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On the Convergence Criterion in Three-Period-Lived Overlapping Generations Models *

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Abstract

This article considers a three-period-lived pure exchange overlapping generations economy and clarifies the role of market complementarities in the scope for Kehoe & Levine's convergence criterion in order to reduce the size of continuation equilibria and establish uniqueness and determinacy. The argument is based upon the price-relatedness of dated goods and the way the law of demand, gross substitutability and the scope for asymmetric complementarities come into play when three periods lifespans are considered. The nature of these restrictions is clarified in the context of stationary economies and the way it relates to equilibrium continuation and Kehoe & Levine's determinacy is made precise. A detailed articulation between complementarities and determinacy is finally provided in the context of Samuelson intermediate economies. The key role of dated goods spaced one period apart and entering in an additive way is emphasised in the determinacy result while the importance of asymetric complementarities between isolated goods spaced two periods apart is also pointed out.

KEYWORDS: continuation equilibria and Kehoe & Levine's convergence criterion, determinacy, indeterminacies, robustness, overlapping generations, gross substitutability, complementarity, aggregate demand, comparative statics.

Contents

1. Introduction

The consideration of economies with overlapping generations instead of the more standard economies with finitely many individuals[1] results in the possible failure of the determinacy of competitive equilibrium allocations[2]. Indeterminacy is however negatively perceived in involving a specification of the fundamental data that does not any longer

[1] See, *e.g.*, Kehoe & Levine [1985] and Kehoe, Levine & Romer [1990].

[2] Geanokoplos [2008] worthily surveys the properties of competitive equilibria in overlapping generations economies.

suffice to uniquely determine the observed outcome[3]. David Levine, in collaboration with various co-authors and noticeably Timothy Kehoe, has made substantial contributions in this area[4]. In stationary environments, Kehoe & Levine [1985] suggested a criterion that serves to select one equilibrium among a continuum, thereby allowing economic predictions and comparative statics analysis. Specifically, having assumed that the economy has attained a steady state and following a small unanticipated perturbation, the requirements for determinacy sum up to a set of acceptable equilibrium *continuations* that is a singleton, acceptable continuations being those converging back to the nearby steady state.

Kehoe & Levine [1985] have restricted their attention to two-period lives because, as suggested by Balasko, Cass & Shell [1980], any endowment economy with finitely-lived individuals trading finitely many commodities can be converted into an equivalent economy with two-period lifetimes (see also Kehoe & Levine [1984a, sect. 11]). Yet, in one commodity frameworks longer lifetimes must be directly faced with[5]. Kehoe & Levine [1990] have examined an economy in which the only departure from Gale [1973]'s framework is that live spans three, rather than two, periods. The authors provided parameterized examples in which there are indeterminate equilibria with and without nominal debt (fiat money), simultaneously drawing the reader's attention to the fact that the examples in question violate the gross substitutes property. As a matter of fact, the examples provided build on some goods in different periods being gross complements for some price ranges. However, the complementarities in play are neither emphasized nor discussed.The current paper is aimed to shed some light on the relation between gross complementarity among goods and (in)determinacy of equilibria in the Kehoe & Levine [1990]'s framework.

The setting is first detailed and the market demand is characterised. More specifically, a precise account of the workings of the gross substitutability assumption, the law of demand but also of the various ranges of complementarities — asymmetric vs symmetric — is first presented. An equilibrium is then carefully analysed, where the various types — intermediated or balanced — of stationary equilibria are discussed, together with their Samuelson or balanced dimension, etc. Their determinacy properties are related with the complementarity facets of the market demand function.

Some important conclusions list as follows: it is first established that the gross substitutability property, while being sufficient, is not necessary for determinacy in Samuelson economies. More specifically, a necessary condition for determinacy is established when the dependencies of the aggregate demand with respect the one period forward price and the one period backward price result in a positive sum. Interestingly, these indeed enter in an additive way, meaning that complementarities between goods spaced one period apart are allowed as long as they end up being overcome by substitutabilities between such goods. That necessary condition becomes sufficient when some gross substitutability of dated goods spaced two periods forward is satisfied, *i.e.*, more specifically when the aggregate demand depends positively on the two periods forward price, the gross substitutability between dated goods spaced two periods backward, *i.e.*, the sensitivity of the aggregate demand on the two periods backward price, being oppositely not involved in the determinacy result. Some more complex configurations are also put into evidence when this latter condition is not satisfied and some complementarities emerge with an aggregate demand that negatively depend upon the two periods forward price. A final complex determinacy configuration is, *e.g.*, available that involves the comparison between expressions

[3]By its very troublesome nature, indeterminacy has attracted a lot of attention and given rise to a huge literature: Burke [2009], Kim & Spear [2021], Burke [2022] or Gorokhosky & Rubinchik [2022] are some recent worthy examples.

[4]Kehoe & Levine [1984a], Kehoe & Levine [1985], Kehoe, Levine, Mas-Colell & Woodford [1991].

[5]With one composite commodity available and a single individual born each period, a framework usual in applied macroeconomics studies, the Balasko, Cass & Shell [1980] trick can not be used as the reduction of the lifespan goes with an increases of the number of commodities available and (heterogeneous) individuals born each period.

where the gross substitutability or complementarity properties of goods spaced one period apart sum up with the ones of goods spaced two periods apart.

The scope for local indeterminacies is also detailed. Unsurprisingly, the most canonical configuration builds upon the invalidation of the above mentioned necessary condition for determinacy, *i.e.*, when the sum of the dependencies of the aggregate demand with respect to the one period forward price and the one period backward price becomes negative. Otherwise stated, local indeterminacies emerge when complementarities between goods spaced one period apart overcome substitutabilities between such goods. Two main configurations are pointed out: a first that involves larger orders for the gross substitutability / complementarity properties between forward dated goods spaced two periods apart than for backward dated goods spaced two periods apart and a second that considers an opposite occurrence for these two periods apart dated goods but eventually rests upon the complex comparison between the expressions where the gross substitutability or complementarity properties of goods spaced one period apart sum up with the ones of goods spaced two periods apart. Interestingly, and in line with the Kehoe and Levine's example of a two-dimensional indeterminacy with valued outside money, another class of *larger dimension* indeterminacy conclusions keeps on being available when the sum of the dependencies of the aggregate demand with respect the one period forward price and the one period backward price is positive — subtitutabilities between goods spaced one period apart overcome complementarities between such goods — but unequivocally comes together a gross complementarity property for forward goods spaced two periods apart and also rests expressions where the gross substitutability or complementarity properties of goods spaced one period apart sum up with the ones of goods spaced two periods apart.

The rest of the article is organised as follows: the framework together with the gross substitutability and gross complementarity facets of market demand are analysed in Section 2 while equilibria and their determinacy properties are considered in Section 3 and some comments are gathered in a concluding section. The formal proofs are given in the appendix together with the sketch of an alternative approach based upon aggregate savings functions and gross rates of interests.

2. Market Demand & Gross Complementarities

2.1 The Framework

Consider a simple exchange overlapping generations economy that expands over an infinite number of discrete periods denoted by t; time has neither beginning nor end, it extends infinitely into the future as well as into the past[6]. In each period a new individual is born; the one born in period t will be referred to variously as the agent t or generation t. Individuals' lives span any *three* consecutive periods. The demography thus entails three living individuals in every period.

There is a single completely perishable consumption commodity. Agents are endowed with a strictly positive amount of that consumption commodity in each period of her life. Let $e^t = (e^t_t, e^t_{t+1}, e^t_{t+2}) \in \mathbb{R}^3_{++}$ be the lifetime endowment of agent t. In parallel to this, let $x^t = (x^t_t, x^t_{t+1}, x^t_{t+2}) \in \mathbb{R}^3_{++}$ denotes her consumption plan. Individuals derive utility from consuming goods during their lifetime; utility functions are assumed to be twice continuously differentiable, strictly quasi-concave, and strictly monotonic.

Two interpretations of the market structure are used in the literature. One in which all markets are open simultaneously at a moment *out of time*, and in which, irrespective of the period of their birth, all agents participate in an exchange of contracts for the dated

[6]The economy has no inception date, strictly speaking. The time period $t = 0$ will be considered when undertaking a comparative statics thought experiment.

delivery of commodities; p_t being then the present value price of the commodity delivered in period t. In the alternative interpretation markets meet sequentially, according to their dates, and agents freely borrow or lend; the term of exchange between goods t and $t+1$ being then the gross interest rate on savings between periods t and $t+1$. Following Kehoe & Levine, the former interpretation is retained as it is more appropriate for addressing substitutability in demand[7].

Given the price system $p = (\cdots, p_{t-1}, p_t, p_{t+1} \cdots)$, agent t maximizes her lifetime utility under the budget constraint:

$$p_t x_t^t + p_{t+1} x_{t+1}^t + p_{t+2} x_{t+2}^t \leq p_t e_t^t + p_{t+1} e_{t+1}^t + p_{t+2} e_{t+2}^t. \tag{2.1}$$

The solving of this problem leads to a demand function $x^t : \mathbb{R}_{++}^3 \times \mathbb{R}_{++} \to \mathbb{R}_{++}^3$ with a domain $(p_t, p_{t+1}, p_{t+2}, p_t e_t^t + p_{t+1} e_{t+1}^t + p_{t+2} e_{t+2}^t)$. The individual demand functions are fully characterized by homogeneity of degree zero in prices (and wealth), Walras's law, and Slutsky conditions. With regard to the latter in particular, the emphasis being thereafter put on *gross* substitutability rather than on the Slutsky decomposition per se, the demand dependance on wealth will be left implicit, and demand regarded as a function of prices only. The demand of generation t as a function of prices is here a *smooth* map $x^t : \mathbb{R}_{++}^3 \to \mathbb{R}_{++}^3$ with domain (p_t, p_{t+1}, p_{t+2}).

2.2 Price-Relatedness, Gross Substituability & Complementarity

Prices being determined in competitive markets, the emphasis is put on market, *i.e.*, aggregate, demand. At each period, the market demand adds up the demands of the living generations. Its domain is thus the union of those of the proper individual demand functions. At time t, agents alive are generations $t-2$, $t-1$, and t, the market demand function being a map $X_t : \mathbb{R}_{++}^5 \to \mathbb{R}_{++}$ with a domain $(p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2})$ that is defined by

$$\begin{aligned} X_t(p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2}) = {} & x_t^{t-2}(p_{t-2}, p_{t-1}, p_t) \\ & + x_t^{t-1}(p_{t-1}, p_t, p_{t+1}) + x_t^t(p_t, p_{t+1}, p_{t+2}). \end{aligned} \tag{2.2}$$

The market demand functions X_t and X_s, $s \neq t$, are *price-related* on the condition that the intersection of their domains is non-empty. In such a case, goods t and s are said to be price-related at the level of the economy.

The structure of generational overlap, by which the individuals are linked, gives rise to price-relatedness between market demands that do not exist at the individual level. As the individuals' life span is finite, and event though there is an infinite number of goods, the span of such linkage is also finite. For instance, good t is only price-related with goods $s \in \{t-2, t-1, t+1, t+2\}$. The price-relatedness of goods t and s will be denoted (t, s) and described in terms of gross substitutability. The market demand function X_t satisfies the gross substitutability property at prices $(p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2})$ if:

$$\frac{\partial X_t}{\partial p_s}(p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2}) > 0 \quad \text{for all} \quad s \in \{t-2, t-1, t+1, t+2\}$$

Whenever $\partial X_t / \partial p_s > 0$ holds, it will be said that good t is a gross substitute for good s at prices $(p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2})$. When, symmetrically, at those prices, good s is a gross substitute for good t, *i.e.*, $\partial X_s / \partial p_t > 0$, goods t and s are said to be gross substitutes at prices $(p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2})$. As neither the assumptions made on preferences nor the maximizing principle guarantees such a symmetry, good t may well be a gross substitute

[7]For completeness, an articulation with a gross interest rates and aggregate savings is nonetheless sketched in Section 3.2, the detailed formal argument being available in a companion work available as ?.

for good s when, at the same time, good s is a gross *complement* for good t. Lastly, if the *own* price effect is negative at price $(p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2})$, that is if

$$\frac{\partial X_t}{\partial p_t}(p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2}) < 0,$$

the demand function X_t satisfies the Law of Demand.

The market demand functions being homogenous of degree zero,

$$-\frac{\partial X_t}{\partial p_t} p_t = \sum_s \frac{\partial X_t}{\partial p_s} p_s.$$

Thus, if X_t exhibits gross substitutability, then it satisfies the law of demand. The converse implication is obviously not true.

Now, take t as a period of reference and consider market demand functions which are price-related with X_t; these are X_s, $s \in \{t-2, t-1, t+1, t+2\}$. The domain of X_t makes immediately clear that the pairs of dated goods $(t-2, t)$, $(t-1, t)$, $(t, t+1)$, and $(t, t+2)$ are price-related[8]. In terms of gross substitutability, (t, s) is summarized by the pair of price derivatives $\left(\partial X_t / \partial p_s, \partial X_s / \partial p_t\right)$, each one being evaluated at relevant prices, so that (t, s) is equivalent to (s, t). Considering a price system $p = (\cdots, p_{t-1}, p_t, p_{t+1}, \cdots)$, it becomes possible to ascertain the set of gross substitutes of good t and to check upon the validity of the law of demand at p.

The previous considerations will actually be applied to *stationary* economies, *i.e.*, economies in which individuals are identical up to the calendar time of their birth. In such circumstances the demand functions are time invariant, $X_t = X$ for all t. Of course, stationarity does not imply that $\cdots, (t-1, t), (t, t+1), (t+1, t+2), \cdots$ are equivalent in terms of substitutability. Otherwise stated, the fact that goods $t-1$ and t are gross substitutes does not indicates that the same holds true for immediately subsequent goods t and $t+1$. The reason is that the time invariance of the demand functions is not synonymous with time invariance of price derivatives at a point. Such stationary-like property will yet happen along any sequence of prices such that $p_{t+1} = \delta p_t$. This merely derives from the fact that the derivatives of the demand functions are homogeneous of degree minus one. Thus, in any stationary economy, all the relevant effects of price changes, while evaluated at a price system such that $p_{t+1} = \delta p_t$, are *fully* described by the 5-uple of price derivatives (the indexes are detailed for ease of reading)

$$\frac{\partial X_t}{\partial p_{t-2}}, \frac{\partial X_t}{\partial p_{t-1}}, \frac{\partial X_t}{\partial p_t}, \frac{\partial X_t}{\partial p_{t+1}}, \frac{\partial X_t}{\partial p_{t+2}}.$$

In terms of gross substitutability/law of demand properties, $\partial X_t / \partial p_t < 0$ indicates that market demand verifies the law of demand. In such a case, $\partial X_t / \partial p_{t-1} > 0$ (< 0) means that any good is a gross substitute (gross complement) for the good of the *prior* period, whereas $\partial X_{t-1} / \partial p_t > 0$ (< 0) means that any good is a gross substitute (gross complement) for the good of the *next* period. Provided that both derivatives have the same sign, goods spaced by *one* period apart are either gross substitutes or gross complements at p[9].

Remark 2.1. In a two-periods lifespan overlapping generations economy, the market demand of each dated good[10] would depend upon three neighbours prices and each price appears as an argument in three consecutive market demand functions. For instance, and for

[8]Observe that it also shows that the pairs of goods $(t-2, t-1)$, $(t-1, t+1)$, and $(t+1, t+2)$ are also price-related. The latter price-relatedness are preferably associated with periods of reference $t-1$ and $t+1$, respectively.

[9]The interpretation, of course, goes for goods spaced by *two* periods apart, $\cdots, (t-2, t), (t-1, t+1), (t, t+2), \cdots$.

[10]Except for $t = 0$.

p_t, in $X_{t-1}(p_{t-2},p_{t-1},p_t)$, $X_t(p_{t-1},p_t,p_{t+1})$ and $X_{t+1}(p_t,p_{t+1},p_{t+2})$. Otherwise stated, taking t as a period of reference and considering market demand functions which are price-related with X_t, these are X_s, $s \in \{t-1, t+1\}$ The aggregate law of demand (LOD) is *fulfilled* if $\partial X_t/\partial p_t < 0$; *e.g.*, dated goods t and $(t-1)$ are gross substitutes if both $\partial X_{t-1}/\partial p_t > 0$ and $\partial X_t/\partial p_{t-1} > 0$.

Remark 2.2. In order to ease the comparison, the current aggregate demand approach mimics Kehoe & Levine [1985]'s presentation in terms of present-value prices. While the market structure has hitherto been understood as if all markets were simultaneouly open before the beginning of time, in an alternative interpretation, markets meet sequentially according to their dates so that agents may freely save or borrow while facing a lifetime budget constraint, the term of exchange between goods t and $t+1$ being interpreted as the one-period interest factor. Letting $R_{t+1} = p_t/p_{t+1}$ denote the (gross) interest rate from period t to period $t+1$, a more complex alternative could have then proceeded from the consideration of aggregate savings functions and gross interest rates. Appendix A sketches a quick articulation between this alternative and the current approach[11]: the aggregate demand relates to aggregate savings as a function of gross interest rates through a relationship $X_t(p_{t-2},p_{t-1},p_t,p_{t+1},p_{t+2}) = R_t S_{t-1}(R_{t-1},R_t,R_{t+1}) - S_t(R_t,R_{t+1},R_{t+2}) + \mathscr{E}_t$ for $\mathscr{E}_t$ the total endowment of the three living generations at date t and $S_t = s_t^{t-1} + s_t^t$ the aggregate savings function at date t, the life-cycle savings patterns of generation t being described by the pair (s_t^t, s_{t+1}^t) .

3. Equilibrium, Complementarities & Determinacy

3.1 Equilibrium Definition and Kehoe & Levine's Determinacy

3.1.1 The Set of Equilibria

This section explores the properties of the set of equilibria. An equilibrium is a sequence of positive prices $(\cdots, p_{t-1}, p_t, p_{t+1}, \cdots)$ that solves, for every t, the sequence of market clearing equations

$$X(p_{t-2},p_{t-1},p_t,p_{t+1},p_{t+2}) = e_1 + e_2 + e_3, \tag{3.1}$$

where e_i is the individuals' endowment in their i-th period of life. As X is homogeneous of degree zero, an equilibrium is stationary if prices grow steadily, *i.e.*, $p_{t+1} = \delta p_t$, $\delta > 0$, a price $p_t = \delta^t$ being called a steady state equilibrium. In order to characterise the steady states equilibria a disaggregated standpoint is however required. First letting $x_i(\cdot) - e_i$ denote the excess demand function of agents in their i-th period of life, remark that the sequence of market clearing equations (3.1) reformulates to:

$$x_3\big(p_{t-2},p_{t-1},p_t\big) - e_3 + x_2\big(p_{t-1},p_t,p_{t+1}\big) - e_2 + x_1\big(p_t,p_{t+1},p_{t+2}\big) - e_1 = 0, \quad \text{for all } t,$$

but transposes to $x_1(1,\delta,\delta^2) - e_1 + x_2(1,\delta,\delta^2) - e_2 + x_3(1,\delta,\delta^2) - e_3 = 0$ along a steady state equilibrium. Further letting $x_i := x_i(1,\delta,\delta^2)$:

$$(x_1 - e_1) + (x_2 - e_2) + (x_3 - e_3) = 0, \tag{3.2a}$$
$$(x_1 - e_1) + \delta(x_2 - e_2) + \delta^2(x_3 - e_3) = 0, \tag{3.2b}$$

[11] A detailed and extensive presentation is available in Barinci, Cho & Drugeon [2024] in the context of productive economies.

where (3.2b) refers to the Walras's law. It is finally obtained, solving between (3.2a) and (3.2b), that:

$$(1-\delta)[\delta(x_3-e_3)-(x_1-e_1)]=0, \tag{3.3}$$

where $\delta(x_3-e_3)-(x_1-e_1)$ is nothing more but the aggregate savings[12] or debt in each period at the steady state. There are two types of steady states: the *balanced* steady states in which $\delta \neq 1$ and $\delta(x_3-e_3)=x_1-e_1$, and the *intermediated* steady state in which $\delta=1$ and $x_3-e_3 \neq x_1-e_1$[13]. The existence of the latter indeed supposes the presence of some social contrivance, as for instance outside money (hence monetary) or an infinite-lived financial intermediary (hence intermediated), that expand the range of reliable exchanges between generations. Furthermore, making use of the terminology coined by Gale [1973], an economy will be referred as *Samuelson* whenever aggregate savings are positive, *i.e.*, $x_3-e_3>x_1-e_1$, and as *classical* whenever they are negative, *i.e.*, $x_3-e_3<x_1-e_1$.

3.1.2 Equilibrium Continuation, the Convergence Criterion & Kehoe & Levine's Determinacy

Consider now the Kehoe & Levine [1985, p. 443] comparative statics thought experiment: suppose that the economy attained a steady state equilibrium and that a small unanticipated shock occurs at time $t=0$. That shock can change the endowments (life-cycle earnings), the preferences or merely agents' expectations. After that perturbation individuals maximize and trade according to their revised expectations, and consequently the economy will follow a new equilibrium path from period 0 onward. Call the latter an equilibrium *continuation.* Specifically, an equilibrium continuation of the steady state path $(\cdots,\delta^{-2},\delta^{-1})$ is a sequence of prices beginning at 0, $(p_0,p_1,\cdots)$, and such that markets from time 0 onward clear. What about the set of such equilibrium continuations?

In order to save on notations as it does neither change endowments nor preferences, consider a sunspot shock. The aggregate demands at 0 and 1 differ from the general form since prices differ from what generations -2 and -1 had anticipated. Let m^{-2} and m^{-1} be their respective money savings carried out from time -1 ($m^i<0$ is a nominal debt). The demand function of agent -2, for whom 0 is the last period of life, is merely:

$$x_0^{-2}(p_0,m^{-2})=e_3+\frac{m^{-2}}{p_0}.$$

The twosome demand function of agent -1, $x^{-1}(p_0,p_1;m^{-1})$, is the solution of maximizing, for fixed past consumption, $u(\cdot,x_0^{-1},x_1^{-1})$ subject to $p_0x_0^{-1}+p_1x_1^{-1}=p_0e_2+p_1e_3+m^{-1}$. Thus, making use of the subscript to mark the idiosyncrasy of the market demand at dates $t=0$ and $t=1$ (from date 2 on the aggregate functions are given by X):

$$\begin{aligned}
X_0(p_0,p_1,p_2)&=x_0^{-2}(p_0;m^{-2})+x_0^{-1}(p_0,p_1;m^{-1})+x_0^0(p_0,p_1,p_2)\\
X_1(p_0,p_1,p_2,p_3)&=x_1^{-1}(p_0,p_1;m^{-1})+x_1^0(p_0,p_1,p_2)+x_1^1(p_1,p_2,p_3).
\end{aligned}$$

An *equilibrium continuation* is a sequence of positive prices such that

$$X_0(p_0,p_1,p_2)=e_1+e_2+e_3,\quad t=0, \tag{3.4a}$$
$$X_1(p_0,p_1,p_2,p_3)=e_1+e_2+e_3,\quad t=1, \tag{3.4b}$$
$$X(p_{t-2},p_{t-1},p_t,p_{t+1},p_{t+2})=e_1+e_2+e_3,\quad t\geq 2. \tag{3.4c}$$

[12] With the aggregate savings / gross interest rates approach of appendix A that is also mentioned in remark 2.2, this is indeed equivalent to $(1-R)S(R,R,R,R)$, for S the stationary aggregate savings function and R the stationary value of the rate of interest.

[13] Kehoe & Levine refer to balanced steady states as real and to the intermediated ones as monetary.

The equilibrium continuation which will be achieved is indeterminate in the true sense of the word. It is however determined in the following sense: once (p_0,p_1) is given, the future evolution of prices, *i.e.*, $(p_2,p_3,p_4,\ldots)$, is determinate by equations (3.4a)-(3.4c). While there is undoubtedly a huge multiplicity of equilibrium continuations, Kehoe & Levine [1985] suggested a *convergence criterion* to reduce the size of the equilibrium continuations set. On the basis of that criterion, the relevant set consists of those equilibria *reverting back* to a steady state, with the understanding that the initial conditions are sufficiently close to the steady state under concern. Whenever that criterion produces an unique equilibrium continuation, the latter is said to be locally determinate in the Kehoe and Levine sense.

To proceed, let

$$X'_i := \frac{\partial X}{\partial p_{t+i}}(1,\delta,\delta^2,\delta^3,\delta^4), \qquad i \in \{-2,-1,0,1,2\}.$$

Under the regularity assumption $X'_2 \neq 0$, the implicit function theorem ensures that the equilibrium paths near steady states follow the fourth-order difference equation implicitly defined by (3.4c). Making use of the homogeneity of degree minus one of the market demand derivatives, the first-order approximations of the market clearing equations can be written as

$$X'_{-2}q_{t-2} + \delta X'_{-1}q_{t-1} + \delta^2 X'_0 q_t + \delta^3 X'_1 q_{t+1} + \delta^4 X'_2 q_{t+2} = 0, \tag{3.5}$$

where $q_s := p_s/\delta^s - 1$. The equilibrium dynamics near a steady state are governed by the roots of the characteristic equation:

$$X'_{-2} + \delta X'_{-1}\lambda + \delta^2 X'_0\lambda^2 + \delta^3 X'_1\lambda^3 + \delta^4 X'_2\lambda^4 = 0. \tag{3.6}$$

Yet, depending on the type of the involved steady state, economic theory restricts either one of two characteristic roots. As a matter of fact, the homogeneity of degree zero of X, which implies:

$$X'_{-2} + \delta X'_{-1} + \delta^2 X'_0 + \delta^3 X'_1 + \delta^4 X'_2 = 0, \tag{3.7}$$

entails that $\lambda = 1$ is a characteristic root for the intermediated steady state. In addition, if $\delta \neq 1$, Walras's law, which implies

$$X'_{-2} + X'_{-1} + X'_0 + X'_1 + X'_2 = 0, \tag{3.8}$$

reveals that $\lambda = 1/\delta$ is another characteristic root for the balanced steady state.

REMARK 3.1. In a two-period-lived framework, date t market demand function has domain (p_{t-1},p_t,p_{t+1}) with $X(p_{t-1},p_t,p_{t+1}) = x_t^{t-1}(p_{t-1},p_t) + x_t^t(p_t,p_{t+1})$. The pendant of the linearized equation (3.5) and its associated characteristic polynomial (3.6) would respectively be available as $X'_{-1}q_{t-1} + \delta X'_0 q_t + \delta^2 X'_1 q_{t+1} = 0$ and $X'_{-1}\lambda + \delta X'_0\lambda^2 + \delta^2 X'_1\lambda^2 = 0$.

The subsequent analysis is aimed at clarifying the role of gross complementarity in the equilibrium indeterminacy examplified by Kehoe & Levine [1990] and will specialise to intermediated economies[14] that admittedly and since Samuelson [1958] provide the most specific account of overlapping generations economies. rather than two periods, the conceivable market demand complementarities between goods are enlarged.

[14]For completeness, the case of balanced equilibria is also briefly discussed in appendix C.

3.2 The Intermediated Steady State for Gross Substitutes Economies

Consider the intermediated steady state $\delta = 1$, $x_3 - e_3 \neq x_1 - e_1$ and assume that generations -2 and -1 carried out fiat money from time -1, *i.e.*, $m^{-2} \neq 0$ and $m^{-1} \neq 0$. Determinacy then requires that the polynomial (3.6) has a single root located inside the unit circle.

The gross substitutes configuration first allows for a simple understanding based upon the nature of the economy under scrutiny:

Proposition 3.1. Consider an economy in which goods are all gross substitutes.

(i) If aggregate savings are positive (Samuelson case) at the intermediated steady state, then the equilibrium is determinate.

(ii) If aggregate savings are negative (classical case), then there exists one dimension of indeterminacy.

One may then notice the asymmetry between Samuelson and classical economies. Under a gross substitutability between all goods and for positive aggregate savings, the intermediated equilibrium is determinate without any further qualification. In contradistinction with this, the sole consideration of negative aggregate savings opens the road for local indeterminacy. The following section will analyse how this strong determinacy result for Samuelson economies will get questioned when gross complementarities are explicitly considered between the dated goods.

3.3 The Intermediated Steady State and Complementarities in Samuelson economies

Proposition 3.1 shows that gross substitutability guarantees the determinacy of the equilibrium nearby the golden rule steady state insofar as fiat money is positively valued. Focusing on such economies, the universal gross substitutability assumption is now relaxed; market demand functions exhibiting gross complementarities at the steady state, *i.e.*, $X'_i < 0$ for some i, are allowed. The following proposition characterises the various conceivable cases.

Proposition 3.2. Consider a Samuelson economy in which aggregate savings are positive at the intermediated steady state. Then the equilibrium exhibits

(i) determinacy if:

a) $[X'_{-1} + X'_1 > 0]$,

b) $[X'_2 > 0]$ or $[X'_2 < 0 \text{ and } -X'_{-2} < |X'_2|]$ or $[X'_2 < 0, -X'_{-2} > |X'_2| \text{ and } (X'_{-1}+X'_2)X'_2 > (X'_1 + X'_{-2})X'_{-2}]$;

(ii) one dimension of indeterminacy if:

a) $[X'_{-1} + X'_1 < 0]$,

b) $[|X'_2| > |X'_{-2}|]$ or $[|X'_2| < |X'_{-2}| \text{ and } (X'_{-1} + X'_2)X'_2 < (X'_1 + X'_{-2})X'_{-2}]$;

(iii) two dimensions of indeterminacy if:

a) $[X'_{-1} + X'_1 > 0 \text{ and } X'_2 < 0]$,

b) $[-X'_{-2} > |X'_2| \text{ and } (X'_{-1} + X'_2)X'_2 < (X'_1 + X'_{-2})X'_{-2}]$;

(iv) non-convergence if:

a) $[X'_{-1} + X'_1 < 0]$,

B) $\left[\left|X_2'\right| < \left|X_{-2}'\right| \text{ and } (X_{-1}' + X_2')X_2' > (X_1' + X_{-2}')X_{-2}'\right]$.

It is first obvious from Proposition 3.2(I) and its relation with Proposition 3.1 that the gross substitutability property, while being sufficient, is not necessary for determinacy in Samuelson economies. More specifically, the holding of a positive sign for the sum of the aggregate demand dependencies with respect to the one period forward price and the one period backward price, *i.e.*, the satisfaction of $X_{-1}' + X_1' > 0$, emerges as a necessary condition for determinacy. Interestingly, this does not rule out complementarities for one period apart goods: indeed, these are allowed for a given pair of one period spaced apart goods as long as the occurrence of $X_{-1}' < 0$ or $X_1' > 0$ is eventually overcome by the substitutability properties of the remaining one period space apart goods, *i.e.*, the *additive* simultaneous satisfaction of $X_{-1}' + X_1' > 0$. Now, it is first worthwhile emphasising that this necessary condition on the sum becomes sufficient when the aggregate demand dependency with respect to the two period forward price is positive, *i.e.*, under the holding of $X_2' > 0$, that identifies a gross substitutability for dated goods spaced two periods forward, Differing in this regard from the influence of the substitutability properties between goods spaced one period apart that always come into in an additive way, the remaining backward component of the substitutability /complementarity properties between two periods spaced apart goods, namely the occurrence of $X_{-2} > 0$ or $X_{-2} < 0$, is irrelevant in the determinacy result. Under the opposite configuration $X_2' < 0$, it is interesting to notice that the determinacy result keeps on being available for $X_{-2}' < 0$ under the extra qualification that $-X_2' > -X_{-2}'$, that means when dated goods spaced two periods apart are *symmetrically* gross complements. A final determinacy configuration is even available when that extra qualification is not satisfied: it however involves the rather complex comparison between the joint substitutability / complementarity properties of goods spaced one period apart together with the ones spaced two periods apart, namely $X_{-1}' + X_2'$ and $X_1' + X_{-2}'$, these two coefficients being in turn respectively *weighted* by X_2' and X_{-2}'.

The scope for local indeterminacies is gathered in Proposition 3.2(II)-(III). Proposition 3.2(III) is of interest by establishing that a class of *large dimension* indeterminacy conclusions is available under the above restriction for the subsitutability/complementarity properties of goods spaced one period apart, namely $X_{-1}' + X_1' > 0$. While such a configuration is unequivocally associated with a gross complementarity property between forward goods spaced two periods apart, namely $X_2' > 0$, it also rests upon the rather complex comparison between the joint substitutability / complementarity properties of goods spaced one period apart together with the ones spaced two periods apart that was mentioned above, albeit now in an opposite direction. Interestingly, it is also the one that motivated this study, for Kehoe and Levine's example of a two-dimensional indeterminacy with valued outside money precisely belongs to this case (see Kehoe & Levine [1990, Matrix (4.1), p. 231]) that here appears as 3.2(III)B). Proposition 3.2(II) however admittedly depicts the most common configurations that build upon a gross complementarity for an overall complementarity between goods spaced one period apart, namely the satisfaction of $X_{-1}' + X_1' < 0$, two ranges of indeterminate configurations being distinguished. A first that involves larger orders for the gross substitutability / complementarity properties of forward dated goods spaced two periods apart than for backward dated goods spaced two periods apart, namely $|X_2'| > |X_{-2}'|$. A second considers an opposite occurrence for these two periods apart dated goods but eventually rests upon the complex comparison between the joint substitutability / complementarity properties of goods spaced one period apart together with the ones spaced two periods apart that was mentioned above.[15]

REMARK 3.2. In a two-period-lived framework, calculus indicates that the dynamics around the intermediated steady state $\delta = 1$ are governed by $\lambda = X_{-1}'/X_1'$. Making use of the

[15]In Proposition 3.2(II) and in Proposition 3.2(IV), the occurrence of a symmetric gross complementarity of goods spaced two periods apart, $X_{-2} > 0$ and $X_2 < 0$, or more broadly $X_{-2} + X_2 < 0$, is associated with a violation of the Law of demand at the steady state, that is with goods of the Giffen type.

Jacobian matrix of the individual demand function, one gets $X'_{-1} = x'_{21}$ and $X'_1 = x'_{12}$. Now, from Walras's law, $x'_{11} + x'_{21} = -(x_1 - e_1)$ and $x'_{12} + x'_{22} = -(x_2 - e_2)$; where it is recalled that $x_2 - e_2 = -(x_1 - e_1)$ is the amount of nominal debt transferred from generation to generation in the steady state. Substitutions finally deliver $\lambda = 1 + (x_2 - e_2)/X'_1$. Thus, for instance, in a Samuelson economy where $x_2 > c_2$, if each good is a gross substitutability for the good of the subsequent period and $X'_1 > 0$, the (steady state) equilibrium is determinate. If, on the opposite, gross complementarity prevails, the equilibrium is indeterminate.

4. Concluding Comments

This study has clarified the scope for Kehoe & Levine's convergence criterion in reducing the size of equilibrium continuation and establishing determinacy in a three periods lifespan pure exchange economy. The argument was based upon aggregate demand functions and the role of an overall gross substitutability property for goods spaced one period apart was emphasised in the determinacy result while the asymmetric role of complementaries between goods spaced two periods apart was clarified in the scope for indeterminacies. On separated grounds, an interesting study by Lahiri & Puhakka [1998] has provided an insightful consideration of habit persistence preferences in a pure exchange two-period-lived lifespan overlapping generations economy that singles out an extra role for complementarities in an otherwise standard framework. The regards in which their results relate to the current range of asymmetric complementarities would remain to be clarified and will be the object of future research.

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A. From Demand to Savings and from Present Value Prices to Gross Interest Rates

While the market structure has hitherto been understood as if all markets were simultaneouly open before the beginning of time, in an alternative interpretation, markets meet sequentially according to their dates so that agents may freely save or borrow. The current appendix briefly clarifies the link between the derivatives of the aggregate demand with respect to present value prices and those of the aggregate savings with respect to interest rates, both at the steady state (for a detailed analysis, see Barinci, Cho & Drugeon [2024]).

Taking advantage of the homogeneity of degree zero of $X_t(\cdot)$:

$$X_t(p_{t-2},p_{t-1},p_t,p_{t+1},p_{t+2}) = X_t\left(\frac{p_{t-2}}{p_{t+2}},\frac{p_{t-1}}{p_{t+2}},\frac{p_t}{p_{t+2}},\frac{p_{t+1}}{p_{t+2}},1\right). \tag{A.1}$$

Let $R = (\cdots, R_{t-1}, R_t, R_{t+1}, \cdots)$, with $R_{t+1} = p_t/p_{t+1}$, be the (gross) interest rate path associated with the price system $p = (\cdots, p_{t-1}, p_t, p_{t+1}, \cdots)$. The market demand can then be expressed as a function of interest rates:

$$\overline{X}_t(R_{t-1},R_t,R_{t+1},R_{t+2}) := X_t(R_{t-1}R_tR_{t+1}R_{t+2}, R_tR_{t+1}R_{t+2}, R_{t+1}R_{t+2}, R_{t+2}, 1). \tag{A.2}$$

Making use of (A.1), (A.2), and the homogeneity of the demand function, one shows that:

$$\begin{bmatrix}\partial\bar{X}_t/\partial R_{t-1}\\ \partial\bar{X}_t/\partial R_t\\ \partial\bar{X}_t/\partial R_{t+1}\\ \partial\bar{X}_t/\partial R_{t+2}\end{bmatrix} = \begin{bmatrix} R_tR_{t+1}R_{t+2} & 0 & 0 & 0\\ R_{t-1}R_{t+1}R_{t+2} & R_{t+1}R_{t+2} & 0 & 0\\ R_{t-1}R_tR_{t+2} & R_tR_{t+2} & R_{t+2} & 0\\ R_{t-1}R_tR_{t+1} & R_tR_{t+1} & R_{t+1} & 1\end{bmatrix}\begin{bmatrix}\partial X_t/\partial p_{t-2}\\ \partial X_t/\partial p_{t-1}\\ \partial X_t/\partial p_t\\ \partial X_t/\partial p_{t+1}\end{bmatrix}. \tag{A.3}$$

Turning the attention to savings, let s_t^t be the savings, *i.e.*, the net addition of assets, of agent t from period t to $t+1$. Given strictly positive interest rates (R_{t+1}, R_{t+2}), the maximizing behavior of agent t is summarised by the individual savings functions $s_t^t(\cdot) = e_t^t - \overline{x}_t^t(\cdot)$ and $s_{t+1}^t(\cdot) = \left(\overline{x}_{t+2}^t(\cdot) - e_{t+2}^t\right)/R_{t+1}$ where $\overline{x}^t(\cdot)$ is the solution of the problem of maximizing u subject to a single lifetime constraint $R_{t+1}R_{t+2}x_t^t + R_{t+2}x_{t+1}^t + x_{t+2}^t = R_{t+1}R_{t+2}e_t^t + R_{t+2}e_{t+1}^t + e_{t+2}^t$. In period t, aggregate savings are $S_t = s_t^{t-1} + s_t^t$. Making use of the budget constraints of generations $t-2$, $t-1$ and t, namely:

$$\begin{aligned}
&\overline{x}_t^{t-2}(R_{t-1}, R_t) - e_t^{t-2} = R_t s_{t-1}^{t-2}(R_{t-1}, R_t),\\
&\overline{x}_t^{t-1}(R_t, R_{t+1}) - e_t^{t-1} = R_t s_{t-1}(R_t, R_{t+1})^{t-1} - s_t^{t-1}(R_t, R_{t+1}),\\
&\overline{x}_t^t(R_{t+1}, R_{t+2}) - e_t^t = -s_t^t(R_{t+1}, R_{t+2}),
\end{aligned}$$

one obtains:

$$\bar{X}_t(R_{t-1}, R_t, R_{t+1}, R_{t+2}) = R_t S_{t-1}(R_{t-1}, R_t, R_{t+1}) - S_t(R_t, R_{t+1}, R_{t+2}) + \mathscr{E}_t, \tag{A.4}$$

for $\mathscr{E}_t := e_t^{t-2} + e_t^{t-1} + e_t^t$. Observe that savings *income*, *i.e.*, the first term on the r.h.s. of (A.4), is interesting *per se* while considering a variation of R_t. Taking derivatives of both sides of (A.4), delivers

$$\frac{\partial\overline{X}_t}{\partial R_{t-1}} = \frac{\partial R_t S_{t-1}}{\partial R_{t-1}} \tag{A.5a}$$

$$\frac{\partial\overline{X}_t}{\partial R_t} = \frac{\partial R_t S_{t-1}}{\partial R_t} - \frac{\partial S_t}{\partial R_t} \tag{A.5b}$$

$$\frac{\partial\overline{X}_t}{\partial R_{t+1}} = \frac{\partial R_t S_{t-1}}{\partial R_{t+1}} - \frac{\partial S_t}{\partial R_{t+1}} \tag{A.5c}$$

$$\frac{\partial\overline{X}_t}{\partial R_{t+2}} = -\frac{\partial S_t}{\partial R_{t+2}} \tag{A.5d}$$

In the current stationary economy, saving functions are time invariant, $S_t = S$. Therefore, along a steady sequence of interest rates $(\ldots, R_{t-1}, R_t, R_{t+1}, \ldots) = (\ldots, R, R, R, \ldots)$

$$\cdots, \frac{\partial S_{t-1}}{\partial R_{t-1}} = \frac{\partial S_t}{\partial R_t},\ \frac{\partial S_{t-1}}{\partial R_t} = \frac{\partial S_t}{\partial R_{t+1}},\ \frac{\partial S_{t-1}}{\partial R_{t+1}} = \frac{\partial S_t}{\partial R_{t+2}}, \cdots.$$

It follows that in a stationary economy, the r.h.s. of the system (A.5) has only four *unknowns*, namely

$$\left(\frac{\partial S_{t-1}}{\partial R_{t-1}}, \frac{\partial S_{t-1}}{\partial R_t}, \frac{\partial S_{t-1}}{\partial R_{t+1}}; \frac{\partial(R_t S_{t-1})}{\partial R_t}\right).$$

Solving, one gets:

$$\begin{bmatrix} \partial S_{t-1}/\partial R_{t-1} \\ \partial S_{t-1}/\partial R_t \\ \partial S_{t-1}/\partial R_{t+1} \\ \partial R_t S_{t-1}/\partial R_t \end{bmatrix} = \begin{bmatrix} 1/R & 0 & 0 & 0 \\ 0 & 0 & -1 & -R \\ 0 & 0 & 0 & -1 \\ 1/R & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial \bar{X}_t/\partial R_{t-1} \\ \partial \bar{X}_t/\partial R_t \\ \partial \bar{X}_t/\partial R_{t+1} \\ \partial \bar{X}_t/\partial R_{t+2} \end{bmatrix}. \tag{A.6}$$

Now, making use of (A.3) at the steady state, (A.6) finally yields the sought link between the marginal properties of market demand as a function of present value prices and aggregate savings and savings *income* as functions of interest rates:

$$\begin{bmatrix} \partial S_{t-1}/\partial R_{t-1} \\ \partial S_{t-1}/\partial R_t \\ \partial S_{t-1}/\partial R_{t+1} \\ \partial (R_t S_{t-1})/\partial R_t \end{bmatrix} = \begin{bmatrix} R^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & (1+R)/R \\ 0 & 0 & 0 & 1 \\ R^2(1+R) & R^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial X_t/\partial p_{t-2} \\ \partial X_t/\partial p_{t-1} \\ \partial X_t/\partial p_{t+1} \\ \partial X_t/\partial p_{t+2} \end{bmatrix}. \tag{A.7}$$

For instance, $\partial S_{t-1}/\partial R_{t+1}$, the stationary sensitivity of today's savings with respect to after tomorrow's interest rate equals $\partial X_t/\partial p_{t+2}$. It follows that savings would be an increasing function of the two period forward interest rate provided that today's good be a gross substitute for To sum up, an approach based upon gross rates of interest, aggregate savings function and aggregate savings income, though of interest, significantly complexifies a direct comparison with the Kehoe & Levine [1985]'s line of argument.

B. Proof of Propositions 3.1 & 3.2

As $\lambda = 1$ is a root of the polynomial (3.6), the later restates as $(1-\lambda)Q(\lambda)$, for $Q(\lambda)$ a third-order characteristic polynomal, the determinacy result being unequivocally associated with a one stable root for $Q(\lambda)$.

Letting then, from a more general perspective, the equilibrium dynamics of an economy be described by a system: $y_{t+1} = G(y_t)$, $y_t \in \mathbb{R}^3_+$, steady states equilibria are the roots of $\bar{y} - G(\bar{y}) = 0$. The characterisation of the local dynamics nearby a given steady equilibrium proceeds from the appraisal of an associated linear map $\lambda_{t+1} = \mathcal{J}\lambda_t$, for $\mathcal{J} := DG(\bar{y})$ the Jacobian matrix of $G(\cdot)$ evaluated at $\bar{y}$ and $\lambda_t := y_t - \bar{y}$ the deviation from the steady state. The eigenvalues of the matrix $\mathcal{J}$ are the zeroes of the following third order polynomial:

$$\begin{aligned} Q(\lambda) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \\ &= -\lambda^3 + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)z + \lambda_1\lambda_2\lambda_3 \\ &= -\lambda^3 + T\lambda^2 - M\lambda + D, \end{aligned} \tag{1}$$

for T, M and D that respectively denote the trace, the sum of the principal minors of order two and the determinant of the Jacobian matrix $\mathcal{J} := DG(\bar{y})$.

Barinci & Drugeon [2017] build a typology[16] of the roots of $Q(\lambda) = 0$ based upon three critical loci respectively defined from the occurrences of a root of a real root of $+1$ — $Q(+1) = 0$ —, of a real root of -1 — $Q(-1) = 0$ — and of a pair of complex conjugate eigenvalues with unitary modulus. These are associated with the following surfaces in a (T, M, D) three-dimensional space:

$$\begin{aligned} &1 + T - M + D = 0, \\ &1 + T + M + D = 0, \\ &M - 1 - (T - D)D = 0, \qquad |T - D| < 2, \end{aligned}$$

[16]See also Brooks [2004] for a more specialised argument that focuses on stability conditions and Barinci & Drugeon [1999] for a more abstract approach based upon the geometry of three-dimensional spaces.

For a given D, the depiction of these three critical surfaces is going to be facilitated by the consideration of a collection of generic sections along the D coordinate, henceforth denoted as $(TM)_D$, any of the aforementioned critical loci being then represented through a straight-line or a segment.

It is more precisely established that the typology of the number of stable roots can advantageously get summarised in the following set of figures in the $(TM)_D$ space where the number of stable eigenvalues appears between parenthesis and where it is understood that the points above (below) the locus $Q(+1) = 0$ are associated with the occurrence of $Q(+1) < 0$ (respectively $Q(+1) > 0$). Likewise, the points above (below) the locus $Q(-1) = 0$ are associated with the occurrence of $Q(-1) > 0$ (respectively $Q(-1) < 0$). Finally remark that the darker segment features the third critical locus with a pair of complex eigenvalues with unitary modulus, namely $M - 1 - (T - D)D = 0$ for $|T - D| < 2$ and that it takes place within areas with $Q(+1)Q(-1) > 0$ for $|D| < 1$ but oppositely within areas with $Q(+1)Q(-1) < 0$ for $|D| > 1$.

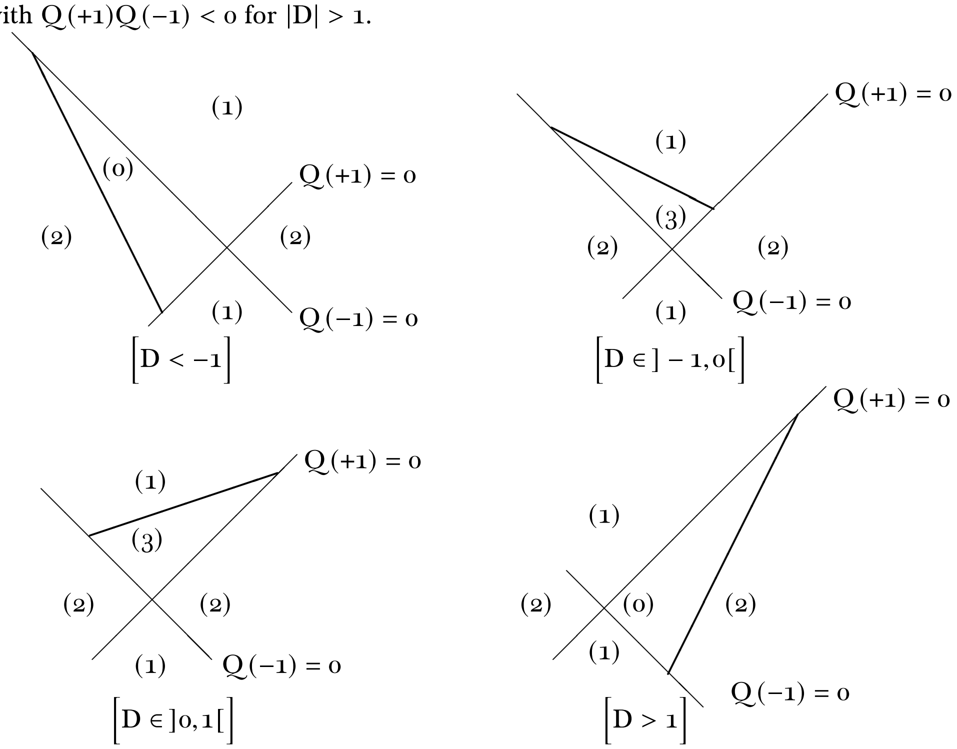

Figure B.1: Critical loci and typologies of stable eigenvalues for $|D| < 1$ and $|D| > 1$

The obtention of Kehoe & Levine's determinacy having been argued to uniformly rest upon a single root within the unit circle, the following conclusions become available:

(I) The holding of $Q(-1)Q(+1) < 0$ is necessary for $Q(\lambda) = 0$ to assume one single root that is less than one in absolute value.

(II) The restriction $Q(-1)Q(+1) < 0$ turns out to be also sufficient for $Q(\lambda) = 0$ to assume a single root that is less than one in absolute value if:

- A) $Q(+1) > 0$;
- B) $Q(-1) < 0$ provided that $\left|D\right| > 1$;
- C) $Q(-1) < 0$ provided that $\left|D\right| < 1$ if the auxiliary condition $M - 1 - (T - D)D > 0$ holds.

A straightforward application of the just depicted typology will now allow for delineating the determinacy requisites on the parameters of the model in the neighbourhood of a steady state equilibrium. The characteristic polynomial associated to the linearised form of the dynamical system is here available as:

$$Q(\lambda) = -\lambda^3 - \left(1 + \frac{X'_1}{X'_2}\right)\lambda^2 + \left(\frac{X'_{-1}}{X'_2} + \frac{X'_{-2}}{X'_2}\right)\lambda + \frac{X'_{-2}}{X'_2}.$$

To proceed, consider the three critical coefficients involved in the typology together with the auxiliary condition mentioned in (II)C):

$$\begin{aligned}
Q(+1) &= \frac{2X'_{-2} + X'_{-1} - X'_1 - 2X'_2}{X'_2},\\
Q(-1) &= -\frac{X'_{-1} + X'_1}{X'_2},\\
|D| - 1 &= \left|\frac{X'_{-2}}{X'_2}\right| - 1,\\
M - 1 - (T - D)D &= \frac{-(X'_1 + X'_{-2})X'_{-2} + (X'_{-1} + X'_2)X'_2}{(X'_2)^2},
\end{aligned}$$

the signs of which are key in determining the number of the stable roots.[17] Gross substitutes and gross complements economies will be considered in their turn:

A) Considering first local gross substitutability economies, that is economies satisfying the gross substitutes property at the steady state, $X'_i > 0$ for all i. If so, $Q(-1) < 0$ and $D = X'_{-2}/X'_2 > 0$ both unambiguously hold. It follows that for $Q(+1) > 0$ (resp. $Q(+1) < 0$) the equation $Q(\lambda) = 0$ has 1 (resp. 2) stable root(s). A final statement therefore follows from the sign of the numerator of $Q(+1)$. To see this let $J = [x'_{ij}]$, $i, j = 1, 2, 3$, represents the Jacobian matrix of the *individual* demand function x; x'_{ij} being the derivative of the demand function of the i-th period of life with respect of its j-th argument evaluated at the steady state. The derivatives of the aggregate demand function X are related with the terms of the matrix J as follows (see eq. (2.2)):

$$\begin{aligned}
X'_{-2} &= x'_{31}\\
X'_{-1} &= x'_{21} + x'_{32}\\
X'_0 &= x'_{11} + x'_{22} + x_{33}\\
X'_1 &= x'_{12} + x'_{23}\\
X'_2 &= x'_{13}
\end{aligned}$$

The homogeneity of degree zero of the demand functions implies $\sum_j x'_{ij} = 0$ for all i. Making use of the equalities above, calculus gives:

$$2X'_{-2} + X'_{-1} - X'_1 - 2X'_2 = \sum_i x'_{i1} - \sum_i x'_{i3}.$$

Now, differentiating Walras's law with respect to prices gives $\sum_i x'_{ij} = -(x_j - e_j)$ for all j. Consequently

$$\sum_i x'_{i1} - \sum_i x'_{i3} = x_3 - e_3 - (x_1 - e_1).$$

[17] Following the savings / gross interest rates approach mentioned in section 3.2 and appendix A, the three critical values of the characteristic polynomial together with the auxiliary condition mentioned in the typology would derive as $Q(+1) = ((1+R)[S'_0 - S'_1 + S'_2] - S)/S'_2$, $Q(-1) = ((R-1)[S'_0 + S'_1 + S'_2] + S)/S'_2$, $|D| - 1 = |S'_0/S'_2| - 1$ and $M - 1 - (T - D)D = -(S + S'_1 - S'_0)/S'_2 - 1 - [(S'_2 - S'_1)/S'_2 - S'_0/S'_2]S'_0/S'_2$.

This lastly establishes that

$$Q(+1) \lesseqgtr 0 \Longleftrightarrow x_3 - e_3 \lesseqgtr x_1 - e_1.$$

This first line of results is summarised in Proposition 3.1.

B) Considering now locally gross complements economies, that is economies allowing for gross complementarities at the steady state, *i.e.*, $X'_i < 0$ for at least one i. First remark that the argument of Proposition 3.1 that is detailed in A) happens to strongly simplify the whole study : the sign of the numerator of $Q(+1)$ summing up to the one of aggregate savings, it will always be positive in the Samuelson economy that is currently considered. This also implies that the sign of $Q(+1)$ directly derives from the one of its denominator X'_2: positive for $X'_2 > 0$ and negative for $X'_{+2} < 0$. The sign of $Q(-1)$, as for itself, will also depend upon the one of $-(X'_{-1} + X'_{+1})$. First consider the configuration with $X'_2 > 0$: it is associated with $Q(-1) < 0$ and $Q(+1) > 0$, hence the holding of $Q(-1)Q(+1) < 0$ for $X'_{-1} + X'_{+1} > 0$. In opposition to this, it will be associated with $Q(-1) > 0$ and $Q(+1) > 0$, hence the holding of $Q(-1)Q(+1) > 0$ for $X'_{-1} + X'_{+1} < 0$. Second consider the configuration with $X'_2 < 0$: it is associated with $Q(-1) > 0$ and $Q(+1) < 0$, hence the holding of $Q(-1)Q(+1) < 0$ for $X'_{-1} + X'_{+1} > 0$. In oppostion to this, it will be associated with $Q(-1) < 0$ and $Q(+1) < 0$, hence the holding of $Q(-1)Q(+1) > 0$ for $X'_{-1} + X'_{+1} < 0$. Interestingly, the sign of X_{-2}, and thus the scope for, *e.g.*, complementarities for backward goods spaced two periods apart, $X_2 < 0$, is entirely irrelevant for the determination of the sign of $Q(-1)$ and $Q(+1)$.

Focusing first on the determinacy issue and thus the scope for a unique root inside the unit circle, it is clear for Figure B.1 that this occurrence is unequivocally associated with the satisfaction of $Q(-1)Q(+1) < 0$ and thus some substitutability property for the sum of the goods spaced one period apart, *i.e.*, $X'_{-1} + X_{+1} > 0$. The simplest area is the bottom one that corresponds to the occurrence of $Q(-1) < 0$ and $Q(+1) > 0$ for $X'_2 > 0$. Other simple determinacy configurations are available on Figure B.1 that correspond to the upper area where $Q(-1) < 0$ and $Q(+1) > 0$ when $|D| = |X_{-2}/X_2| > 1$. Finally, some determinacy configurations remain available in the upper area with $Q(-1) < 0$ and $Q(+1) > 0$ when $|D| = |X_{-2}/X_2| < 1$. These are however more involving on a formal basis in that these further require the satisfaction of the auxiliary condition $M - 1 - (T - D)D > 0$ and thus the satisfaction of $X'_{-1} + X'_2)X'_2 > (X'_1 + X'_{-2})X'_{-2}$. The details of Proposition 3.2(i). The other configuration with an odd number of stable roots, namely 3 and two degrees of indeterminacy in Proposition 3.2(i) come as a complement for the upper area when the auxiliary condition is not satisfied.

The remaining configurations are associated with $Q(-1)Q(+1) > 0$ and an even number of roots, namely 0 or 2, inside the unit circle. These are also associated with the eastern and western areas on Figure B.1. The details of Proposition 3.2(ii) and Proposition 3.2(iv) are established following the same lines of reasoning as above.[18]

C. Determinacy for the Balanced Steady States

Consider now the balanced steady states in which $\delta \neq 1$, $\delta(x_3 - e_3) = x_1 - e_1$. For the sake of conciseness, the purpose will be restricted to gross substitutes economies. As two characteristic roots are now economically restricted, the characteristic polynomial in (3.6)

[18]The reader may notice some form of symmetry that was absent from the configurations with $Q(-1)Q(+1) < 0$. This springs from the fact that, depending upon the sign of D when it satisfies $|D| > 1$, the auxiliary condition may appear in the eastern area or in the western area. In opposition to this, for $Q(-1)Q(+1) < 0$ and $|D| < 1$, the auxiliary condition would always appear in the upper area of Figure B.1.

can be written $(1-\lambda)(1/\delta-\lambda)P(\lambda)$, where

$$P(\lambda) = \lambda^2 + \left(1 + \frac{1}{\delta} + \frac{1}{\delta}\frac{X'_1}{X'_2}\right)\lambda + \frac{1}{\delta^3}\frac{X'_{-2}}{X'_2}.$$

The assumed gross substitutability property entails that $P(+1) > 0$. Thus, if $P(-1) < 0$, the equation $P(\lambda) = 0$ will have one root with norm lesser that one. If, on the opposite, $P(-1) > 0$, the stable roots are either zero or two depending on the sign of $(1/\delta^3)(X'_{-2}/X'_2) - 1$. As the steady states are balanced, the interesting cases are those associated with $m^{-2}+m^{-1} = 0$. As a matter of fact, if life spans only two periods as in Gale [1973], without fiat money at the *inception* of the economy, thanks to a price normalization, the balanced steady state is locally the only non divergent equilibrium. Kehoe & Levine [1990, p. 230] rightly emphasized that in the three-period-lived model indeterminacy can happen even starting without outside money. It is easily show that such a possibility is consistent with the gross substitutability property (assumed above). To see this notice that $m^{-2} = m^{-1} = 0$ is obviously not necessary for $m^{-2}+m^{-1} = 0$; two consecutive generations meeting twice along their lifetime, one can be in debt to the other. Whenever $m^i \neq 0$, the price normalization is not permitted, and determinacy is associated with a single stable root. Now consider, for instance, an inefficient balanced steady state $\delta > 1$. A two-dimensional indeterminacy will therefore occur if the polynomial $P(\cdot)$ has two stable roots.