogeneous agents and borrowing constraints. It has been shown that the Maximum Income Monotonocity (MIM) assumption is no longer sufficient to rule out non-convergent Ramsey equilibria, even if the turnpike property applies. Furthermore, the existence of such equilibria is compatible with the second order external effects on felicity functions. This clearly establish that nonmarket interdependences may have noticeable positive (as opposite to normative) influence on competitive market mechanisms.

5. Appendix

5.1 Proof of Proposition 2.1

To simplify the exposition, let

$$u^{h\prime}(c^h, c^{-h}) = \frac{\partial u^h}{\partial c^h}(c^h, c^{-h}),$$

for every $h = 1, 2, \ldots, H$, and $c \in \mathbb{R}^H_+$.

We present here a proof being based on the arguments presented in the proof of Theorem 4.1 and Proposition 4.4 in Becker et al. (1991). Following Becker et al. (1991), the main idea is to construct a Tâtonnement Map on the set of feasible aggregate capital sequences, which is compact.

First, fix $\epsilon > 0$ small enough such that $\epsilon < \sum_{h=1}^{H} k^h$ and for every h:

$$f'(\epsilon) > \frac{1}{\delta_h} \times \sup_{c \in \mathbb{R}_+^H, \tilde{c}^{-h} \in \mathbb{R}_+^{H-1}} \frac{u^{h\prime}(c^h, c^{-h})}{u^{h\prime}(c^h, \tilde{c}^{-h})}. \tag{31}$$

Given a level of aggregate capital K_t, by the strict concavity of f, there is a unique pair (r_t, w_t) such that K_t maximizes one-period profits at rental rate r_t and wage w_t. Precisely, $r_t = f'(K_t)$ and $w_t = \frac{1}{H}\left(f(K_t) - f'(K_t)K_t\right)$. Each sequence $\{K_t\}_{t=0}^{\infty}$ generates a sequence $\{(r_t, w_t)\}_{t=0}^{\infty}$. Given $\mathbf{K} = \{K_t\}_{t=0}^{\infty}$, let $B^h(\mathbf{K})$ the set of consumption sequences $\{c_t^h\}_{t=0}^{\infty}$ such that there exists a sequence of investment $\{x_t^h\}_{t=0}^{\infty}$ satisfying $x_0^h = k^h$ and

$$c_t^h + x_{t+1}^h = r_t x_t^h + w_t l^h \text{ for all } h.$$

Our proof differs from the one of Becker et al. (1991) *only* at this stage. Define

$$B(\mathbf{K}) = \Pi_{h=1}^{H} B^h(\mathbf{K}),$$

the cartesian product of $\{B^h(\mathbf{K})\}_h$. Recall that $B(\mathbf{K})$ is a compact subset of a Fréchet space.[10] For each sequence $(\mathbf{c}^1, \mathbf{c}^2, \dots, \mathbf{c}^H) \in B(\mathbf{K})$, let $\{\hat{c}^h\}_{t=0}^{\infty}$ as the solution of the following problem:

$$\begin{aligned}
&\max \sum_{t=0}^{\infty} \delta_h^t u^h(\tilde{c}_t^h, c_t^{-h}) \\
&\text{s.c. } \tilde{c}_t^h + x_{t+1}^h = r_t x_t^h + w_t l^h, \\
&\tilde{c}_t^h, x_t^h \geq 0, \forall t, \\
&x_0^h = k^h.
\end{aligned}$$

The agent h maximizes her or his inter-temporal utility, taking $\{c_t^{-h}\}_{t=0}^{\infty}$ as given. Let $T\big(\mathbf{c}^1, \mathbf{c}^2, \dots, \mathbf{c}^H\big) = \big\{(\hat{c}_t^1, \hat{c}_t^2, \dots, \hat{c}_t^H)\big\}_{t=0}^{\infty}$. By the strict concavity of utility functions u^h, with $1 \leq h \leq H$, in respect to the product topology, the operator T is a continuous function from $B(\mathbf{K})$ to $B(\mathbf{K})$, which is compact. Hence, there exists a fixed point $(\mathbf{c}^{*1}, \mathbf{c}^{*2}, \dots, \mathbf{c}^{*H}) \in B(\mathbf{K})$ such that $T(\mathbf{c}^{*1}, \mathbf{c}^{*2}, \dots, \mathbf{c}^{*H}) = (\mathbf{c}^{*1}, \mathbf{c}^{*2}, \dots, \mathbf{c}^{*H})$.

[10] A Fréchet space is a locally convex metrizable topological vector space (TVS).

The sequence $(\mathbf{c}^{*1}, \mathbf{c}^{*2}, \ldots, \mathbf{c}^{*H})$ solves the following problem:

$$\begin{aligned}
&\max \sum_{t=0}^{\infty} \delta_h^t u^h(c_t^h, c_t^{*-h}) \\
&\text{s.c. } c_t^h + x_{t+1}^h = r_t x_t^h + w_t l^h, \\
&c_t^h, x_t^h \geq 0, \forall t, \\
&x_0^h = k^h.
\end{aligned}$$

This is the optimal consumption with externality, given capital sequence $\mathbf{K}$. Consider the fixed point of operator T and for each h, let the *capital supply function* $x^h(\mathbf{K})$ be defined as:

$$c_0^{*h} + x_1^h(\mathbf{K}) = r_0 k^h + w_0 l^h,$$
$$c_t^{*h} + x_{t+1}^h(\mathbf{K}) = r_t x_t^h(\mathbf{K}) + w_t l^h,$$

for every $t \geq 1$. Let $K_t(\mathbf{K}) = \sum_{h=1}^H x_t^h(\mathbf{K})$ and defined the tâtonnement map Φ such that

$$\Phi_t(\mathbf{K}) = \max\{\epsilon, K_t(\mathbf{K})\}.$$

This map has a fixed point $\overline{\mathbf{K}}$. Denote by $\mathbf{c}^h(\overline{\mathbf{K}})$ the corresponding consumption sequence of agent h and let $\mathbf{c}(\overline{\mathbf{K}}) = \sum_h \mathbf{c}^h(\overline{\mathbf{K}})$. We prove that for every t, $\overline{K}_t = K_t(\overline{\mathbf{K}})$. From now on, we follows exactly the same arguments as in the proof of Proposition 4.4 in Becker et al. (1991).

Assume that for some t_0, $K_{t_0}(\overline{\mathbf{K}}) = \epsilon$. We will prove that from some moment s_0, the sequence $\{K_t(\overline{\mathbf{K}})\}_{t=s_0}^{\infty}$ is decreasing, and that leads us to a contradiction. Indeed, fix s_0 the smallest time such that $\overline{K}_{s_0-1} > \epsilon = \overline{K}_{s_0}$. From the no-arbitrage inequality, we have

$$\frac{u^{h\prime}(c_{s_0-1}^h, c_{s_0-1}^{-h})}{u^{h\prime}(c_{s_0}^h, c_{s_0}^{-h})} \geq \delta_h f'(\epsilon).$$

Combining this inequality with (31), we have

$$\frac{u^{h\prime}(c_{s_0-1}^h, c_{s_0-1}^{-h})}{u^{h\prime}(c_{s_0}^h, c_{s_0}^{-h})} \geq \frac{u^{h\prime}(c_{s_0-1}^h, c_{s_0}^{-h})}{u^{h\prime}(c_{s_0}^h, c_{s_0}^{-h})}.$$

This implies $u^{h\prime}(c_{s_0}^h, c_{s_0}^{-h}) \geq u^{h\prime}(c_{s_0}^h, c_{s_0}^{-h})$, and $c_{s_0}^h \geq c_{s_0-1}^h$. Since this inequality is verified for every h, we have $c_{s_0}(\overline{\mathbf{K}}) \geq c_{s_0-1}(\overline{\mathbf{K}})$. Using same arguments as Becker et al. (1991), by induction, we verify that for every $t \geq s_0$, $c_t(\overline{\mathbf{K}}) \leq c_{t+1}(\overline{\mathbf{K}})$ and $K_t(\overline{\mathbf{K}}) \geq K_{t+1}(\overline{\mathbf{K}})$. The sequence $\{K_t(\mathbf{K})\}_{t=0}^{\infty}$ is decreasing, with a direct consequence that $\overline{K}_t = \epsilon$ for every $t \geq s_0$.

The same arguments as Becker et al. (1991) in page 454-455 lead us to a contradiction. Hence, $K_t(\mathbf{K}) > \epsilon$ for every $t \geq 0$. We obtain $\overline{K}_t = K_t(\overline{\mathbf{K}})$, for every $t \geq 0$. Let $\overline{r}_t = f'(\overline{K}_t)$, $\overline{w}_t = \frac{1}{H} \times (f(\overline{K}_t) - \overline{K}_t f'(\overline{K}_t))$, the sequence $(\mathbf{c}^1, \mathbf{c}^2, \ldots, \mathbf{c}^H)$ be the fixed point of operator T in $B(\overline{\mathbf{K}})$ and $x^h_{t+1} = \overline{r}_t x^h_t + \overline{w}_t l^h$, with $x^h_0 = k^h$. It is easy to verify that $\overline{K}_t = \sum_{h=1}^H x^h_t$ for every $t \geq 0$, and $\{\overline{r}_t, \overline{w}_t, \overline{K}_t, L_t, (c^h_t, x^h_t)_{h\in\mathcal{H}}\}_{t=0}^\infty$ is a Ramsey equilibrium.

5.2 Proof of Proposition 2.2

(i) First, consider Euler inequalities. For $h = 1$:

$$\delta_1 r_{t+1} u^{h\prime}(c^1_{t+1}, c^{-1}_{t+1}) \leq u^{1\prime}(c^1_t, c^{-1}_t).$$

Since $\{c^1_t\}_{t=0}^\infty$ is bounded from above, with a direct consequence that $\inf_{t\geq 0} u^{1\prime}(c^1_t, c^{-1}_t) > 0$, we have:

$$\begin{aligned}(\delta_1)^T \Pi_{t=0}^{T-1} r_{t+1} &\leq \Pi_{t=0}^{T-1} \frac{u^{1\prime}(c^1_t, c^{-1}_t)}{u^{1\prime}(c^1_{t+1}, c^{-1}_{t+1})} \\ &= \frac{u^{1\prime}(c^1_0, c^{-1}_0)}{u^{1\prime}(c^1_T, c^{-1}_T)} \\ &< \infty.\end{aligned}$$

Hence

$$\limsup_{T\to\infty} \left((\delta_1)^T \times \Pi_{t=0}^{T-1} r_{t+1}\right) < \infty.$$

A consequence of this inequality is that

$$\liminf_{t\to\infty} \delta_1 r_{t+1} \leq 1.$$

This inequality implies

$$\limsup_{t\to\infty} k_t \geq \overline{K}.$$

(ii) Assume the existence of t such that $K_t < \min\{K_0, \underline{K}\}$. Since $K_0 \geq \min\{K_0, \underline{K}\}$, there exists some T such that $K_T \geq K_{T+1}$ and $K_{T+1} \leq \min\{K_0, \underline{K}\}$. For every h, by the very definition of $\underline{K}$, we obtain the following inequality:

$$\begin{aligned}\delta_h f'(K_{T+1}) &\geq \delta_h f'(\underline{K}) \\ &\geq \delta_H f'(\underline{K}) \\ &\geq \frac{u^{h\prime}(c^h_T, c^{-h}_T)}{u^{h\prime}(c^h_T, c^{-h}_{T+1})}.\end{aligned}$$

Hence, for every h,

$$
\begin{aligned}
u^{h\prime}(c^h_{T+1}, c^{-h}_{T+1}) &\leq \frac{1}{\delta_h f'(K_{T+1})} \times u^{h\prime}(c^h_T, c^{-h}_T) \\
&\leq \frac{1}{\frac{u^{h\prime}(c^h_T, c^{-h}_T)}{u^{h\prime}(c^h_T, c^{-h}_{T+1})}} \times u^{h\prime}(c^h_T, c^{-h}_T) \\
&= u^{h\prime}(c^h_T, c^{-h}_{T+1}).
\end{aligned}
$$

This implies $c^h_{T+1} \geq c^h_T$, for every h. Combining this with $K_{T+1} < K_T$, we have $K_{T+2} < K_{T+1}$. By induction, we can prove that the sequence $\{K_{T+t}\}_{t=0}^{\infty}$ is decreasing, and $\lim_{t\to\infty} k_t \leq \underline{K} < \overline{K}$, in contradiction with (i).

(iii) This is a direct consequence of (i). Indeed, since $K_t \geq \min\{K_0, \underline{K}\}$ for every t, the sequence of wage is bounded from below: $\inf_t w_t > 0$. Hence, for every h, $\limsup_{t\to\infty} c^h_t > 0$.

(iv) Assume that for some $h \geq 2$, there exists T_0 such that $x^h_t > 0$ for every $t \geq T_0$. Then the Euler equation is satisfied:

$$
\delta_h r_{t+1} u^{h\prime}(c^h_{t+1}, c^{-h}_{t+1}) = u^{h\prime}(c^h_t, c^{-h}_t).
$$

This implies

$$
\begin{aligned}
(\delta_h)^T \Pi_{t=0}^{T-1} r_{t+1} &= \Pi_{t=0}^{T-1} \frac{u^{h\prime}(c^h_t, c^{-h}_t)}{u^{h\prime}(c^h_{t+1}, c^{-h}_{t+1})} \\
&= \frac{u^{h\prime}(c^h_0, c^{-h}_0)}{u^{h\prime}(c^1_T, c^{-1}_T)}.
\end{aligned}
$$

Recall that

$$
\limsup_{T\to\infty} (\delta_1)^T \Pi_{t=0}^{T-1} r_{t+1} < \infty.
$$

Since $\delta_h < \delta_1$, we have

$$
\begin{aligned}
\lim_{T\to\infty} \frac{u^{h\prime}(c^h_0, c^{-h}_0)}{u^{h\prime}(c^1_T, c^{-1}_T)} &= \lim_{T\to\infty} \left[\left(\frac{\delta_h}{\delta_1}\right)^T \times (\delta_1)^T \Pi_{t=0}^{T-1} r_{t+1} \right] \\
&= 0.
\end{aligned}
$$

This implies

$$
\lim_{T\to\infty} u^{h\prime}(c^1_T, c^{-1}_T) = \infty,
$$

which is equivalent to $\lim_{T\to\infty} c^h_T = 0$, a contradiction with (iii).

5.3 Proof of Lemma 3.1

The compactness of $\Pi(k^1)$ is clear, combining with the strict concavity of u^1 in respect to the first argument, this implies the existence and the unicity of $\hat{\mathbf{x}}^1$ as well as the continuity of function T^*. Hence, the existence of a fixed point is garuanted.

5.4 Proof of Lemma 3.2

Using directly the definition in Section 2.2. The claim straightforwardly follows from the marginal conditions of the agents maximization problems P^f and P^h. The statement about prices merely arises from the market clearing conditions $x_t^{1\star} = K_t^\star$ and $\ell := \sum_{h\in H} \ell^h = L_t^\star$.

5.5 Proof of Proposition 3.1

This is a direct consequence of the *myopic* condition and equations (16) and (17).

5.6 Proof of Proposition 3.2

The proof makes use of the linear analysis of the dynamics near the fixed point $(\overline{x}, \overline{c})$; it therefore requires $\mathcal{E}$ to be such that the absolute value $|k^1 - \overline{x}|$ is small enough. The differentiation of the system (23)-(24), evaluated at $(\overline{x}, \overline{c})$, yields:

$$\begin{aligned} dx_{t+1} &= -dc_t + g'(\bar{x})x_t, \\ -\eta dc_{t+1} + (\delta_1 r'(\overline{x})\theta + w'(\overline{x})\ell\varepsilon)dx_{t+1} &= -\eta dc_t + w'(\overline{x})\ell\varepsilon dx_t. \end{aligned}$$

The Jacobian matrix is determined as:

$$J = \frac{1}{\eta}\begin{pmatrix} \eta g'(\overline{x}) & -\eta \\ (1-g'(\overline{x}))\, w'(\overline{x})\varepsilon - g'(\overline{x})\delta r'(\overline{x})\theta & \eta + \delta r'(\overline{x})\theta + w'(\overline{x})\varepsilon \end{pmatrix},$$

where

$$\begin{aligned} \theta &:= \frac{\partial u^1}{\partial c}\left(\overline{c}, ((w(\overline{x})\ell^h)_{h\in\mathcal{H}^\imath}\right), \\ \eta &:= -\frac{\partial^2 u^1}{\partial c \partial c}\left(\overline{c}, ((w(\overline{x})\ell^h)_{h\in\mathcal{H}^\imath}\right), \\ \varepsilon &:= \sum_{h\in\mathcal{H}^\imath} \frac{\partial^2 u^1}{\partial c \partial c^h}\left(\overline{c}, w(\overline{x})\ell^h\right)\frac{\ell^h}{\ell}. \end{aligned}$$

The characteristic polynomial for the Jacobian matrix J is $\lambda^2 - T\lambda + D = 0$, where

$$\begin{aligned} T &= 1 + g'(\overline{x}) - \frac{1}{\eta}(\delta_1 r'(\overline{x})\theta + w'(\overline{x})\ell\varepsilon), \\ D &= g'(\overline{x}) - \frac{1}{\eta}w'(\overline{x})\ell\varepsilon. \end{aligned}$$

Now, choose the primitives of an economy $\tilde{\mathcal{E}}$ in such a way that

$$2\eta\big(1 + g'(\overline{x})\big) = \delta_1 r'(\overline{x})\theta + 2w'(\overline{x})\ell\varepsilon. \tag{32}$$

Then, one of the eigenvalues of J is equal to -1. The stationary equilibrium of $\tilde{\mathcal{E}}$ is not hyperbolic. The flip bifurcation theorem (see, e.g., Ruelle (1989), Theorem 12.1) ensures the generic existence of an economy $\mathcal{E}$ in a suitable neighborhood of $\tilde{\mathcal{E}}$ which possesses a Ramsey equilibrium exhibiting a cycle of period two.

5.7 Proof of Corollary 3.1

Note that $\varepsilon > \underline{\varepsilon}$ merely means that the right-hand-side in (32) is strictly positive. Now, choose the primitives in such a way that $g'(\overline{x}) > 0$, and at the same time (32) holds true. The claim follows from Proposition 3.2.

5.8 Proof of Proposition 3.3

We focus on the set of economies that satisfy $w'(\overline{x})\ell > 1 + g'(\overline{x})$.

Choose the primitives such that $g'(\bar{x}) > 0$ and $\varepsilon > \underline{\varepsilon}$. Rewrite the steady state non-hyperbolicity condition (32) as follows:

$$\eta = \frac{\delta r'(\bar{x})\theta + w'(\bar{x})\ell\varepsilon}{2\big(1 + g'(\bar{x})\big)}.$$

Observe that

$$\eta - \varepsilon = \frac{\delta r'(\bar{x})\theta}{2(1 + g'(\bar{x}))} + \left(\frac{w'(\bar{x})\ell}{1 + g'(\bar{x})} - 1\right)\varepsilon. \tag{33}$$

Since $w'(\overline{x})\ell > 1 + g'(\overline{x})$, there exists $\overline{\varepsilon} > \underline{\varepsilon}$ such that if $\epsilon > \overline{\varepsilon}$, we obtain $\eta > \varepsilon$. The claim lastly follows from the flip bifurcation theorem in Ruelle (1989).

5.9 Proof of Proposition 3.4

Lower bound: $\lambda > \frac{\sigma}{1-s}(1+\delta_1)-1 =: \underline{\lambda}$ means that $w'\ell > 1+g'$, thus implies the existence of $\varepsilon > \underline{\varepsilon}$ such that $\eta > \varepsilon$.

Upper bound: $\lambda < \frac{\sigma}{1-s} =: \bar{\lambda}$ merely entails that $g' > 0$. The restrictions upon σ ensures that $(\underline{\lambda}, \bar{\lambda}) \subseteq [0,1]$.

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