WORKING PAPERS

2025-12

06/04/2025

Hoping for the best while preparing for the worst in the face of uncertainty: a new type of incomplete preferences

Pierre Bardier, Bach Dong-Xuan and Van-Quy

CEPS
Centre for Economics at Paris-Saclay

Hoping for the best while preparing for the worst in the face of uncertainty: a new type of incomplete preferences

Pierre Bardier* Bach Dong-Xuan† Van-Quy Nguyen‡

January 2025

Abstract

We propose and axiomatize a new model of incomplete preferences under uncertainty, which we call *hope-and-prepare preferences*. Act f is considered more desirable than act g when, and only when, both an optimistic evaluation, computed as the welfare level attained in a best-case scenario, and a pessimistic one, computed as the welfare level attained in a worst-case scenario, rank f above g. Our comparison criterion involves multiple priors, as best and worst cases are determined among sets of probability distributions, and is, generically, less conservative than *Bewley preferences* and *twofold multi-prior preferences*, the two ambiguity models that are closest to ours.[1]

Keywords: Decision theory; Incomplete preference; Multiple-selves; Non-obvious manipulability.

JEL classification: D01; D81; D90

*Paris School of Economics, École Normale Supérieure de Paris (Paris Sciences et Lettres). E-mail: pierre.bardier@ens.fr

†Center for Mathematical Economics, Bielefeld University. E-mail: bach.dong@uni-bielefeld.de

‡Université Paris-Saclay, Univ Evry, EPEE, France. National Economics University, Faculty of Mathematical Economics, Vietnam. E-mail: nguyenvanquy0301@gmail.com

[1]These preferences are introduced in Bewley (2002) and Echenique et al. (2022), respectively.

1 Introduction

> *"Hoping for the best, prepared for the worst, and unsurprised by anything in between."*
>
> \- Maya Angelou, *I Know Why the Caged Bird Sings.*

The complexity of economic decisions is likely to result in agents' inability or unwillingness to decide over the uncertain options they are supposed to compare. In this regard, the restrictiveness of the assumption that individual preferences be complete was early acknowledged,[2] and was recently highlighted by empirical studies.[3] We propose and characterize a new *incomplete* decision criterion according to which, in the face of Knightian uncertainty (Knight (1921)), agents *both hope for the best and prepare for the worst.*

We study preferences over acts $f : S \to X$, which are mappings from states of the world to outcomes, and we introduce and axiomatize preferences $\succ$ admitting the following representation:

$$f \succ g \iff \begin{cases} \min_{p \in C} \int u(f) dp > \min_{p \in C} \int u(g) dp \\ \max_{p \in D} \int u(f) dp > \max_{p \in D} \int u(g) dp \end{cases}, \tag{1}$$

where u is a numerical representation of preferences over outcomes, and C and D are sets of probability distributions over the states, interpreted as sets of different *scenarios*.[4] Thus, a decision maker (DM) following such a criterion ranks an act f above an act g if and only if f provides a higher expected utility than g in the worst-case scenario in C as well as in the best-case scenario in D.

Our criterion is based on the conjunction of an optimistic (or ambiguity-seeking) assessment and of a pessimistic (or ambiguity-averse) assessment.[5] We then interpret a DM with such a preference as hoping for the best scenario to realize, while also preparing for the worst one to happen, when evaluating each option: we thus refer to a preference relation admitting such a representation as a *hope-and-prepare preference.* As a brief illustration, think

[2] For instance, Aumann (1962) wrote: "*Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint.*" Schmeidler (1989), commenting on his characterization of the maxmin criterion, depicted the completeness axiom as "*the most restrictive and demanding assumption.*"

[3] See Cettolin and Riedl (2019), Nielsen and Rigotti (2022).

[4] The function $u : X \to \mathbb{R}$ is non-constant, affine and unique up to positive affine transformation. The sets C and D are unique, non-disjoint, compact and convex.

[5] C and D being non-disjoint, the expected utility in the best-case scenario is higher.

of a company considering launching a new product. Typically, such a dual policy of decision making would favor investment or production strategies that present promising profit opportunities, in case the product captures an important market share, *and* a substantial safeguard, in case the product does not.

We shall give special attention in our analysis to the *concordant case* in which $C = D$ —for which we also provide an axiomatization. Acts are then evaluated according to *the interval of all expected utility levels that they induce across all possible scenarios.* More precisely, an act f is preferred to an act g if and only if *any expected utility level that is attainable from g but not from f is below any expected utility level that is attainable from f, and there exists at least one level that is indeed attainable from g but not from f.* This intuitive criterion for comparing *ranges* of expected utility levels works as a strict version of *the strong set order*, which is, arguably, the most common way to compare intervals.

Importantly, hope-and-prepare preferences treat the optimistic and the pessimistic assessments symmetrically: therefore, they do not systematically display a particular attitude toward ambiguity, which is consistent with extensive empirical evidence (see Trautmann and van de Kuilen (2015) for a survey).

The conjunction of a best-case evaluation and of a worst-case evaluation at play in our criterion is akin to the one at play in the notion of *obvious manipulation* (Troyan and Morrill (2020)), defined for revelation games in which the uncertainty faced by an agent concerns others' messages. Accordingly, the significant practical relevance of the notion of obvious manipulation provides support for our criterion within uncertain strategic environments. This notion gives an explanation, for instance, of untruthful reporting strategies that have been consistently observed in the *Immediate Acceptance mechanism*, used to match students with schools.[6]

The scope for applications of our criterion goes beyond strategic interactions. The idea that *both* worst-case and best-case scenarios serve as reference points is recognized for various social and economic domains where ambiguity is present. In this regard, let us simply mention the evaluation of financial assets (Bossaerts et al. (2010), Schröder (2011), Ahn et al. (2014)), or the evaluation of different medical treatments by physicians and patients (Back et al. (2003), Taylor et al. (2017)); we discuss a third example in more detail.

It is not unusual for practitioners, reporters or fans to evaluate "young prospects" participating in the annual *Draft* in North-American sports leagues —we take the example of the

[6]See Pathak and Sönmez (2008) and Dur et al. (2018).

National Basketball Association league (NBA)— according to "ceiling and floor scenarios."[7] This can be formulated in our framework. There is potentially a myriad of parameters that the agent considers relevant for the evaluation of prospects: a state is a particular configuration of parameters.[8] In this complex environment, the agent faces ambiguity and must compare prospects on the basis of a set C of probability distributions over configurations of parameters. Loosely speaking, each player is identified with an act f, indicating their overall performance in each state, which is then evaluated according to a utility function u, and, for every scenario $p \in C$, the agent can compute the expectation of $u(f)$ according to p. Then, the incompleteness of a criterion such as ours reflects the *necessity to have sufficient conviction* when declaring that a player is more promising than an other one. On the other hand, a criterion should not be *too incomplete*; let us illustrate this point by comparing our criterion to two alternative ones. Given u and C, the agent could require, for a "player f" to be declared more promising than a "player g", that, for each scenario in C, the expected utility associated with g be lower than the expected utility level associated with f (Bewley (2002)). One could even require that any expected utility level attainable from g be lower than any expected utility level attainable from f (Echenique et al. (2022)) —that the "ceiling" of g be lower than the "floor" of f. Both of these conditions are stronger than condition (1), expressing a more demanding notion of sufficient conviction. However, it may very well be the case that only "generational talents" such as Victor Wembanyama,[9] (who was present in the 2023 Draft) be distinguished from other players on the basis of these more conservative criteria, and that for rather homogeneous cohorts such as the 2024 cohort, the agent fail to rank any player above an other one.[10] In practical terms, according to our criterion, a "player f" is declared more promising than a "player g" if and only if anything that g could achieve and that f could not is considered worse than anything f could achieve. With hope-and-prepare preferences, which, in this case, compare players on the basis of the associated ranges of expected utility, in a way that is reminiscent of the strong set order, the *trade-off between decisiveness and conviction* is addressed in a way that is more favorable to

[7]See, for example, James Hansen, "*What makes an NBA Draft prospect high ceiling or high floor?*", SLC Dunk, June 2023, and Kyle Boone, "*NBA Draft 2024 ceiling and floor scenarios: The best or worst case projections for five top prospects*", CBS Sports, June 2024. We note that the use of the expressions "ceiling" and "floor" suggests that any case lying "in between" is considered possible.

[8]A state may thus encompass the roasters of coaches and players, at the beginning of the season and after the winter "trade" period, of each franchise, their financial capacities, the performance of players already in the league, the progression of each of these prospects, the approach to officiating favored by the league's executives, *etc.*

[9]See, for example, Sam Harris, "*Why 'alien' Wembanyama is France's next big thing - literally*", BBC Sports, July 2024.

[10]See, for example, Adam Finkelstein, "*No stars have revealed themselves in the 2024 NBA Draft, but history tells us they're hiding in plain sight*", CBS Sports, June 23 2024.

decisiveness.

The two original axioms involved in our characterization are interpreted along this line: we propose in Axiom 6 a relatively strong sufficient condition for *incomparability* —so that Axiom 6 is satisfied by the vast majority of incomplete criteria defined on single acts proposed in the literature— and a relatively weak sufficient condition for *comparability* in Axiom 7. Our axiomatization maintains the assumption that preferences are complete over constant acts, deemed as the simplest ones. Axiom 6 underscores the role of constant acts as benchmarks for decision making: if the DM is unable to compare the act g to the constant act x whenever she is unable to compare x to f, then she is not able to compare f and g. According to Axiom 7, if *i)* the DM cannot compare f to the constant x, while she declares x more desirable than g and, on the other hand, *ii)* she cannot compare g to the constant act y, while she declares f more desirable than y, then she declares f more desirable than g. Thus, *two specific aligned pieces of evidence are enough* to conclude that an act is better than an other one, and Axiom 7 may be seen as formulating a minimal departure from the completeness of a standard expected utility preference relation —we refer the reader to Section 3.1.1 for a more precise discussion.

Furthermore, in order to account for typical situations in which agents *have to* choose between two options, even if they lack conviction to express a clear preference between them in the first place, we study the completion of hope-and-prepare preferences.[11] We demonstrate that the *invariant biseparable complete extension* of a hope-and-prepare preference admits an *asymmetric*[12] *α-maxmin expected utility* (α-MEU) representation —and a standard α-maxmin representation if the hope-and-prepare preference is concordant. Notably, the asymmetric α-MEU retains much of the tractability of the standard α-MEU —which is beneficial for applications— while remaining flexible enough to accommodate mixed ambiguity attitudes (Chandrasekher et al. (2022)).[13] Importantly, in the representation we obtain, *the weight α does not depend on the considered acts, and is unique whenever the extended hope-and-prepare preference is incomplete.*

Finally, answering two natural questions of comparative statics that emerge from the proposition of a new type of incomplete preference under ambiguity, we compare the degree of incompleteness of our criterion to that of Bewley preferences (Bewley (2002)) and of twofold preferences (Echenique et al. (2022)), and we provide a way to compare the ambiguity

[11] From a theoretical point of view, studying a completion of an incomplete preference relation enables to use standard mathematical tools, for example for utility maximization and welfare analysis.

[12] "Asymmetric" refers to the fact that best and worst cases may be taken on different sets of scenarios.

[13] Specifically, it captures ambiguity-averse behavior for *large/moderate-likelihood* events, ambiguity-seeking behavior for *small-likelihood* events, and *source-dependent* ambiguity attitudes (Chandrasekher et al. (2022)).

attitudes of two hope-and-prepare preferences.

Our paper is organized as follows: we define the formal framework and introduce our criterion in Section 2. In Section 3, we give the main representation result and explore the case in which the sets of scenarios used in the two assessments are equal. In Section 4, we investigate the completion of our criterion. Section 5 is dedicated to the comparative statics questions mentioned above. Section 6 provides an illustrative comparison of concordant hope-and-prepare preferences to Bewley preferences, in the context of the aggregation of opinions of experts. The conclusions are presented in Section 7. All proofs can be found in the appendix.

1.1 Related literature

A DM hoping for the best while also preparing for the worst responds to uncertainty by combining opposite ambiguity attitudes. In this perspective, one may interpret a DM with a hope-and-prepare preference as *requiring that her optimistic (ambiguity loving) self and her pessimistic (ambiguity averse) self be unanimous for her to rank some act above an other one*. The idea that the DM consists of multiple (strategic) selves appears frequently in behavioral economics, in particular in models of dynamic choice or choice within risky environments.[14] In recent works, Chandrasekher et al. (2022) and Xia (2020) provided axiomatizations for preferences involving two selves, called by the former *dual-self expected utility*. Their representation differs from ours in that the agent's final decision is to be interpreted as the result of a specific *leader-follower game* between an optimistic self and a pessimistic self, whereas, in our representation, it is induced by a requirement of unanimity imposed by the agent herself on the assessments of her two selves.[15]

Our representation is also motivated by the concept of obvious manipulation proposed in the context of mechanism design by Troyan and Morrill (2020). A revelation mechanism is said to be *non-obviously manipulable* if, for any agent and any potential untruthful report from her, revealing her own type leads to a more desirable outcome in both of the following cases: when the others' reports are the most favourable to her, and when they are the least favourable. In our model, in the same spirit, an option —such as an untruthful report in the previous example— is only *abandoned* for an alternative if this alternative leads to preferred

[14] Thaler and Shefrin (1981), Bénabou and Pycia (2002), Fudenberg and Levine (2006), Brocas and Carrillo (2008).

[15] At a hight-level, the difference of our approach with that of *"Preparing for the Worst but Hoping for the Best: Robust (Bayesian) Persuasion"* (Dworczak and Pavan (2022)) is similar to the difference with Chandrasekher et al. (2022): in the criteria studied in both of these papers, one of the (pessimistic or optimistic) evaluations constrains the other. It is not the case with hope-and-prepare preferences in which both evaluations are treated symmetrically.

outcomes in both the best and the worst scenarios among given sets of probability measures. The relation of our contribution to the concept of non-obvious manipulation mirrors that of Echenique et al. (2022) to the concept of obvious dominance, due to Li (2017): informally, when the set of scenarios according to which all acts are evaluated is the simplex, act f is preferred to act g by a *twofold multi-prior preference* if and only if f obviously dominates g, and, on the other hand, f is preferred to g by a *hope-and-prepare preference* if and only if f dominates g in the sense of Troyan and Morrill (2020).[16]

Hope-and-prepare preferences define a partial order on acts. Pioneering work by Aumann (1962), Bewley (2002) and Dubra et al. (2004) studied the representation of incomplete preferences under risk and uncertainty. Incomplete preferences in non-deterministic environments have been the object of a growing literature: see, for example, Nascimento and Riella (2011) Galaabaatar and Karni (2012), Efe et al. (2012), Faro (2015), Minardi and Savochkin (2015), Hill (2016), Karni (2020), Cusumano and Miyashita (2021) and Echenique et al. (2022). The closest model of incomplete preference to ours, apart from those studied in Bewley (2002) and Echenique et al. (2022), both compared to ours in the introduction, is introduced in Nascimento and Riella (2011). As a special case of their main result, they study a criterion in which the DM considers several sets of scenarios, in each of which the performance of an act is evaluated according to the worst-case expected utility level. Then, an act is preferred to an other one if and only if it performs better in each set of scenarios. Hope-and-prepare preferences enable to capture a different type of ambiguity attitude, through the consideration of the optimistic assessment. We discuss in more details how our work relates to Bewley (2002), Nascimento and Riella (2011) and Echenique et al. (2022) in the next sections.

In line with Hurwicz's approach for decision making under complete ignorance (Hurwicz (1951)), the α-MEU model was proposed to capture the idea that, under ambiguity, worst and best expected utility levels, over *one* set of probability measures, can serve as sufficient statistics for the DM: she then computes an α-weighted average of these levels (Marinacci (2002), Kopylov (2002), Ghirardato et al. (2004)).[17] Among the recent explorations of (variants of) the α-maxmin model,[18] the one of Frick et al. (2022) is particularly important for the way we characterize the *asymmetric* α-maxmin model as representing the completion of

[16]With this same set of scenarios, one can also recover the concept of strategy-proofness from *Bewley preferences* (Bewley (2002)).

[17]Let us mention two alternatives to the standard α-MEU model. The *geometric* α-MEU model (Binmore (2009)) uses a geometric weighted average. More recently, Grant et al. (2020) introduced and characterized a general aggregation of best-case and worst-case expected utility representations, referred to as *ordinal Hurwicz expected utility*.

[18]Chateauneuf et al. (2007), Eichberger et al. (2011), Gul and Pesendorfer (2015), Frick et al. (2022), Klibanoff et al. (2022), Hartmann (2023), Hill (2023), Chateauneuf et al. (2024).

hope-and-prepare preferences.[19] In the *objective and subjective rationality* framework proposed by Gilboa et al. (2010), they show that the *invariant biseparable complete extension* of a Bewley preference admits a standard α-MEU representation. We show that the asymmetric α-maxmin representation characterizes the invariant biseparable complete extension of a hope-and-prepare preference relation. It reduces to standard α-maxmin in the case of a concordant hope-and-prepare preference relation. Beyond the fact that we consider the completion of a new type of preferences, our result has three salient feature: the representation is asymmetric in general, α does not depend on the considered acts, and is unique.[20]

2 Setup and representation

2.1 Model

Our analysis is conducted in the classical framework proposed by Anscombe and Aumann (1963). Uncertainty is modeled through a set S of *states of the world*, endowed with an algebra Σ of subsets of S called *events*, and a non-empty set of consequences X, which is a non-singleton convex subset of a real vector space. *A simple act* is defined as a function $f : S \to X$ which takes finitely many values and is measurable with respect to Σ; we denote by $\mathcal{F}$ the set of all simple acts. The *mixture* of two simple acts f and g, for any $\alpha \in [0, 1]$, denoted by $\alpha f + (1 - \alpha)g$, is then defined by setting, for each $s \in S$, $[\alpha f + (1 - \alpha)g](s) = \alpha f(s) + (1 - \alpha)g(s)$. With the usual slight abuse of notation, for all $x \in X$, we use x to denote the constant act defined by $f_x(s) = x$ for all $s \in S$. We use Δ to denote the set of all finitely additive probability distributions on (S, Σ), endowed with the weak* topology.[21] We refer to a measure $p \in \Delta$ as a *scenario* according to which simple acts are evaluated.[22]

We consider a DM whose preference is represented by a binary relation $\succ \subseteq \mathcal{F} \times \mathcal{F}$. It is a partial ranking over acts and we use the standard notation $f \succ g$ to denote $(f, g) \in \succ$. If $f \not\succ g$ and $g \not\succ f$, we write $f \bowtie g$, and say that f and g are *incomparable*.[23] We interpret $f \succ g$ as reflecting the fact that the DM considers that f is more desirable than g *with sufficient conviction*. In other words, in each pairwise comparison, one act (g in the previous notation) has the role of a *default* that would be abandoned only if the DM had enough

[19]Recall that "asymmetric" refers to the fact that best and worst cases may be taken on different sets of scenarios.

[20]More precisely, α is unique whenever the considered hope-and-prepare preference relation is not complete.

[21]The set of finitely additive bounded measures on (S, Σ) is the dual of the set of all measurable real-valued bounded functions on (S, Σ). Thus the weak* topology on Δ is defined according to the following convergence notion: we say that a sequence $\{p_n\}$ of elements of Δ converges to $p \in \Delta$ if for all measurable bounded function $\varphi : S \to \mathbb{R}$, $\int \varphi dp_n$ converges to $\int \varphi dp$.

[22]From now on, we refer to simple acts as "acts".

[23]Accordingly, we say that f and g are *comparable* if either $f \succ g$ or $g \succ f$.

reasons to believe that the alternative performs better.

We denote the set of vectors whose k elements are non-negative by $\mathbb{R}^k_+$, the set of vectors whose k elements are positive by $\mathbb{R}^k_{++}$, for any natural number k. For a given set A, $|A|$ denotes the cardinality of A.

2.2 Hope-and-prepare preferences

2.2.1 Definition

Our representation involves multiple priors:[24] the DM has a set of relevant beliefs according to which she evaluates acts.

Definition 1. A binary relation $\succ$ is a *hope-and-prepare* preference if

$$f \succ g \iff \begin{cases} \min_{p\in C} \int u(f)dp > \min_{p\in C} \int u(g)dp \\ \max_{p\in D} \int u(f)dp > \max_{p\in D} \int u(g)dp \end{cases},$$

where u is a non-constant affine function defined on X, and C and D are two compact and convex subsets of Δ with $C \cap D \neq \emptyset$.

The representation is *concordant* if $C = D$.

We sometimes write that $\succ$ admits the representation (u, C, D) to refer to the hope-and-prepare representation given in Definition 1. We obtain in our axiomatization the uniqueness up to affine transformation of u, and the uniqueness of C and D. *We sometimes write, then, as a shortcut, that* $\succ$ *admits the unique representation* (u, C, D).[25]

2.2.2 Discussion

Take such a preference relation $\succ$, with representation (u, C, D). An act f is preferred to an act g if and only if f gives a higher expected utility than g when they are evaluated according to their best-case scenario in D, *and* gives a higher expected utility than g when they are evaluated according to their worst-case scenario in C. This conjunction of an optimistic (or ambiguity-seeking) assessment and of a pessimistic (or ambiguity-averse) assessment models a DM hoping for the best scenario to realize, while also preparing for the worst one, when she evaluates each option.

[24] Etner et al. (2012) and Gilboa and Marinacci (2016) both provide a review of the ways in which ambiguity, and ambiguity attitudes, have been modeled in order to offer alternatives to the traditional Bayesian framework. Multiple prior models stand out as one of the main lines of research.

[25] As opposed to writing that $\succ$ admits representation (u, C, D), where u is unique, *up to affine transformation*, and C and D are unique.

The combination of two such opposite ambiguity attitudes may also be interpreted in the perspective of a DM consisting of multiple selves: for the DM to consider with sufficiently strong conviction that an act f is more desirable than an act g, it is necessary, and sufficient, that her optimistic self and pessimistic self, be unanimous over the ranking of f and g.[26]

The case of concordant preferences calls for further interpretation. Let $\succ$ be a hope-and-prepare preference relation with representation (u, C, C). Then, the DM evaluates any act f in terms of its range $R(f) = \{\int u(f)d\mu : \mu \in C\}$ of possible expected utility levels, which, as C is convex and compact, is a closed interval. Consider an other act $g \in \mathcal{F}$ and suppose that $R(f) = [a, b]$ and $R(g) = [c, d]$. Then f is preferred to g if and only if $a > c$ and $b > d$. This intuitive criterion for comparing ranges of expected utility levels works as an asymmetric version of *the strong set order*, applied to the special case of intervals: f is preferred to g if and only if *any expected utility level that is attainable from g but not from f is below any expected utility level that is attainable from f, and there exists at least one level that is indeed attainable from g but not from f.*

At this point, it is interesting to describe how the way ranges are compared according to concordant hope-and-prepare preferences can be formally related to the way in which they are compared according to concordant *twofold preferences*, introduced and axiomatized by Echenique et al. (2022).

Definition 2. (Echenique et al. (2022)) A binary relation $\succ$ is a (multi-prior) *twofold preference* if

$$f \succ g \iff \min_{p \in C} \int u(f)dp > \max_{p \in D} \int u(g)dp,$$

where u is a non-constant affine function defined on X, C and D are two compact and convex subsets of Δ with $C \cap D \neq \emptyset$. The representation is said concordant if $C = D$.[27]

Consider $\succ_H$ a concordant hope-and-prepare preference and $\succ_T$ a concordant twofold preference with the same representing utility function u on X and the same set of scenarios $C \in \Delta$. For any two real intervals I and I', we say that I *lies above* I' whenever $u > v$ for any $u \in I, v \in I'$. One has $f \succ_T g$ if and only if the interval $R(f)$ lies above the interval $R(g)$, and $f \succ_H g$ if and only if $R(f) \setminus R(g)$ is non-empty and lies above the interval $R(g) \setminus R(f)$, also non-empty.

Accordingly, given a set of scenarios, for a concordant hope-and-prepare preference and a concordant twofold preference, of which the restrictions to outcomes (constant acts) are

[26] In a previous version of this work, hope-and-prepare preferences were called *unanimous dual-self preferences*.

[27] They obtain the uniqueness, up to affine transformation, of u, and the uniqueness of C and D in their axiomatization.

equal, the former is always *more complete* than the latter, in the sense that it is an order extension of it —a general statement, beyond the case of concordant preferences, is given in Proposition 1. This observation is the basis on which we compared concordant hope-and-prepare preferences to concordant twofold preferences in the NBA example of the introduction, and a similar comparison to Bewley preferences should be made (again, see Proposition 1):

Definition 3. (Bewley (2002)) A binary relation $\succ$ is a (multi-prior) *Bewley preference* if

$$f \succ g \iff \int u(f)dp > \int u(g)dp \text{ for all } p \in C,$$

where u is a non-constant affine function defined on X, C is a non-empty compact and convex subset of Δ.[28]

As we highlighted in the NBA example, with hope-and-prepare preferences, the trade-off between decisiveness and conviction is addressed in a way that is more favorable to decisiveness, compared to twofold preferences and to Bewley preferences. That is, our criterion still reflects the necessity for DM to have sufficient conviction when declaring an act more desirable than an other one, while it induces more choices. The two original axioms involved in our characterization are interpreted along this line in Section 3.1.1.

When the preference relation $\succ$ is not concordant, different collections of scenarios are considered under the optimistic evaluation and under the pessimistic one. The fact that hope and preparation in our criterion involve different scenarios can capture the influence of non payoff-relevant elements. In practical decisions, DMs may have subjective views on what constitutes “good” and “bad” states, beyond the outcomes that acts yield in these states. Then, when thinking optimistically, they tend to favor beliefs that assign higher probabilities to “good” states, while, when thinking pessimistically, they tend to focus on beliefs assigning higher probabilities to “bad” states. When hoping, a DM may, for instance, only consider scenarios in which states where the weather is “nice” have a high probability, even if the weather does not affect their monetary payoffs.

Taking the dual-self interpretation more literally, a DM with non-concordant preferences does not simply combine the evaluations of two selves with opposite ambiguity attitudes, but also with different beliefs. Yet, both selves must agree on at least one belief; this common prior may represent the scenario that the DM considers as the most plausible, or, more generally, a reference scenario.

[28]Similarly to the expression we used for our criterion, we will say that the twofold preference $\succ_T$ admits the unique representation (u, C_T, D_T), and that the Bewley preference $\succ_B$ admits the unique representation (u, C_B) to refer to the fact that u is unique up to affine transformation, and C_T, D_T and C_B are unique.

In addition, the special case in which the discrepancy between the sets of scenarios takes the form of inclusion reflects the difference between the degree of preference for uncertainty of the optimistic evaluation and the degree of aversion to uncertainty of the pessimistic one (see Proposition 2 in Section 5). Let $\succ$ be a hope-and-prepare preference relation with representation (u, C, D), such that $D \subset C$: the pessimistic evaluation is more pessimistic than the optimistic evaluation is optimistic.

A special case of the preferences studied in Nascimento and Riella (2011) is, as ours, defined by the conjunction of different assessments. These preferences are pre-orders,[29] in contrast to hope-and-preferences, in general. We thus adapt their definition —see Theorem 4 in Nascimento and Riella (2011)— by replacing weak inequalities by strict inequalities.

Definition 4. (Nascimento and Riella (2011)) A binary relation $\succ$ is a *N&R preference* if

$$f \succ g \iff \min_{p \in C} \int u(f)dp > \min_{p \in C} \int u(g)dp \text{ for all } C \in \mathcal{C},$$

where u is a non-constant affine function defined on X, $\mathcal{C}$ is a class of non-empty compact and convex subsets of Δ.

Our approach and criterion differ from those of Nascimento and Riella (2011) in several aspects. In terms of methodology, in contrast to them, we provide an axiomatization directly on the domain of simple acts.[30] In addition, in our representation result, the pair of sets of probability measures is unique.[31]

Furthermore, while the set $\mathcal{C}$ may be infinite in their model —the DM then takes decisions based on the unanimity of an arbitrary, potentially infinite, number of selves— our criterion requires the conjunction of merely two evaluations.

From a behavioral perspective, N&R preferences are based on the unanimity of a collection of MEU representations, while hope-and-prepare preferences capture a mixture of different ambiguity attitudes. Moreover, we emphasized the importance of the special case of concordant hope-and-prepare preferences, defined by a very intuitive comparison of ranges of expected utility: this has no counterpart in their model.

[29] That is, reflexive and transitive binary relations.

[30] Their result is obtained on the set of lotteries on simple acts.

[31] In their representation, $\mathcal{C}$ is not unique. The authors obtain the uniqueness of the closure of the convex hull of $\mathcal{C}$, where the set of subsets of the simplex is endowed with the Hausdorff topology.

3 Representation results

3.1 Characterization of hope-and-prepare preferences

3.1.1 Axioms

We now proceed to an axiomatic characterization of hope-and-prepare preferences based on the following seven axioms. Axioms 1, 2, 3 and 5 express common requirements, Axiom 4 was proposed in Echenique et al. (2022), Axioms 6 and 7 are original to this work.

Let us insist on the fact that all these axioms (except Axiom 2), in which constant acts play a central role, are grounded on the basic idea that constant acts, because they are simpler, are relevant reference points for decision making.

Axiom 1. Relation $\succ$ is asymmetric and transitive, and the restriction of $\succ$ to X is non-trivial and negatively transitive.[32]

Axiom 2. For all triple $(f, g, h) \in \mathcal{F}^3$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succ h\}$ and $\{\alpha \in [0,1] : h \succ \alpha f + (1-\alpha)g\}$ are open.

Axiom 3. For all $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$, $f \succ g$ if and only if $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$.

The interpretation of the assumptions in Axiom 1 is well-known. In particular, on X, $\succ$ is the asymmetric part of a complete and transitive relation. Axiom 2 is the standard Archimedean continuity condition adopted in models of decision under uncertainty. Axiom 3 is the independence axiom proposed by Gilboa and Schmeidler (1989) in their seminal paper as a weakening of the independence axiom at play in the characterization of subjective expected utility.

Axiom 4. For all $x \in X$, the sets $\{f \in \mathcal{F} : f \succ x\}$ and $\{f \in \mathcal{F} : x \succ f\}$ are convex.

Axiom 4 is identical to Axiom 4 in Echenique et al. (2022), which states that comparisons to a given constant act should not be sensitive to hedging. Recall that since $\succ$ is asymmetric and incomplete, in general, $\{f \in \mathcal{F} : f \succ x\}$ is interpreted as the set of acts for which the DM has sufficient conviction to consider them more desirable than the constant act x. The convexity of $\{f \in \mathcal{F} : f \succ x\}$ is interpreted in terms of uncertainty aversion: an act obtained through hedging between two acts that provide sufficient evidence to be declared more desirable than the constant act x is also considered more desirable than x with sufficient conviction. The convexity of $\{f \in \mathcal{F} : x \succ f\}$ is interpreted in terms of preference

[32] Negative transitivity of $\succ$ means that for all $x, y, z \in X$, if $x \not\succ y$ and $y \not\succ z$ then $x \not\succ z$.

for uncertainty: when the DM has sufficient conviction to declare two uncertain acts less desirable than the constant act x, then the DM also considers with sufficient conviction that an act obtained through hedging between the two is less desirable than x.

Axiom 5. For all $f, g \in \mathcal{F}$, if $f(s) \succ g(s)$ for all $s \in S$, then $f \succ g$.

According to Axiom 5, if the outcome of an act is considered more desirable than the outcome of an other act in each state of the world, then the first act is preferred to the second one. In other words, according to a preference relation satisfying Axiom 5, the state-wise dominance of an act f over an act g provides sufficient conviction to rank f above g. In the perspective of the trade-off between decisiveness and conviction, we see this property as an intuitive limitation of incomparability. While it is imposed in most approaches close to ours, the strong degree of conservatism, or indecisiveness, of twofold preferences is rooted in the fact that they violate it.[33]

We propose in Axiom 6 a relatively strong sufficient condition for incomparability — equivalently, a relatively weak necessary condition for comparability— so that Axiom 6 is satisfied by almost all (the asymmetric part of) the incomplete criteria comparing single acts mentioned in Section 1.1, that is, almost all the incomplete criteria defined in a classical Anscombe-Aumann framework mentioned in Section 1.1. More precisely, the (asymmetric part of) the criteria proposed in Bewley (2002), Nascimento and Riella (2011), Efe et al. (2012), Faro (2015), Cusumano and Miyashita (2021) and Echenique et al. (2022) all satisfy Axiom 6 (see Appendix A).[34] On the other hand, we impose a relatively weak sufficient condition for comparability in Axiom 7.

We jointly discuss these axioms after we briefly present them.

Axiom 6. For all $f, g \in \mathcal{F}$, if *for all* $x \in X$, $f \bowtie x$ implies $g \bowtie x$, then $f \bowtie g$.

Axiom 6 underscores the role of constant acts as benchmark acts based on which comparisons of more complex acts are made: for the DM to express a preference between the acts f and g, *it is necessary that there exists a constant act x that the DM prefers to either f or g, while she cannot compare x with the other act.*[35]

Axiom 7. For all $f, g \in \mathcal{F}$, and for all $x, y \in X$, if $f \bowtie x$, $x \succ g$, $g \bowtie y$ and $f \succ y$ then $f \succ g$.

[33] See Cusumano and Miyashita (2021) and Echenique et al. (2022).

[34] There is one incomplete criterion mentioned in Section 1.1 that is defined on single acts and that may not satisfy Axiom 6, the one proposed in Hill (2016).

[35] Note that we do not impose that whenever there exists $x \in X$ such that $f \bowtie x$ and $g \bowtie x$, then $f \bowtie g$ (which is Axiom 5 in Echenique et al. (2022)). Actually, our criterion does not satisfy this property in general.

While the DM cannot compare f to the constant x, she declares x more desirable than g. On the other hand, while she cannot compare g to the constant act y, she declares f more desirable than y. Axiom 7 implies that *in the presence of such consonant conclusions as to the comparison of f and g, the DM considers f, with sufficient conviction, more desirable than g.*

As we already highlighted, given the complexity involved in the evaluation of uncertain acts, constant acts, which are the simplest acts, are likely to be used as comparison devices. A straightforward way to use them in comparing two acts, when preferences may be incomplete, then consists in looking for a constant act that is incomparable to one of them and comparable to the other one. Each such constant act then provides *a piece of evidence* as to the comparison between the two uncertain acts —the question is then to determine what are sufficient pieces of evidence.

Consequently, given two acts f and g, the DM we model compares to g all constant acts that are incomparable to f, and vice versa. This process gives rise to three possible cases:

(i) for all $x \in X$ such that $f \bowtie x$, $g \bowtie x$;

(ii) for all $x \in X$ such that $g \bowtie x$, $f \bowtie x$; and

(iii) there are $x, y \in X$ such that [$f \bowtie x$ and g and x are comparable] and [$g \bowtie y$ and f and y are comparable].

In the first two cases, there is no piece of evidence on which the DM may base her comparison: Axiom 6 implies that f and g are incomparable.

In the last case, there are four possible situations; it suffices to consider the following two, to which the other ones are symmetric:

(a) [$f \bowtie x$ and $x \succ g$] combined with [$y \succ f$ and $g \bowtie y$]; and

(b) [$f \bowtie x$ and $x \succ g$] combined with [$f \succ y$ and $g \bowtie y$].

In case (a), the first piece of evidence favors f while the second favors g. In contrast, in case (b), the two pieces of evidence go in the same direction, favoring f: according to Axiom 7, this is sufficient to conclude that f is more desirable than g.

There is a sense in which Axiom 7 expresses, for an asymmetric and incomplete preference relation, a minimal departure from the completeness of weak orders for which all acts admit a *certainty equivalent*.[36] Using the previous formulation, for these weak orders, one piece of

[36] That is, binary relations $\succsim$ which are reflexive, transitive and complete, such that, for all $f \in \mathcal{F}$, there is $x \in X$ such that $f \sim x$.

evidence is sufficient: if $f \in \mathcal{F}$ has a certainty equivalent $x \in X$ and x is strictly preferred to $g \in \mathcal{F}$, then f is strictly preferred to g. For an asymmetric and incomplete preference relation, Axiom 7 involves no more than one piece of evidence based on a constant act incomparable to f *and* one piece of evidence based on a constant act incomparable to g.

Axiom 7 is violated by twofold preferences, Bewley preferences and N&R preferences in general. Through the satisfaction of both Axioms 6 and 7, in particular, hope-and-prepare preferences address the trade-off between decisiveness and conviction in a new way.

We sometimes refer to the classical Axioms 2, 3 and 5 as continuity, certainty independence and monotonicity.

3.1.2 First characterization theorem

Theorem 1. *A binary relation $\succ$ satisfies Axioms 1-7 if and only if there exist*

- *a non-constant affine function $u : X \to \mathbb{R}$, unique up to positive affine transformation,*
- *a unique pair (C, D) of non-disjoint convex compact subsets of Δ,*

such that, for all $f, g \in \mathcal{F}$,

$$f \succ g \Leftrightarrow \begin{cases} \min_{p \in C} \int u(f)dp > \min_{p \in C} \int u(g)dp \\ \max_{p \in D} \int u(f)dp > \max_{p \in D} \int u(g)dp \end{cases},$$

that is, $\succ$ admits the hope-and-prepare representation (u, C, D), where C and D are unique, and u is unique up to positive affine transformation.

We now give a brief sketch of the proof and highlight some interesting properties of $\succ$ that we derive.[37] First of all, Axioms 1-3 guarantee that there exists a non-constant affine function $u : X \to \mathbb{R}$, unique up to affine transformation, representing $\succ$ on X.

The proof consists in defining two binary relations on $\mathcal{F}$, denoted $\succ_p$ and $\succ_o$, such that for any $f, g \in \mathcal{F}$, $f \succ g$ if and only if $f \succ_p g$ and $f \succ_o g$ —we provide the precise definitions of these relations below. In that perspective, the following two lemmas are crucial.

Lemma. *For all $f \in \mathcal{F}$, the set $\{x \in X : x \bowtie f\}$ is non-empty.*

Lemma. *For all $f \in \mathcal{F}$, and $x, y, z \in X$, if $x \bowtie f$, $f \succ y$, and $z \succ f$, then $z \succ x \succ y$.*

This second result has an interesting interpretation. While the DM cannot assert with sufficient conviction that f is more desirable than the constant act x, she considers with

[37]The following lemmas are not presented here in the order in which they are proved.

sufficient conviction that f is more desirable than the constant act y and worse than the constant act z. We show that in such a case, the DM considers, with sufficient conviction, that z is more desirable than x, and that x is more desirable than y.

From the original relation $\succ$, we define two preference relations on $\mathcal{F}$ as follows:

$$g \succ_p f \iff g \succ x \text{ and } x \bowtie f \text{ for some } x \in X,$$
$$g \succ_o f \iff x \bowtie g \text{ and } x \succ f \text{ for some } x \in X.$$

The subscripts p and o are used to denote respectively a pessimistic and an optimistic assessment, based on $\succ$, where these two terms are justified given the way the incomparability to a constant act is treated. Let us describe the interpretation of $\succ_p$: this relation is pessimistic in the sense that for the default act f, whenever there is a constant act x such that f cannot be compared with sufficient conviction to x, while g is considered more desirable than x with sufficient conviction, then $\succ_p$ declares f to be worse than g.

We then proceed by showing that $\succ_p$ and $\succ_o$ are asymmetric and negatively transitive. This enables us to define $\sim_p$ by $f \sim_p g$ if and only if $f \nsucc_p g$ and $g \nsucc_p f$, for all $f, g \in \mathcal{F}$, and to define $\succsim_p$ by $f \succsim_p g$ if and only if either $f \succ_p g$ or $f \sim_p g$, for all $f, g \in \mathcal{F}$. We define in the same way $\sim_o$ and $\succsim_o$. Then it is clear that $\succsim_p$ and $\succsim_o$ are weak orders,[38] and we show that they are continuous and monotone, that they satisfy the classical properties of certainty independence, and, respectively, aversion to ambiguity and preference for ambiguity.[39]

As a consequence, $\succsim_p$ can be represented by the function $f \mapsto \min_{p \in C} \int u_p(f)dp$, and $\succsim_o$ can represented by the function $f \mapsto \max_{p \in D} \int u_o(f)dp$, where C and D are non-empty convex compact subsets of Δ, and u_p and u_o are two affine functions on X. We conclude that there is no loss of generality in assuming $u_p = u_o = u$, and that $C \cap D \neq \emptyset$, using the separating hyperplane theorem on these subsets of Δ endowed with the weak* topology.

Note that in this sketch of proof, the relation between $\succ$ and the two weak orders $\succ_p$ and $\succ_o$ is established before the minmax and maxmax representations of $\succ_p$ and $\succ_o$: Axioms 1-3 and Axioms 5-7 are necessary and sufficient for a general representation that we describe in Appendix B.

[38] They are non-trivial asymmetric and negatively transitive binary relations.

[39] Definitions of these properties for weak orders are provided in the appendix.

3.2 Characterization of concordant hope-and-prepare preferences

3.2.1 Axioms

The necessary and sufficient conditions identified in Echenique et al. (2022) for the identity $C = D$ to hold in their twofold multiprior preference representation are also necessary and sufficient in our representation.[40] Before introducing them, let us specify that, as suggested in the sketch of the proof of Theorem 1, when $\succ$ satisfies Axioms 1-3, we define on X the relation $\gtrsim$ by $x \gtrsim y$ if and only if $y \not\succ x$ for all $x, y \in X$. Clearly, $\gtrsim$ on X is asymmetric and negatively transitive; and $\bowtie$ is equivalent to $\sim$, the symmetric part of $\gtrsim$, on X.

We use the notion of complementary acts (Siniscalchi (2009)) to identify comparisons that are, under Axioms 1-7, characteristic of the uncertainty aversion of the agent's pessimistic evaluation, and of the preference for uncertainty of her optimistic evaluation, respectively. Two acts f and g are *complementary* if they perfectly hedge against each other in the sense that their equal-weight-mixture is equivalent to a constant act:

$$\frac{1}{2}f(s) + \frac{1}{2}g(s) \sim \frac{1}{2}f(s') + \frac{1}{2}g(s') \text{ for all } s, s' \in S.$$

Axiom 8. If f and g in $\mathcal{F}$ are complementary, then $f \succ \frac{1}{2}f + \frac{1}{2}g$ implies $\frac{1}{2}f + \frac{1}{2}g \succ g$.

Consider two complementary $f, g \in \mathcal{F}$, and a preference $\succ$ with representation (u, C, D) on $\mathcal{F}$. Assume $f \succ \frac{1}{2}f + \frac{1}{2}g$, as in Axiom 8 and let $x \in X$ denote a constant act such that $x \sim \frac{1}{2}f + \frac{1}{2}g$. It cannot be the case that $g \succ \frac{1}{2}f + \frac{1}{2}g$, because this would imply $f \succ x$ and $g \succ x$, and thus $\frac{1}{2}f + \frac{1}{2}g \succ x$ —a contradiction.

In other words, if $f \succ \frac{1}{2}f + \frac{1}{2}g$, then either $\frac{1}{2}f + \frac{1}{2}g \bowtie g$ or $\frac{1}{2}f + \frac{1}{2}g \succ g$. Axiom 8 requires that the second case hold, and this requirement is interpreted as a consequence of the simplicity of constant acts. Indeed, by transitivity, in this second case, one has, by transitivity, $f \succ g$, so that Axiom 8 states that whenever $f \succ \frac{1}{2}f + \frac{1}{2}g$, one has $f \succ g$, that is, it should always be easier for the DM to assess whether f is more desirable than the essentially constant act $\frac{1}{2}f + \frac{1}{2}g$ than to assess whether f is more desirable than g.

The interpretation of Axiom 9 is similar: it states that for complementary acts $f, g \in \mathcal{F}$, it should always be easier for the DM to assess whether the essentially constant act $\frac{1}{2}f + \frac{1}{2}g$ is more desirable than g than to assess whether f is more desirable than g.

Axiom 9. If f and g in $\mathcal{F}$ are complementary, then $\frac{1}{2}f + \frac{1}{2}g \succ g$ implies $f \succ \frac{1}{2}f + \frac{1}{2}g$.

[40] The proof of the following result is a direct adaptation of the proof of Proposition 1 in their paper.

3.2.2 Second characterization theorem

Theorem 2. *The following statements hold:*

(i) A hope-and-prepare preference $\succ$, with unique representation (u, C, D), satisfies Axiom 8 if and only if $D \subseteq C$.

(ii) A hope-and-prepare preference $\succ$, with unique representation (u, C, D), satisfies Axiom 9 if and only if $C \subseteq D$.

In particular, a binary relation $\succ$ satisfies Axioms 1-9 if and only if there exist

- *a non-constant affine function $u : X \to \mathbb{R}$, unique up to positive affine transformation,*
- *a unique convex compact subset of Δ, denoted C, such that, for all $f, g \in \mathcal{F}$,*

$$f \succ g \iff \begin{cases} \min_{p \in C} \int u(f)dp > \min_{p \in C} \int u(g)dp \\ \max_{p \in C} \int u(f)dp > \max_{p \in C} \int u(g)dp \end{cases}.$$

When $\succ$ admits a concordant representation, acts are evaluated according to the minimum and the maximum expected utility level attained on a common set of scenarios. On the other hand, when $\succ$ satisfies both Axiom 8 and 9, for any simple complementary acts f and g, $f \succ \frac{1}{2}f + \frac{1}{2}g$ if and only if $\frac{1}{2}f + \frac{1}{2}g \succ g$. In other words, for complementary acts, it is always as easy to determine whether their equal-weight-mixture is more desirable than one of them as it is to determine whether one of them is more desirable than the mixture.

As a recall, according to a concordant hope-and-prepare preference relation, acts are evaluated according to the interval of all expected utility levels that they induce across all scenarios in a given set. More precisely, an act f is preferred to an act g if and only if *any expected utility level that is attainable from g but not from f is below any expected utility level that is attainable from f, and there exists at least one level that is indeed attainable from g but not from f.*

4 Complete extension of hope-and-prepare preferences

In this section, we will explore the extension of hope-and-prepare preferences to complete preferences. We will focus on the *invariant biseparable* complete extension; these define a broad class of complete preferences that nests the majority of preferences studied in the literature.

We refer to an asymmetric complete and negatively transitive binary relation on $\mathcal{F}$ satisfying Axioms 2, 3 and 5 as *invariant biseparable*.[41] When it is, in addition, a weak order, it satisfies the axioms characterizing expected utility, apart from the independence axiom, which is weakened to the certainty independence property introduced in Gilboa and Schmeidler (1989).

Definition 5. A preference relation $\succ$ on $\mathcal{F}$ admits an *asymmetric α-MEU* representation if there exist $\alpha \in [0,1]$, two non-disjoint compact convex subsets C and D of Δ, and a non-constant affine function $u : X \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$,

$$f \succ g \iff \alpha \min_{p \in C} \int u(f)dp + (1-\alpha) \max_{p \in D} \int u(f)dp$$
$$> \alpha \min_{p \in C} \int u(g)dp + (1-\alpha) \max_{p \in D} \int u(g)dp.$$

We will refer to such representation as a (u, C, D, α) representation.

Remarkably, Chandrasekher et al. (2022) show that the asymmetric α-MEU, while retaining the tractability property of the standard α-MEU, is flexible enough to accommodate ambiguity-averse for *large/moderate-likelihood* events but ambiguity-seeking for *small-likelihood events* and *source-dependent* ambiguity attitudes.

Standard α-MEU criteria are obtained if $C = D$ in Definition 5, and the following result, as a particular case, characterizes them as invariant bi-separable extensions of concordant hope-and-prepare preferences.

Theorem 3. *The following conditions are equivalent when $\succ$ is a hope-and-prepare preference with unique representation (u, C, D):*

(i) $\succ^*$ is an invariant biseparable preference and an extension of $\succ$.

(ii) $\succ^*$ admits an α-maxmin expected utility representation (u, C, D, α) in which α is unique whenever $\succ$ is not complete.

[41] Ghirardato et al. (2004) originally used the expression "invariant biseparable preferences" when studying weak-orders. For an asymmetric complete and negatively transitive binary relation $\succ$ on $\mathcal{F}$, as for $\succ_p$ and $\succ_o$ in Section 3, we define $\sim$ by $f \sim g$ if and only if $f \not\succ g$ and $g \not\succ f$, for all $f, g \in \mathcal{F}$, and $\succsim$ by $f \succsim g$ if and only if either $f \succ g$ or $f \sim g$, for all $f, g \in \mathcal{F}$. Then, in the proof of Theorem 3, we show that $\succsim^*$ is an "invariant biseparable preference" in the sense of Ghirardato et al. (2004).

5 Comparison of incomplete criteria

5.1 Degree of incompleteness

We have stated that with hope-and-prepare preferences, in comparison to Bewley preferences and twofold preferences, the trade-off between decisiveness and conviction is addressed in a way that is more favorable to decisiveness. The criterion we use to determine whether a binary relation is more conservative than an other one pertains to their respective *degree of incompleteness.*

Definition 6. Given two preference relations $\succ_1$ and $\succ_2$ on $\mathcal{F}$, we say that $\succ_1$ is more conservative than $\succ_2$ if $\succ_2$ is an extension of $\succ_1$, that is, for all $f, g \in \mathcal{F}$,

$$f \succ_1 g \text{ implies } f \succ_2 g.$$

The next proposition identifies necessary and sufficient conditions under which a hope-and-prepare preference relation is an extension of a Bewley or of a twofold preference relation.

Proposition 1. *Let $\succ_H$ be a hope-and-prepare preference with unique representation (u, C_H, D_H). Let $\succ_T$ be a twofold multiprior preference with unique representation (u, C_T, D_T). Let $\succ_B$ be a Bewley preference with unique representation (u, C_B). Then,*

(i) the preference relation $\succ_B$ is more conservative than $\succ_H$ if and only if $C_H \cup D_H \subseteq C_B$;

(ii) the preference relation $\succ_T$ is more conservative than $\succ_H$ if and only if $C_H \subseteq C_T$ and $D_H \subseteq D_T$.

Remark 1. A direct consequence of this proposition and Proposition 4 in Echenique et al. (2022) is that if $C_H \cup D_H \subseteq C_B \subseteq C_T \cap D_T$, in particular if $C_H = D_H = C_B = C_T = D_T$, then $\succ_T$ is more conservative than $\succ_B$, which is more conservative than $\succ_H$.

5.2 Ambiguity attitudes

We are able to compare ambiguity attitudes displayed by different hope-and-prepare preferences using the classical comparative statics notions of Ghirardato and Marinacci (2002).

Definition 7. Given two preference relations $\succ_1$ and $\succ_2$ on $\mathcal{F}$,

(i) $\succ_1$ is more ambiguity averse than $\succ_2$ if, for all $f \in \mathcal{F}$ and $x \in X$, $f \succ_1 x$ implies $f \succ_2 x$.

(ii) $\succ_1$ is more ambiguity loving than $\succ_2$ if, for all $f \in \mathcal{F}$ and $x \in X$, $x \succ_1 f$ implies $x \succ_2 f$.

An agent is more ambiguity averse than an other one if she is less inclined to choose an uncertain act f over a constant act x. On the other hand, an agent is more uncertainty loving than an other one if she is more inclined to stick to an uncertain act f than to switch to a constant act x. The next result characterizes ambiguity attitudes for hope-and-prepare preferences.

Proposition 2. *Let $\succ_1$ and $\succ_2$ be two hope-and-prepare preference relations with unique representation (u, C_1, D_1) and (u, C_2, D_2), respectively. Then,*

(i) $\succ_1$ is more ambiguity averse than $\succ_2$ if and only if $C_2 \subseteq C_1$.

(ii) $\succ_1$ is more ambiguity loving than $\succ_2$ if and only if $D_2 \subseteq D_1$.

For a hope-and-prepare representation (u, C, D), the two sets of priors C and D represent the level of pessimism and optimism related to the DM's ambiguity attitudes. More precisely, the relationship $C_2 \subseteq C_1$ means that, in the worst scenario, the level of welfare attained by the agent if she has preference relation $\succ_1$ is lower than the one attained if she has preference relation $\succ_2$. Similarly, $D_2 \subseteq D_1$ means that, in the best scenario, the level of welfare attained by the agent if she has preference relation $\succ_1$ is higher than the one attained if she has preference relation $\succ_2$.

Based on Proposition 2 *(i)*, by comparing the concordant preference $\succ$ with representation (u, C, C) to the non-concordant preference $\succ_1$ with representation (u, C_1, C), with $C_1 \subset C$, we can say that $\succ_1$ *is more ambiguity averse than it is ambiguity loving*. Similarly, the non-concordant representation $\succ_2$ with representation (u, C, D_2), with $D_2 \subset C$, *can be said to be more ambiguity loving than it is ambiguity averse*. Then, a DM with concordant preferences is as ambiguity loving as she is ambiguity averse, or, in other words, *her pessimistic evaluation is as pessimistic as her optimistic evaluation is optimistic*.

We end this subsection by briefly discussing the relation between the degree of conservatism of a hope-and-prepare preference relation and the attitude towards ambiguity that it displays. It is easy to see that if $\succ_1$ and $\succ_2$ are hope-and-prepare preferences, and if $\succ_1$ is more conservative than $\succ_2$, then $\succ_1$ is both more ambiguity averse and more ambiguity loving than $\succ_2$. Does the converse statement hold ? This question is all the more natural that if $\succ_1$ and $\succ_2$ are twofold preferences, then $\succ_1$ is more conservative that $\succ_2$ if, *and only if*, $\succ_1$ is more ambiguity averse and more ambiguity loving than $\succ_2$.[42] An example in Appendix C shows that the answer is negative for hope-and-prepare preferences.

[42] See Corollary 1 in Echenique et al. (2022).

6 Aggregating the opinion of experts with hope-and-prepare preferences

Numerous economic decisions under uncertainty, such as those related to fiscal policy and those addressing climate change, often hinge on the guidance provided by groups of experts, who frequently hold conflicting "opinions." We propose a simple illustration, in the context of the aggregation of conflicting opinions among experts, in which the fact that the planner's decisions are taken according to a hope-and-prepare preference relation rather than according to a Bewley one reflects her preference for decisiveness.

Due to the complexity of the issue at hand, the opinions of experts may encompass several probability distributions (scenarios) over payoff-contingent states. Following Danan et al. (2016), we assume that experts have Bewley preferences, which expresses, given a set of plausible scenarios, the need of experts to have a strong conviction in order to report to the planner (the DM) that an option is better than an other one:[43]

> "[...] a given individual may also consider more than one model to be plausible—or have an imprecise belief. For such an individual, which of two policies yields the highest expected utility may depend on the model considered. When a policy yields a higher expected utility than another one for all plausible models, we say that the individual unambiguously prefers the former policy to the latter. Unambiguous preferences are thus robust to belief imprecision."

Let $N = \{1, 2, \ldots, n\}$ be a finite set of experts. Expert $j \in N$ has a preference $\succ_j$ on $\mathcal{F}$. We use $\succ_0$ to denote the DM's preference on $\mathcal{F}$. We suppose that, for all $i \in N$, expert i's preference is a Bewley preference with unique representation (u, C_i). We thus assume in particular that there is no diversity of preferences over outcomes, which is a distinctive element of the theory of the aggregation of opinions, compared to the theory of the aggregation of preferences. We study how $\succ_0$ should depend on $(\succ_j)_{j\in N}$ and impose the two following conditions:

Axiom 10 (**Pareto**)**.** For all $f, g \in F$, if $f \succ_i g$ for all $i \in N$, then $f \succ_0 g$.

Axiom 11 (**Caution for incomparability**)**.** For all $f \in F$ and $x \in X$, if there exists $i \in N$ such that $f \bowtie_i x$, then $f \bowtie_0 x$.

The Pareto condition is the standard one. It asserts that the DM should follow the comparisons expressed by experts when they are *unanimous*: if all experts prefer act f to act g,

[43] In particular, given this set of scenarios, the condition under which they have sufficient conviction that an option is better than an other one is stronger than if they had a concordant hope-and-prepare preference relation.

the DM should also favor f over g. Caution for incomparability focuses on situations without clear comparisons. Based on the idea that the reason why the DM wants to rely on the opinions of experts to take decisions is that the issue at hand is *crucial* to her, it states that if some experts struggle to compare act f to a constant act x, while comparisons involving at least one constant act are simpler, the DM should treat these acts as incomparable.

Interestingly, the result below demonstrates that these two axioms imply that the rule for aggregating the beliefs of experts is the same when the DM adopts a Bewley criterion as when she adopts a concordant hope-and-prepare one: the set of plausible scenarios considered by the DM is the same. The relative merits of the two criteria then depend on the view of the DM as to the trade-off between conviction and decisiveness. If the DM wants to be more decisive, she should adopt the hope-and-prepare criterion. Conversely, if the DM prioritizes having stronger conviction in her decisions, the Bewley criterion, as considered by Danan et al. (2016), would be more suitable.[44]

For all $P \subseteq \Delta$, we use $\mathrm{co}(P)$ to denote the convex hull of P.

Proposition 3. *Suppose for all $i \in N$, $\succ_i$ is a Bewley preference with unique representation (u, C_i), and either of the following holds:*

- *$\succ_0$ is a hope-and-prepare preference with unique representation (u, C_0, C_0), or*
- *$\succ_0$ is a Bewley preference with unique representation (u, C_0).*

Then,

(i) Pareto is satisfied if and only if $C_0 \subseteq \mathrm{co}\left(\bigcup_{i=1}^n C_i\right)$.

(ii) Caution for incomparability is satisfied if and only if $\mathrm{co}\left(\bigcup_{i=1}^n C_i\right) \subseteq C_0$.

In particular, when both conditions are met, $C_0 = \mathrm{co}\left(\bigcup_{i=1}^n C_i\right)$.

7 Conclusion

We provided a new perspective on the analysis of incomplete preferences under uncertainty by introducing and characterizing a new decision criterion involving multiple priors —which constitutes a translation and a generalization of the domination concept at work in the notion of non-obvious manipulability. It is based on a requirement of unanimity between an

[44]This result shows in particular that a concordant hope-and-prepare preference relation, which, given a set of scenarios, is more complete than a Bewley preference relation, is still compatible with the limitation of comparability embedded in our caution axiom.

optimistic and a pessimistic evaluation reflecting the behavior of a DM who *hopes for the best while she also prepares for the worst.* When both of these evaluations are computed according to the same set of scenarios, hope-and-prepare preferences compare ranges of expected utility according to a strict version of the well known and commonly used strong set order.

Comparing hope-and-prepare preferences to the two closest incomplete criteria proposed in this framework —Bewley and twofold preferences— we argued, and made visible in our axiomatization, that the trade-off between decisiveness and conviction is addressed in a way that is more favorable to decisiveness.

We showed that an *invariant biseparable completion* of a hope-and-prepare preference relation necessarily admits a *unique asymmetric* α-maxmin representation.

Finally, we proved that the rule for aggregating the opinions of experts is, under two conditions imposed on the decisions of the DM, the same whether she has a Bewley criterion or a concordant hope-and-prepare one: then, we interpreted the adoption of one or the other criteria as a choice between the strength of her conviction and her ability to take decisions.

Appendix

A Discussion of Axiom 6

It is obvious that Axiom 5 in Echenique et al. (2022) implies our Axiom 6: twofold preferences satisfies Axiom 6. Let us prove that Bewley preferences also satisfy Axiom 6.

Let $\succ$ be a Bewley preference relation with representation (u, C). Let $x, y \in X$ and $f, g \in \mathcal{F}$ satisfying the assumptions of Axiom 6. Consider $x \in X$ such that $f \bowtie x$. By definition of Bewley preferences, there are $p, p' \in C$ such that $\int u(f)dp' \geq u(x) \geq \int u(f)dp$. Therefore, the set $\{x \in X | f \bowtie x\}$ is the set $\{x \in X | \max_{p \in C} \int u(f)dp \geq u(x) \geq \min_{p \in C} \int u(f)dp\}$. Since C is a compact set, there are $\overline{x} \in X$ and $\underline{x} \in X$ such that $u(\overline{x}) = \max_{p \in C} \int u(f)dp$ and $u(\underline{x}) = \min_{p \in C} \int u(f)dp$. Then, $f \bowtie \overline{x}$ and $f \bowtie \underline{x}$. Then, $g \bowtie \overline{x}$ and $g \bowtie \underline{x}$, which implies that there are $p_1, p_2 \in C$ such that

$$u(\underline{x}) = \min_{p \in C} \int u(f)dp \geq \int u(g)dp_1, \text{ and}$$
$$u(\overline{x}) = \max_{p \in C} \int u(f)dp \leq \int u(g)dp_2.$$

That is, $g \not\succ f$ and $f \not\succ g$, *i.e.* $g \bowtie f$, which ends the proof.

Other incomplete preferences: Basic adaptations of this simple proof lead to the conclusion that the (asymmetric part of the) criteria proposed by Nascimento and Riella (2011), Efe et al. (2012), Faro (2015), and Cusumano and Miyashita (2021) satisfy Axiom 6. Let us note that some of these criteria allow for indifference: generically denoting them by $\succsim$, we derive a representation of the associated asymmetric part, denoted $\succ$, by using the representation of $\succsim$ and defining $\succ$ by $[f \succ g]$ if and only if $[[f \succsim g]$ and not$[g \succsim f]]$ for all admissible acts $f, g \in \mathcal{F}$.

B Intermediary representation result

When $\succ$ satisfies Axioms 1-3 and Axioms 5-7, we can still define the pessimistic and optimistic relations $\succ_p$ and $\succ_o$ on $\mathcal{F}$,

$$\begin{aligned} g \succ_p f &\iff g \succ x \text{ and } x \bowtie f \text{ for some } x \in X, \\ g \succ_o f &\iff x \bowtie g \text{ and } x \succ f \text{ for some } x \in X, \end{aligned}$$

and obtain that $f \succ g$ if and only if $f \succ_p g$ *and* $f \succ_o g$.

We denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions, ensuring that $u(f) \in B_0(\Sigma)$ for any function $u : X \to \mathbb{R}$. We take the terminology used in Ghirardato et al. (2004). Accordingly, $I : B_0(\Sigma) \to \mathbb{R}$ is said to be *constant-linear* if, for all $\varphi \in B_0(\Sigma)$, $a \in \mathbb{R}_+$, and $b \in \mathbb{R}$, $I(a\varphi + b) = aI(\varphi) + b$, where, with a slight abuse of notation, we use b to denote the constant function $\phi : s \in S \mapsto b \in \mathbb{R}$. It is said *monotonic* if it weakly preserves the usual partial order of $B_0(\Sigma)$.

Theorem 4. *A binary relation $\succ$ satisfies Axioms 1-3 and Axioms 5-7 if and only if there exist*

- *a non-constant affine function $u : X \to \mathbb{R}$, unique up to positive affine transformation,*
- *a unique pair of monotonic constant linear functionals $I_p, I_o : B_0(\Sigma) \to \mathbb{R}$, with $I_p(u(h)) \leq I_o(u(h))$ for all $h \in \mathcal{F}$,*

such that, for all $f, g \in \mathcal{F}$,

$$f \succ g \iff \begin{cases} I_p(u(f)) > I_p(u(g)) \\ I_o(u(f)) > I_o(u(g)) \end{cases} .$$

The proof follows from the proof of Theorem 1 and Lemma 6, used in the proof of Theorem 3.

C Degree of conservatism and ambiguity attitudes of hope-and-prepare preferences

Let $\succ_1$ and $\succ_2$ be two hope-and-prepare preferences with representations (u, C_1, D_1) and (u, C_2, D_2), respectively. We identify conditions under which $C_2 \subseteq C_1$ and $D_2 \subseteq D_1$, but $\succ_1$ is not more conservative than $\succ_2$. Consider $f, g \in \mathcal{F}$ such that there are $s_1, s_2 \in S$ satisfying

$$\begin{cases} u(f(s_1)) > u(g(s_1)) \\ u(f(s_2)) < u(g(s_2)) \\ u(f(s)) = u(g(s)) \text{ for all } s \neq s_1, s_2. \end{cases}$$

Assume that the utility function u is such that $u(f(s_1)) = u(g(s_2)) = 1$ and $u(f(s_2)) = u(g(s_1)) = 0$. Define p_1, p_2 and p_3 as follows

$$\begin{cases} p_1(s_1) = \frac{1}{3}, p_1(s_2) = \frac{2}{3}, p_1(s) = 0 \ \forall s \neq s_1, s_2 \\ p_2(s_1) = 1, p_2(s_2) = 0, p_2(s) = 0 \ \forall s \neq s_1, s_2 \\ p_3(s_1) = \frac{2}{5}, p_3(s_2) = \frac{3}{5}, p_3(s) = 0 \ \forall s \neq s_1, s_2. \end{cases}$$

Now let $C_1 = C_2 = \{p_2\}$, $D_1 = \text{co}(\{p_1, p_2\})$ and $D_2 = \text{co}(\{p_1, p_3\})$, where co denotes the operator taht associates with any subset of Δ its convex hull in Δ. One readily obtains:

$$\min_{p \in C_1} \int u(f) dp = 1 > 0 = \min_{p \in C_1} \int u(g) dp,$$
$$\max_{p \in D_1} \int u(f) dp = 1 > \frac{2}{3} = \max_{p \in D_1} \int u(g) dp,$$
$$\max_{p \in D_2} \int u(f) dp = \frac{2}{5} < \frac{2}{3} = \max_{p \in D_2} \int u(g) dp,$$

that is, $f \succ_1 g$ but $f \not\succ_2 g$.

D Proofs

D.1 Proof of Theorem 1

Only-if part. Assume that $\succ$ satisfies Axioms 1-7.

Consider the restriction of $\succ$ to X and, for all $x, y \in X$, define $\gtrsim$ by $x \gtrsim y$ if and only if $y \not\succ x$. Clearly, $\gtrsim$ is complete and transitive on X; and $\bowtie$ is equivalent to $\sim$ on X. By Axiom 3, for all $x, y, z \in X$, $x \gtrsim y$ if and only if $\alpha x + (1-\alpha)z \gtrsim \alpha y + (1-\alpha)z$.

Thus, by continuity of $\succ$, there exists an affine function $u : X \to \mathbb{R}$, unique up to affine transformation, such that $x \succsim y$ if and only if $u(x) \geq u(y)$. Also, u is non-constant as $\succ$ is non-trivial.

Let us now introduce intermediary results on which our proof is based.

Lemma 1. *For all $f \in \mathcal{F}$, and $x, y, z \in X$, if $f \bowtie x$, and $f \succ y$, and $z \succ f$, then $z \succ x \succ y$.*

Proof. We prove $x \succ y$, as $z \succ x$ is similarly shown. Assume $x \not\succ y$, by contradiction. Having $y \succ x$ would contradict $f \bowtie x$ by the transitivity of $\succ$. Thus, $y \sim x$. There are three possibilities:

Case 1: There exists $x' \in X$ such that $f \bowtie x'$ and $x \succ x'$. Then $y \succ x'$ since $y \sim x$. So $f \succ y$, and $y \succ x'$ which implies $f \succ x'$, a contradiction.

Case 2: There exists $x' \in X$ such that $f \bowtie x'$ and $x' \succ x$. Then, $f \bowtie x'$, $x \sim y$, $f \succ y$, and $x' \succ x$. Applying Axiom 7, one gets $f \succ x$, a contradiction.

Case 3: For all $x' \in X$ such that $f \bowtie x'$, $x' \sim x$. Applying Axiom 6 to f and y, one gets $f \bowtie y$, a contradiction. □

Lemma 2. *$\bowtie$ satisfies certainty independence.*

Proof. Let $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0, 1)$, the following equivalence relations hold:

$$\begin{aligned} & f \bowtie g \\ \iff & f \not\succ g \text{ and } g \not\succ f \\ \iff & \alpha f + (1-\alpha)x \not\succ \alpha g + (1-\alpha)x \text{ and } \alpha g + (1-\alpha)x \not\succ \alpha f + (1-\alpha)x \\ \iff & \alpha f + (1-\alpha)x \bowtie \alpha g + (1-\alpha)x. \end{aligned}$$

The first and the third ones follow from the definition of $\bowtie$, and the second from the fact that $\succ$ satisfies certainty independence (Axiom 3). □

Lemma 3. *For all $f \in \mathcal{F}$ and $x, y \in X$, if $f \succ x$ and $x \succsim y$, then $f \succ y$.*

Proof. Let $f \in \mathcal{F}$ and $x, y \in X$ such that $f \succ x$ and $x \succsim y$. By contradiction, assume that $f \not\succ y$, then either $y \succ f$ or $y \bowtie f$. If $y \succ f$, then $y \succ x$ by transitivity, which contradicts the assumption that $x \succsim y$. If $y \bowtie f$, it follows from Lemma 1 that $y \succ x$; a contradiction. □

Lemma 4. *For all $f, g \in \mathcal{F}$ with $f(s) \succsim g(s)$ for all $s \in S$, and for all $x \in X$, if $x \bowtie f$, then $g \not\succ x$; and if $x \bowtie g$, then $x \not\succ f$.*

Proof. Suppose that $f(s) \succsim g(s)$ for all $s \in S$, and suppose by contradiction that there is $x \in X$ such that $x \bowtie f$ and $g \succ x$.[45]

[45] The conclusion that $x \not\succ f$ when $x \bowtie g$ follows easily from the same argument by contradiction.

As $\succ$ is non-trivial, there are y and z in X such that $y \succ z$. From Axiom 3 and Lemma 2, $\alpha f(s) + (1-\alpha)z \succsim \alpha g(s) + (1-\alpha)z$ for all $s \in S$ and all $\alpha \in (0,1)$. Let $f^\alpha = \alpha f + (1-\alpha)z$, $g^\alpha = \alpha g + (1-\alpha)z$, and $x^\alpha = \alpha x + (1-\alpha)z$. Note that $f^\alpha(s) \succsim g^\alpha(s)$ for all $s \in S$. Axiom 3 and Lemma 2 imply, for all $\alpha \in (0,1)$,

$$x \bowtie f \iff x^\alpha \bowtie f^\alpha$$
$$g \succ x \iff g^\alpha \succ x^\alpha.$$

Besides, Axiom 2 guarantees that for α close enough to 0, $y \succ g^\alpha(s)$ for all $s \in S$. Now, fix $\alpha \in (0,1)$ such that $y \succ g^\alpha(s)$ for all $s \in S$ and define, for all $\beta \in (0,1)$, $f_\beta \in \mathcal{F}$ by $f_\beta(s) = \beta f^\alpha(s) + (1-\beta)y$ for all $s \in S$. As $u(f^\alpha(s)) \geq u(g^\alpha(s))$ and $u(y) > u(g^\alpha(s))$, $f_\beta(s) \succ g^\alpha(s)$ for all $s \in S$. In addition, by Lemma 2, $x^\alpha \bowtie f^\alpha$ implies $\beta x^\alpha + (1-\beta)y \bowtie f_\beta$. Then, by Axiom 5, $g^\alpha \not\succ \beta x^\alpha + (1-\beta)y$ for all $\beta \in (0,1)$. However, as $g^\alpha \succ x^\alpha$, if β is close enough to 1, Axiom 2 implies that $g^\alpha \succ \beta x^\alpha + (1-\beta)y$, a contradiction. □

Lemma 5. *For all $f \in \mathcal{F}$, the set $\{x \in X : x \bowtie f\}$ is non-empty.*

Proof. By definition of a simple act, for all $f \in \mathcal{F}$, there are x^* and x_* in X such that $x^* \succsim f(s) \succsim x_*$ for all $s \in S$. Since $f(s) \succsim x_*$ for all $s \in S$ and $x_* \bowtie x_*$, Lemma 4 implies that $x_* \not\succ f$. One obtains similarly that $f \not\succ x^*$. Consider the sets $\{\alpha \in [0,1] : f \not\succ \alpha x^* + (1-\alpha)x_*\}$ and $\{\alpha \in [0,1] : \alpha x^* + (1-\alpha)x_* \not\succ f\}$ which are non-empty and closed relative to $[0,1]$ by the continuity of $\succ$. Clearly, their union is $[0,1]$, and the connectedness of $[0,1]$ in turn implies that their intersection is non-empty: there is $\alpha^* \in [0,1]$ such that $\alpha^* x^* + (1-\alpha^*)x_* \bowtie f$. □

From the original relation $\succ$, we define two preference relations as follows:

$$f \succ_p g \iff f \succ x \text{ and } x \bowtie g \text{ for some } x \in X,$$
$$f \succ_o g \iff f \bowtie x \text{ and } x \succ g \text{ for some } x \in X.$$

Step 1. *$\succ_p$ and $\succ_o$ are asymmetric and negatively transitive.*

We only prove that $\succ_p$ has these properties, as the argument for $\succ_o$ is similar.

Assume by contradiction $f \succ_p g$ and $g \succ_p f$ for some $f, g \in \mathcal{F}$. That is, there are some $x, y \in X$ such that $f \succ x$, $x \bowtie g$, $g \succ y$, and $y \bowtie f$. By Lemma 1, one concludes that $y \succ x$ since $f \succ x$ and $y \bowtie f$ and $x \succ y$ as $g \succ y$ and $x \bowtie g$. This is impossible since $\succ$ is asymmetric. As a consequence, $\succ_p$ is asymmetric.

Now, assume by contradiction that for some $f, g, h \in \mathcal{F}$ $f \not\succ_p g$, $g \not\succ_p h$, and $f \succ_p h$. By definition of $\succ_p$, there is $x \in X$ such that $f \succ x$ and $x \bowtie h$. Since $g \not\succ_p h$, the following holds: $f \succ x \succ g$. Let $y \in X$ such that $g \bowtie y$. From Lemma 1, $x \succ y$, implying $f \succ y$. But then, $f \succ_p g$, which is a contradiction. Therefore, $\succ_p$ is negatively transitive.

Step 2. *$f \succ g$ if and only if $f \succ_p g$ and $f \succ_o g$.*

Let us first prove that for $f, g \in \mathcal{F}$ such that $f \succ g$, one has $f \succ_p g$ and $f \succ_o g$, giving the explicit argument exclusively for $f \succ_p g$, as $f \succ_o g$ is proved symmetrically. By contradiction, assume that $f \not\succ_p g$, then for all $x \bowtie g$, one has $f \not\succ x$, that is, either $f \bowtie x$ or $x \succ f$. But if $x \succ f$, then $x \succ g$ by transitivity, which contradicts $x \bowtie g$. Thus, for all $x \in X$, if $x \bowtie g$, then $x \bowtie f$. By Axiom 6, $g \bowtie f$, a contradiction. We have thus proved $f \succ_p g$.

Suppose now $f \succ_p g$ and $f \succ_o g$, and let us show $f \succ g$. By definition of $\succ_p$ and $\succ_o$, there exist $x \in X$ such that $f \succ x$ and $x \bowtie g$, and $y \in X$ such that $y \bowtie f$ and $y \succ g$. Axiom 7 then implies $f \succ g$.

Define $\sim_p$ by $f \sim_p g$ if and only if $f \not\succ_p g$ and $g \not\succ_p f$, for all $f, g \in \mathcal{F}$, and define $\succsim_p$ by $f \succsim_p g$ if and only if either $f \succ_p g$ or $f \sim_p g$, for all $f, g \in \mathcal{F}$. The relations $\sim_o$ and $\succsim_o$ are similarly defined. It is clear that $\succsim_p$ and $\succsim_o$ are complete and transitive. We say that $\succsim_p$ (resp. $\succsim_o$) is continuous if $\succ_p$ (resp. $\succ_o$) is continuous, which is equivalent to the closedness of $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim_p h\}$ and $\{\alpha \in [0,1] : h \succsim_p \alpha f + (1-\alpha)g\}$.

Step 3. *$\succsim_p$ and $\succsim_o$ are continuous and satisfy monotonicity and certainty independence.*[46]

We only provide the proof that $\succsim_p$ is continuous and satisfies monotonicity and certainty independence, where monotonicity, when allowing for indifference, means that for all $f, g \in \mathcal{F}$ such that $f(s) \succsim_p g(s)$ for all $s \in S$, $f \succsim_p g$, and certainty independence means that for all $f, g \in \mathcal{F}$, all $x \in X$, and all $\alpha \in (0,1)$, $f \succsim_p g$ if and only if $\alpha f + (1-\alpha)x \succsim_p \alpha g + (1-\alpha)x$.

We first show that $\succsim_p$ is continuous. Let $f, g, h \in \mathcal{F}$ and $x \in X$; denote A_x the set of $\alpha \in [0,1]$ such that $\alpha f + (1-\alpha)g \succ x$ and $x \bowtie h$. Either A_x is empty or it coincides with $\{\lambda \in [0,1] : \lambda f + (1-\lambda)g \succ x\}$. Then A_x is open by Axiom 2. Therefore, $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succ_p h\} = \cup_{x \in X} A_x$ is open. Similarly, one can show that $\{\alpha \in [0,1] : h \succ_p \alpha f + (1-\alpha)g\}$ is open; thus, $\succsim_p$ is continuous.

Next, we prove that $\succsim_p$ satisfies monotonicity. Let $f, g \in \mathcal{F}$ such that $f(s) \succsim_p g(s)$ for all $s \in S$, which clearly implies $f(s) \succsim g(s)$ for all $s \in S$. Suppose $g \succ_p f$, which means that there exists $x \in X$ such that $g \succ x$ and $x \bowtie f$. This is a direct contradiction as, by Lemma 4, for any $x' \in X$ such that $x' \bowtie f$, $g \not\succ x'$. Thus, $f \succsim_p g$.

[46]The definition of these properties for a weak order are reminded in the following lines.

Lastly, we establish that $\succsim_p$ satisfies certainty independence. Let $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$. We first show that $f \succsim_p g$ implies $\alpha f + (1-\alpha)x \succsim_p \alpha g + (1-\alpha)x$. Since $\succsim_p$ is a weak order, $f \succsim_p g$ is equivalent to $g \not\succ_p f$, which holds if, and only if, for all $y \in X$ such that $g \succ y$, f and y are comparable. Under Axiom 3, it is sufficient to prove that, for all $y \in X$ such that $\alpha g + (1-\alpha)x \succ y$, $\alpha f + (1-\alpha)x$ and y are comparable. Let $y \in X$ such that $\alpha g + (1-\alpha)x \succ y$ and suppose by contradiction $\alpha f + (1-\alpha)x \bowtie y$.

Claim: For such $y \in X$, there is $\underline{z} \in \{z : z \bowtie f\}$ with $u(\underline{z}) = \inf\{u(z) : z \bowtie f\}$ such that $y \succsim \alpha \underline{z} + (1-\alpha)x$.

We have shown in Step 2 that there exists $\underline{z} \in \{z \in X : z \bowtie f\}$ such that $u(\underline{z}) = \inf\{u(z) : z \bowtie f\}$. Since $\bowtie$ satisfies certainty independence (Lemma 2), $\alpha f + (1-\alpha)x \bowtie \alpha \underline{z} + (1-\alpha)x$.

We claim $y \succsim \alpha \underline{z} + (1-\alpha)x$. Indeed, if there exists $z \in X$ such that $\underline{z} \succ z$, then it follows from Axiom 3 and Lemma 1 that $f \succ z_\beta := \beta \underline{z} + (1-\beta)z$ for all $\beta \in (0,1)$. Using Axiom 3 again yields $\alpha f + (1-\alpha)x \succ \alpha z_\beta + (1-\alpha)x$. It then follows from Lemma 1 that $y \succ \alpha z_\beta + (1-\alpha)x$ for all $\beta \in (0,1)$. Letting β tend to 1, one concludes, since u is affine, that $y \succsim \alpha \underline{z} + (1-\alpha)x$. If $z \succsim \underline{z}$ for all $z \in X$, then $\alpha f(s) + (1-\alpha)x \succsim \alpha \underline{z} + (1-\alpha)x$ for all $s \in S$ (otherwise, Axiom 3 implies $\underline{z} \succ f(s)$, which is a contradiction). Since $\alpha f + (1-\alpha)x \bowtie y$, Lemma 4 implies $\alpha \underline{z} + (1-\alpha)x \not\succ y$, which is equivalent to $y \succsim \alpha \underline{z} + (1-\alpha)x$.

Since $\alpha g + (1-\alpha)x \succ y$ and $y \succsim \alpha \underline{z} + (1-\alpha)x$, Lemma 3 implies

$$\alpha g + (1-\alpha)x \succ \alpha \underline{z} + (1-\alpha)x,$$

which is equivalent to $g \succ \underline{z}$ by Axiom 3. Hence, by definition of $\underline{z}$, $g \succ_p f$, a contradiction. Therefore, $\alpha f + (1-\alpha)x$ and y are comparable, which yields $\alpha f + (1-\alpha)x \succsim_p \alpha g + (1-\alpha)x$.

We now show the converse implication. For $\alpha \in (0,1)$, and any two $f, g \in \mathcal{F}$, $\alpha f + (1-\alpha)x \succsim_p \alpha g + (1-\alpha)x$ if, and only if, for all $y \in X$ such that $\alpha g + (1-\alpha)x \succ y$, $\alpha f + (1-\alpha)x$ and y are comparable. Let $y \in X$ such that $g \succ y$, then one has $\alpha g + (1-\alpha)x \succ \alpha y + (1-\alpha)x$ by Axiom 3. Thus, $\alpha f + (1-\alpha)x$ and $\alpha y + (1-\alpha)x$ are comparable, implying that f and y are comparable by Axiom 3. Hence, $g \not\succ_p f$, which is equivalent to $f \succsim_p g$. We have proved that $\succsim_p$ satisfies certainty independence.

Step 4. *An agent with preferences $\succsim_p$ on $\mathcal{F}$ is averse to ambiguity, i.e., for all $f, g \in \mathcal{F}$, $f \sim_p g$ implies $\alpha f + (1-\alpha)g \succsim_p f$. An agent with preferences $\succsim_o$ on $\mathcal{F}$ loves ambiguity, i.e., for all $f, g \in \mathcal{F}$, $f \sim_p g$ implies $f \succsim_o \alpha f + (1-\alpha)g$.*

We only prove that $\succsim_p$ displays ambiguity aversion. Let $f, g \in \mathcal{F}$ such that $f \sim_p g$, *i.e.*, $f \not\succ_p g$ and $g \not\succ_p f$. In other words, for all $x \in X$ with $f \succ x$, x is comparable with g, and for all $x \in X$ with $g \succ x$, x is comparable with f. Let $x \in X$ such that $f \succ x$.

If $x \succ g$, then $f \succ g$, and then, by Step 2, $f \succ_p g$, which is a contradiction; thus, one must have $g \succ x$. This implies $\{x \in X : f \succ x\} \subseteq \{x \in X : g \succ x\}$. Analogously, $\{x \in X : g \succ x\} \subseteq \{x \in X : f \succ x\}$; therefore, $\{x \in X : f \succ x\} = \{x \in X : g \succ x\}$.

Let $\alpha \in (0,1)$, we claim that $\alpha f + (1-\alpha)g \succsim_p f$. Since $\succsim_p$ is a weak order, it is sufficient to prove $f \not\succ_p \alpha f + (1-\alpha)g$, which holds if, for all $x \in X$ such that $f \succ x$, $\alpha f + (1-\alpha)g \succ x$. Yet, we have just proved that $f \succ x$ if and only if $g \succ x$. Axiom 4 then directly entails $\alpha f + (1-\alpha)g \succ x$; which concludes.

Conclusion. It is well-known since Gilboa and Schmeidler (1989) that a weak order defined on $\mathcal{F}$ satisfying the properties stated in Step 3 can be represented by $f \mapsto \min_{p \in C} \int u_p(f)dp$ if it displays ambiguity aversion, such as $\succsim_p$, and by $f \mapsto \max_{p \in D} \int u_o(f)dp$ if it displays love for ambiguity, such as $\succsim_o$, where C and D are unique non-empty convex compact subsets of Δ, u_p and u_o are two affine functions on X, unique up to positive affine transformation. Clearly, for all $x, y \in X$, $x \succsim_p y$ if and only if $x \succsim y$, and $x \succsim_o y$ if and only if $x \succsim y$. Thus, u_p and u_o are positive affine transformations of u, and one may assume that $u_p = u_o = u$. Finally, it remains to prove that $C \cap D \neq \emptyset$.

Claim: C and D are non-disjoint if, and only if, for all $f \in \mathcal{F}$, $\min_{p \in C} \int u(f)dp \leq \max_{p \in D} \int u(f)dp$.

We only prove the *if part*, the other direction being trivial. We proceed by contraposition. Suppose that $C \cap D = \emptyset$. By the separating hyperplane theorem, there exists a bounded measurable function $\varphi : S \to \mathbb{R}$ such that $\min_{p \in C} \int \varphi dp > \max_{p \in D} \int \varphi dp$. Yet, there exists a sequence of simple functions $\{\varphi_n\}$ that converges (in supnorm topology) to φ. Since both $\tilde{\varphi} \mapsto \min_{p \in C} \int \tilde{\varphi} dp$ and $\tilde{\varphi} \mapsto \max_{p \in D} \int \tilde{\varphi} dp$ are continuous, there is $n \in \mathbb{N}$ such that $\min_{p \in C} \int \varphi_n dp > \max_{p \in D} \int \varphi_n dp$. As $a\varphi_n + b$ also satisfies this last inequality for all $a > 0$ and $b \in \mathbb{R}$, one can choose $a > 0$ and $b \in \mathbb{R}$ such that $a\varphi_n(s) + b \in u(X)$ for all $s \in S$, which implies $\varphi_n = u(f)$ for some $f \in \mathcal{F}$:

$$\min_{p \in C} \int u(f)dp > \max_{p \in D} \int u(f)dp.$$

As a consequence, the fact that, for all $f \in \mathcal{F}$, $\min_{p \in C} \int u(f)dp \leq \max_{p \in D} \int u(f)dp$, implies $C \cap D \neq \emptyset$.

Based on this claim, it remains to prove that $\min_{p \in C} \int u(f)dp \leq \max_{p \in D} \int u(f)dp$ for all $f \in \mathcal{F}$ in order to conclude that C and D are not disjoint.

Let us show that the inequality $\min_{p \in C} \int u(f)dp \leq \max_{p \in D} \int u(f)dp$ holds for all $f \in \mathcal{F}$ if and only if, for all $x \in X$, for all $f \in \mathcal{F}$, $f \succ_p x$ implies $f \succ_o x$.

Suppose that for all $x \in X$, for all $f \in \mathcal{F}$, $f \succ_p x$ implies $f \succ_o x$. Suppose, by contradiction, that there is $f \in \mathcal{F}$ such that $\min_{p\in C}\int u(f)dp > \max_{p\in D}\int u(f)dp$. Clearly, one has $u(x_*) \leq \min_{p\in C}\int u(f)dp \leq u(x^*)$, where x_* and x^* are defined as in the proof of Lemma 5. Then, since $u(X)$ is convex, $\min_{p\in C}\int u(f)dp$ belongs to $u(X)$. Similarly, one can deduce that $\max_{p\in D}\int u(f)dp$ lies in $u(X)$. Then, the convexity of $u(X)$ implies that there exists $x \in X$ such that

$$\min_{p\in C}\int u(f)dp > u(x) > \max_{p\in D}\int u(f)dp,$$

which is a contradiction as it implies, as $\min_{p\in C}\int u(x)dp = \max_{p\in D}\int u(x)dp = u(x)$, $f \succ_p x$ and $x \succ_o f$. The other direction of the equivalence is trivial.

It remains to show that, indeed, $f \succ_p x$ implies $\succ_o x$, for all $x \in X$, and all $f \in \mathcal{F}$. Yet, $f \succ_p x$ implies $f \succ x$. Indeed, $f \succ_p x$ if and only if there exists $y \in X$ such that $f \succ y$ and $y \bowtie x$; then Lemma 3 implies $f \succ x$. By Step 2, we conclude that $f \succ_o x$.

We have thus proved that $\min_{p\in C}\int u(f)dp \leq \max_{p\in D}\int u(f)dp$ for all $f \in \mathcal{F}$, and, thus, that C and D are non-disjoint.

If part. Assume that $\succ$ admits a hope-and-prepare representation. One can readily check that Axioms 1 to 5 are satisfied.

For all $f \in \mathcal{F}$, denote $\overline{p}_f \in \arg\max_{p\in D}\int u(f)dp$ and $\underline{p}_f \in \arg\min_{p\in C}\int u(f)dp$. Define also the constant acts $\overline{f} = \int f d\overline{p}_f$ and $\underline{f} = \int f d\underline{p}_f$. Clearly, $f \bowtie \overline{f}$ and $f \bowtie \underline{f}$; moreover,

$$\begin{cases} f \succ x & \iff u(\underline{f}) > u(x), \\ x \succ f & \iff u(x) > u(\overline{f}), \\ f \bowtie x & \iff u(\overline{f}) \geq u(x) \geq u(\underline{f}) \end{cases}. \tag{2}$$

We prove that Axiom 6 is verified by contradiction. Consider $f, g \in \mathcal{F}$ such that for all $x \in X$, $f \bowtie x$ implies $g \bowtie x$. If $f \succ g$, then $u(\overline{f}) > u(\overline{g})$. However, by assumption, $g \bowtie \overline{f}$, which implies $u(\overline{g}) \geq u(\overline{f}) \geq u(\underline{g})$, a contradiction. The same argument applies to prove that $g \succ f$ cannot hold. Therefore, $f \bowtie g$.

Axiom 7 easily obtains from the comparisons in (2). Indeed, let $f, g \in \mathcal{F}$ and $x, y \in X$ such that $f \bowtie x$, $g \bowtie y$, $x \succ g$, and $f \succ y$. Using (2), one gets

$$\begin{cases} u(\overline{f}) \geq u(x) \geq u(\underline{f}), \\ u(\overline{g}) \geq u(y) \geq u(\underline{g}), \\ u(x) > u(\overline{g}), u(\underline{f}) > u(y) \end{cases}. \tag{3}$$

Then $u(\overline{f}) \geq u(x) > u(\overline{g})$ and $u(\underline{f}) > u(y) \geq u(\underline{g})$, that is $f \succ g$, by definition of a hope-and-prepare preference.

D.2 Proof of Theorem 2

By assumption,

$$f \succ g \iff \begin{cases} \min_{p\in C} \int u(f)dp > \min_{p\in C} \int u(g)dp \\ \max_{p\in D} \int u(f)dp > \max_{p\in D} \int u(g)dp \end{cases},$$

where u is an affine function defined on X, unique up to affine transformation, C and D are two unique compact and convex subsets of Δ with $C \cap D \neq \emptyset$. It remains to prove that $\succ$ admitting such a representation satisfies Axiom 8 and 9 if and only if $C = D$. We show in a very similar way to Echenique et al. (2022) that it satisfies Axiom 8 if and only if $D \subseteq C$ —the other inclusion being equivalent to Axiom 9 is shown in a symmetric way.

Only-if part. Suppose by contraposition that $D \nsubseteq C$: there is some $p^* \in D$ such that $p^* \notin C$. Then, by the separating hyperplane theorem and the argument given in the Conclusion step of the proof of Theorem 1, there is an act ψ and $k \in \mathbb{R}$ such that

$$\min_{p\in C} \int u(\psi)dp > k > \int u(\psi)dp^*. \tag{4}$$

By scaling ψ and k appropriately, as u is affine, one can find $f, h \in \mathcal{F}$ and $x \in X$ such that $u(f) = \frac{1}{2}u(\psi), u(h) = -u(\psi)$ and $u(x) = 2k$.[47] Let $g = \frac{1}{2}h + \frac{1}{2}x$:

$$u\Big(\frac{1}{2}f + \frac{1}{2}g\Big) = \frac{1}{4}u(\psi) + \frac{1}{2}\left(-\frac{1}{2}u(\psi) + k\right) = \frac{k}{2},$$

that is, f and g are complementary, and $\frac{1}{2}f(s) + \frac{1}{2}g(s) \sim y$ for some $y \in X$ such that $u(y) = \frac{k}{2}$, for all $s \in S$. Since $u(f) = \frac{1}{2}u(\psi)$, Equation (4) implies

$$\min_{p\in C} \int u(f)\mathrm{d}p = \frac{1}{2} \min_{p\in C} \int u(\psi)\mathrm{d}p > \frac{k}{2} = u(y).$$

In addition, as $D \cap C \neq \emptyset$, $\max_{p\in D} \int u(f)\mathrm{d}p \geq \min_{p\in C} \int u(f)\mathrm{d}p > u(y)$. As a consequence, $f \succ y$.

[47] We abuse notation in a standard way when writing $u(f) = t$, for $t \in \mathbb{R}$, to actually denote $u(f(s)) = t$ for all $s \in S$.

Futhermore, $u(g) = u\left(\frac{1}{2}h + \frac{1}{2}x\right) = -\frac{1}{2}u(\psi) + k$. Since $p^* \in D$, Equation (4) implies

$$\max_{p\in D} \int u(g)\mathrm{d}p \geq \int u(g)\mathrm{d}p^* = -\frac{1}{2}\int u(\psi)dp^* + k > -\frac{k}{2} + k = \frac{k}{2} = u(y),$$

from which $y \not\succ g$. One thus has $\frac{1}{2}f(s) + \frac{1}{2}g(s) \sim y$ for all $s \in S$, $f \succ y$, and $y \not\succ g$, which is a violation of Axiom 8.

If part. Suppose that $D \subseteq C$. Consider two complementary acts $f, g \in \mathcal{F}$ such that $\frac{1}{2}f(s) + \frac{1}{2}g(s) \sim x$ for some $x \in X$, for all $s \in S$, or, $\frac{1}{2}u(f) + \frac{1}{2}u(g) = k$, with $u(x) = k$. Assume $f \succ x$, which is is equivalent to

$$\begin{cases} \min_{p\in C} \int u(f)dp > k \\ \max_{p\in D} \int u(f)dp > k \end{cases} \iff \begin{cases} \frac{1}{2}\min_{p\in C} \int u(f) - u(g)dp > 0 \\ \frac{1}{2}\max_{p\in D} \int u(f) - u(g)dp > 0 \end{cases}$$

$$\iff \begin{cases} \frac{1}{2}\max_{p\in C} \int u(g) - u(f)dp < 0 \\ \frac{1}{2}\min_{p\in D} \int u(g) - u(f)dp < 0 \end{cases}.$$

Since $D \subseteq C$, the last inequalities yield

$$\begin{cases} \frac{1}{2}\max_{p\in D} \int u(g) - u(f)dp < 0 \\ \frac{1}{2}\min_{p\in C} \int u(g) - u(f)dp < 0 \end{cases}.$$

Plugging $u(f) = 2k - u(g)$, one obtains

$$\begin{cases} 2\max_{p\in D} \int u(g) - kdp < 0 \\ 2\min_{p\in C} \int u(g) - kdp < 0 \end{cases} \iff \begin{cases} \max_{p\in D} \int u(g)dp < k \\ \min_{p\in C} \int u(g)dp < k \end{cases}.$$

As $k = u(x)$, this means $x \succ g$. Therefore, $\succ$ satisfies Axiom 8.

D.3 Proof of Proposition 1

i) Let $\succ_H$ be a hope-and-prepare preference with unique representation (u, C_H, D_H), and let $\succ_B$ be Bewley preference with unique representation (u, C_B).

First, suppose that $C_H \cup D_H \subseteq C_B$. If $f \succ_B g$, then for all $p \in C_H \cup D_H$,

$$\int u(f)dp > \int u(g)dp,$$

which implies, as C_H and D_H are not disjoint,

$$\begin{cases} \min_{p\in C_H} \int u(f)dp > \min_{p\in C_H} \int u(g)dp, \\ \max_{p\in D_H} \int u(f)dp > \max_{p\in D_H} \int u(g)dp. \end{cases}$$

Therefore, $f \succ_H g$. Thus, $\succ_B$ is more conservative than $\succ_H$.

Conversely, suppose $\succ_B$ is more conservative than $\succ_H$ and suppose, by contradiction, that there exists $p^* \in C_H \setminus C_B$. By the separation argument we already used in the Conclusion step of the proof of Theorem 1, there are $f \in \mathcal{F}$ and $x \in X$ such that

$$\int u(f)dp^* > u(x) > \max_{p\in C_B} \int u(f)dp.$$

It follows that $x \succ_B f$ but $x \not\succ_H f$, a contradiction. Similarly, suppose there exists $p^* \in D_H \setminus C_B$. Then there are $f \in \mathcal{F}$ and $x \in X$ such that

$$\min_{p\in C_B} \int u(f)dp > u(x) > \int u(f)dp^*.$$

In this case, we have $f \succ_B x$ but $f \not\succ_H x$, an other contradiction. Therefore, $C_H \cup D_H \subseteq C_B$.

ii) Let $\succ_H$ be a hope-and-prepare preference with unique representation (u, C_H, D_H), and $\succ_T$ be a twofold multiprior preference with unique representation (u, C_T, D_T).

First, suppose that $C_H \subseteq C_T$ and $D_H \subseteq D_T$. Since $D_T \cap C_T \neq \emptyset$ and $D_H \cap C_H \neq \emptyset$, $C_H \cap D_T \neq \emptyset$ and $D_H \cap C_T \neq \emptyset$. If $f \succ_T g$, then

$$\min_{p\in C_T} \int u(f)dp > \max_{p\in D_T} \int u(g)dp,$$

which implies

$$\min_{p\in C_H} \int u(f)dp \geq \min_{p\in C_T} \int u(f)dp > \max_{p\in D_T} \int u(g)dp \geq \max_{p\in D_H} \int u(g)dp,$$
$$\max_{p\in D_H} \int u(f)dp \geq \min_{p\in C_T} \int u(f)dp > \max_{p\in D_T} \int u(g)dp \geq \max_{p\in D_H} \int u(g)dp.$$

Since $D_H \cap C_H \neq \emptyset$, one gets

$$\max_{p\in D_H} \int u(f)dp \geq \min_{p\in C_H} \int u(f)dp > \max_{p\in D_H} \int u(g)dp \geq \min_{p\in C_H} \int u(g)dp.$$

Therefore, $f \succ_H g$. Thus, $\succ_T$ is more conservative than $\succ_H$.

Conversely, suppose $\succ_T$ is more conservative than $\succ_H$ and suppose, by contradiction, that there exists $p^* \in C_H \setminus C_T$. There are $f \in \mathcal{F}$ and $x \in X$ such that

$$\min_{p \in C_T} \int u(f)dp > u(x) > \int u(f)dp^*,$$

from which it follows that $f \succ_T x$ but $f \not\succ_H x$, a contradiction. To prove that $D_H \subseteq D_T$, suppose there exists $p^* \in D_H \setminus D_T$. There are $f \in \mathcal{F}$ and $x \in X$ such that

$$\int u(f)dp^* > u(x) > \max_{p \in D_T} \int u(f)dp.$$

In this case, $x \succ_T f$ but $x \not\succ_H f$, an other contradiction.

D.4 Proof of Proposition 2

Clearly, for each $i \in \{1, 2\}$, and all $x \in X$, $f \succ_i x$ if and only if $f \succ_{ip} x$, where $\succ_{ip}$ is the pessimistic relation defined, as in the proof of Theorem 1, by $f \succ_{ip} g$ if and only if $f \succ_i y$ and $y \bowtie_i g$ for some $y \in X$. Thus, $\succ_1$ is more ambiguity averse than $\succ_2$ if and only if $\succ_{1p}$ is more ambiguity averse than $\succ_{2p}$. When proving Theorem 1, we have shown that $\succ_{ip}$ is represented by a maxmin expected utility functional; therefore, $\succ_{1p}$ is more ambiguity averse than $\succ_{2p}$ if and only if $C_2 \subseteq C_1$.

Similarly, for each $i \in \{1, 2\}$, and all $x \in X$, $x \succ_i f$ if and only if $x \succ_{io} f$, where $\succ_{io}$ is the optimistic relation defined, as in the proof of Theorem 1, by $f \succ_{io} g$ if and only if $f \bowtie_i y$ and $y \succ_i g$ for some $y \in X$. As we have proved that $\succ_{io}$ admits a maxmax expected utility representation, one obtains that $\succ_1$ is more ambiguity loving than $\succ_2$ if and only if $D_2 \subseteq D_1$.

D.5 Proof of Theorem 3

We will only prove that (i) implies (ii), the inverse implication being routine.

Lemma 6. *A weak order relation $\succ$ on $\mathcal{F}$ satisfies Axioms 2, 3 and 5 if and only if there exists a monotonic, constant-linear functional $I : B_0(\Sigma) \to \mathbb{R}$ and a non-constant affine function $u : X \to \mathbb{R}$ such that, for all $f, g \in \mathcal{F}$,*

$$f \succ g \iff I(u(f)) > I(u(g)).$$

Moreover, I is unique and u is unique up to positive affine transformation.

Proof. As before, define $\succsim$ by $f \succsim g$ if and only if $g \not\succ f$ for all $f, g \in \mathcal{F}$. Clearly, $\succsim$ is complete and transitive, and $\bowtie$ is an equivalence relation (see Theorem 2.1 in Fishburn (1970)). The weak order $\succsim$ is continuous if, for all $f, g, h \in \mathcal{F}$, $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\}$ are closed. Clearly, $\succsim$ is continuous and non-trivial. It is monotone if and only if, for all $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$. Since Lemma 4 holds, in particular, for a weak order of which the asymmetric part satisfies Axioms 2, 3 and 5, and since $\bowtie$ is an equivalence relation, $\succsim$ is monotone.

Now, we check that that $\succsim$ satisfies certainty independence: for all $f, g \in \mathcal{F}$ and $x \in X$,

$$
\begin{aligned}
f \succsim g &\iff g \not\succ f \\
&\iff \alpha g + (1-\alpha)x \not\succ \alpha f + (1-\alpha)x \\
&\iff \alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x.
\end{aligned}
$$

As a consequence, by Lemma 1 in Ghirardato et al. (2004),[48] there exists a monotonic, constant-linear functional $I : B_0(\Sigma) \to \mathbb{R}$ and a non-constant affine function $u : X \to \mathbb{R}$ such that, for all $f, g \in \mathcal{F}$,

$$
f \succsim g \iff I(u(f)) \geq I(u(g)).
$$

Moreover, I is unique and u is unique up to positive affine transformation. □

Lemma 7. *Suppose that $I, I', I'' : B_0(\Sigma) \to \mathbb{R}$ are monotonic and constant-linear with $I' \leq I''$. Then the following statements are equivalent:*

(i) For all $\phi, \varphi \in B_0(\Sigma)$, if $I'(\phi) > I'(\varphi)$ and $I''(\phi) > I''(\varphi)$, then $I(\phi) > I(\varphi)$.

(ii) There exists $\alpha \in [0,1]$ such that, for all $\varphi \in B_0(\Sigma)$, $I(\varphi) = \alpha I'(\varphi) + (1-\alpha)I''(\varphi)$.

Proof. The following proof closely follows the proof of Lemma A.3 of Frick et al. (2022). We only prove that (i) implies (ii); the other implication is easily checked. By (i), there is an increasing function $W : \{(I'(\varphi), I''(\varphi)) : \varphi \in B_0(\Sigma)\} \to \mathbb{R}$ such that $W(I'(\varphi), I''(\varphi)) = I(\varphi)$.

Let $\varphi \in B_0(\Sigma)$ be such that $I'(\varphi) = I''(\varphi) = k$. We will show that $I(\varphi) = k$. Since I' and I'' are monotonic and constant-linear, $k + \varepsilon = I'(k+\varepsilon) > I'(\varphi) > I'(k-\varepsilon) = k - \varepsilon$ and $k + \varepsilon = I''(k+\varepsilon) > I''(\varphi) > I''(k-\varepsilon) = k - \varepsilon$. Thus, by (i), $k + \varepsilon = I(k+\varepsilon) > I(\varphi) > I(k-\varepsilon) = k - \varepsilon$. Let ε converge to 0, then $I(\varphi) = k$. Thus, $I(\varphi) = k$, which implies that $I(\varphi) = \alpha I'(\varphi) + (1-\alpha)I''(\varphi)$ for all $\alpha \in \mathbb{R}$.

[48] Axiom 2 implies the "Archimedean axiom" in Ghirardato et al. (2004).

Now, consider $\varphi \in B_0(\Sigma)$ such that $I'(\varphi) < I''(\varphi)$. There exists $\alpha(\varphi) \in \mathbb{R}$ such that $I(\varphi) = \alpha(\varphi)I'(\varphi) + (1 - \alpha(\varphi))I''(\varphi)$. By a simple computation, one obtains

$$\alpha(\varphi) = \frac{I(\varphi) - I''(\varphi)}{I'(\varphi) - I''(\varphi)} = -I(\phi) = -W(I'(\phi), I''(\phi)),$$

where $\phi = \frac{\varphi - I''(\varphi)}{I''(\varphi) - I'(\varphi)}$. Clearly, $I'(\phi) = -1$ and $I''(\varphi) = 0$. Thus, $\alpha(\varphi) = -W(-1, 0)$, which is independent of φ. Let $\alpha = -W(-1, 0)$. Then, $I(\varphi) = \alpha I'(\varphi) + (1 - \alpha)I''(\varphi)$ for all $\varphi \in B_0(\Sigma)$.

We now prove that $\alpha \in [0, 1]$. By contradiction, assume that $\alpha < 0$. For any $\varphi \in B_0(\Sigma)$ such that $I'(\varphi) < I''(\varphi)$, we have $I(\varphi) > I''(\varphi)$. There exists $\varepsilon > 0$ such that $I(\varphi) > I''(\varphi) + \varepsilon$. Moreover, $I''(\varphi) + \varepsilon = I'(I''(\varphi) + \varepsilon) > I'(\varphi)$ and $I''(\varphi) + \varepsilon = I''(I''(\varphi) + \varepsilon) > I''(\varphi)$. By (i), $I''(\varphi) + \varepsilon = I(I''(\varphi) + \varepsilon) > I(\varphi)$, which is a contradiction. Thus, $\alpha \geq 0$. One can similarly show that $\alpha \leq 1$. □

Assume that $\succ$ is a hope-and-prepare preference and $\succ^*$ is an invariant biseparable extension of $\succ$. Let $u : X \to \mathbb{R}$ be a non-constant affine function, and let C and D be two compact convex subsets of Δ with $C \cap D \neq \emptyset$ such that

$$f \succ g \iff \begin{cases} \min_{p \in C} \int u(f)dp > \min_{p \in C} \int u(g)dp \\ \max_{p \in D} \int u(f)dp > \max_{p \in D} \int u(g)dp \end{cases}.$$

From the uniqueness result of Theorem 1, u is unique up to positive affine transformation, and C and D are unique.

It follows from Lemma 6 that there exist a monotonic, constant-linear functional $I : B_0(\Sigma) \to \mathbb{R}$ and a non-constant affine function $u' : X \to \mathbb{R}$ such that, for all $f, g \in \mathcal{F}$,

$$f \succ^* g \iff I(u'(f)) > I(u'(g)).$$

Moreover, I is unique and u' is unique up to positive affine transformation.

It trivially follows from the extension property that, for all $x, y \in X$, $u(x) = u(y)$ if and only if $u'(x) = u'(y)$, which implies that u is a positive affine transformation of u'. Thus, one can assume without loss of generality $u = u'$.

Define $I' : B_0(\Sigma) \to \mathbb{R}$ and $I'' : B_0(\Sigma) \to \mathbb{R}$ by $I'(\varphi) = \min_{p \in C} \int \varphi dp$ and $I''(\varphi) = \max_{p \in D} \int \varphi dp$ for all $\varphi \in B_0(\Sigma)$. Clearly, I' and I'' are monotonic, constant-linear functionals; and since $C \cap D \neq \emptyset$, $I'' \geq I'$.

Now, let $\phi, \varphi \in B_0(\Sigma)$ such that $I'(\phi) > I'(\varphi)$ and $I''(\phi) > I''(\varphi)$. We denote by $B_0(\Sigma, u(X))$ the set of all functions in $B_0(\Sigma)$ that take values in $u(X)$. Since $u(X)$ is an interval in $\mathbb{R}$, it is easy to check that there are $a > 0$ and $b \in \mathbb{R}$ such that $\phi = a\phi' + b$ and $\varphi = a\varphi' + b$, where $\phi', \varphi' \in B_0(\Sigma, u(X))$. And, by definition of $u(X)$ and $\mathcal{F}$, there are $f, g \in \mathcal{F}$ such that $\phi' = u(f)$ and $\varphi' = u(g)$. The following equivalences hold:

$$\begin{aligned} & I'(\phi) > I'(\varphi) \\ \iff & \min_{p \in C} \int a\phi' + bdp > \min_{p \in C} \int a\varphi' + bdp \\ \iff & \min_{p \in C} \int u(f)dp > \min_{p \in C} \int u(g)dp. \end{aligned}$$

Similarly, $I''(\phi) > I''(\varphi)$ is equivalent to $\max_{p \in D} \int u(f)dp > \max_{p \in D} \int u(g)dp$. Thus, $f \succ g$. Since $\succ^*$ is an extension of $\succ$, $f \succ^* g$, which is equivalent to $I(u(f)) > I(u(g))$. Thus, $I(\phi') > I(\varphi')$. Because I is constant-linear, one obtains $I(\phi) > I(\varphi)$.

It then follows from Lemma 7 that there exists $\alpha \in [0, 1]$ such that, for all $\varphi \in B_0(\Sigma)$, $I(\varphi) = \alpha I'(\varphi) + (1 - \alpha) I''(\varphi)$. In particular, for all $f \in \mathcal{F}$,

$$I(u(f)) = \alpha \min_{p \in C} \int u(f)dp + (1 - \alpha) \max_{p \in D} \int u(f)dp.$$

Finally, if $\succ$ is incomplete, there exists $f \in \mathcal{F}$ such that $\min_{p \in C} \int u(f)dp < \max_{p \in D} \int u(f)dp$. Thus, for all $\alpha' \neq \alpha$,

$$\alpha \min_{p \in C} \int u(f)dp + (1 - \alpha) \max_{p \in D} \int u(f)dp \neq \alpha' \min_{p \in C} \int u(f)dp + (1 - \alpha') \max_{p \in D} \int u(f)dp.$$

This implies that α is uniquely defined since I is uniquely defined.

D.6 Proof of Proposition 3

Let $C = \text{co}\left(\bigcup_{i=1}^{n} C_i\right)$. Clearly, C is a compact convex set. Let $\succ$ be a Bewley preference relation on $\mathcal{F}$, with representation (u, C).

First, assume that $\succ_0$ is a hope-and-prepare preference with unique representation (u, C_0, C_0).

i) Suppose that Pareto holds. Clearly, for all $f, g \in \mathcal{F}$, $f \succ g$ implies $f \succ_0 g$. Thus, by definition, $\succ$ is more conservative than the social preference $\succ_0$, and, by Proposition 1, $C_0 \subseteq C$. The converse implication is immediate.

ii) Assume that caution for incomparability holds. Suppose, by contradiction, that there is $p^* \in C \setminus C_0$. Then, there are $f \in \mathcal{F}$ and $x \in X$ such that

$$\int u(f)dp^* > u(x) > \max_{p \in C_0} \int u(f)dp.$$

By definition of C, there exist $(p_i)_{i\in N} \in \bigtimes_{i\in N} C_i$ and $(\lambda_i)_{i\in N} \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n \lambda_i = 1$ and $p^* = \sum_{i=1}^n \lambda_i p_i$. Thus, there exists $j \in N$ such that $\int u(f)dp_j > u(x) > \max_{p\in C_0} \int u(f)dp$, which implies $\max_{p\in C_j} \int u(f)dp > u(x) > \max_{p\in C_0} \int u(f)dp$. As $x \succ_0 f$ —this is the right hand side of the inequality above— caution for incomparability implies that either $x \succ_j f$ or $f \succ_j x$. Since $\max_{p\in C_j} \int u(f)dp > u(x)$, one concludes that $f \succ_j x$, which implies $\min_{p\in C_j} \int u(f)dp > u(x)$. Let $\bar{x} \in X$ such that $f \bowtie_j \bar{x}$:

$$u(\bar{x}) \geq \min_{p\in C_j} \int u(f)dp > u(x) > \max_{p\in C_0} \int u(f)dp,$$

which implies $\bar{x} \succ_0 f$. This is in contradiction with caution for incomparability. Therefore, $C \subseteq C_0$.

We now prove the converse implication. Assume $C \subseteq C_0$. Let $f \in \mathcal{F}$ and $x \in X$ such that $f \bowtie_i x$ for some $i \in N$. This implies

$$\max_{p\in C_i} \int u(f)dp \geq u(x) \geq \min_{p\in C_i} \int u(f)dp.$$

Since $C_i \subseteq C$, $C_i \subseteq C_0$. Hence,

$$\max_{p\in C_0} \int u(f)dp \geq \max_{p\in C_i} \int u(f)dp \geq u(x) \geq \min_{p\in C_i} \int u(f)dp \geq \min_{p\in C_0} \int u(f)dp,$$

that is, $f \bowtie_0 x$.

Now, assume that $\succ_0$ is a Bewley preference with unique representation (u, C_0). Statement *i)* is an immediate consequence of Theorem 2 in Danan et al. (2016). Statement *ii)* in this case is proved from the same argument as statement *ii)* in the previous case.

The if part is immediate. Suppose that caution for incomparability holds. Suppose, by contradiction, that there exits $p^* \in C \setminus C_0$. There are $f \in \mathcal{F}$ and $x \in X$ such that

$$\int u(f)dp^* > u(x) > \max_{p\in C_0} \int u(f)dp.$$

Then, there is $j \in N$ such that

$$\max_{p\in C_j} \int u(f)dp > u(x) > \max_{p\in C_0} \int u(f)dp.$$

The right hand side of this inequality implies that $x \succ_0 f$. From caution for incomparability, either $x \succ_j f$ or $f \succ_j x$. By the left hand side of the inequality above, $f \succ_j x$. Let $\bar{x} \in X$ such that $f \bowtie_j \bar{x}$. By caution for incomparability, $f \bowtie_0 \bar{x}$. However, it also holds that

$u(\bar{x}) \geq \min_{p \in C_j} \int u(f)dp > u(x) > \max_{p \in C_0} \int u(f)dp$, which implies $\bar{x} \succ_0 f$; a contradiction. Therefore, $C \subseteq C_0$.

References

Ahn, David, Syngjoo Choi, Douglas Gale, and Shachar Kariv (2014): "Estimating ambiguity aversion in a portfolio choice experiment," Quantitative Economics, 5 (2), 195–223.

Anscombe, F. J. and R. J. Aumann (1963): "A Definition of Subjective Probability," The Annals of Mathematical Statistics, 34 (1), 199–205.

Aumann, Robert J. (1962): "Utility Theory without the Completeness Axiom," Econometrica, 30 (3), 445–462.

Back, Anthony L, Robert M Arnold, and Timothy E Quill (2003): "Hope for the Best, and Prepare for the Worst," Annals of Internal Medicine, 138 (5), 439–443, pMID: 12614110.

Bewley, Truman F. (2002): "Knightian decision theory. Part I," Decisions in Economics and Finance, 25 (2), 79–110.

Binmore, Ken (2009): Rational Decisions, Princeton University Press, stu - student edition ed.

Bossaerts, Peter, Paolo Ghirardato, Serena Guarnaschelli, and William R. Zame (2010): "Ambiguity in Asset Markets: Theory and Experiment," The Review of Financial Studies, 23 (4), 1325–1359.

Brocas, Isabelle and Juan D. Carrillo (2008): "The Brain as a Hierarchical Organization," American Economic Review, 98 (4), 1312–46.

Bénabou, Roland and Marek Pycia (2002): "Dynamic inconsistency and self-control: a planner–doer interpretation," Economics Letters, 77 (3), 419–424.

Cettolin, Elena and Arno Riedl (2019): "Revealed preferences under uncertainty: Incomplete preferences and preferences for randomization," Journal of Economic Theory, 181, 547–585.

Chandrasekher, Madhav, Mira Frick, Ryota Iijima, and Yves Le Yaouanq (2022): "Dual-Self Representations of Ambiguity Preferences," Econometrica, 90 (3), 1029–1061.

Chateauneuf, Alain, Jürgen Eichberger, and Simon Grant (2007): "Choice under uncertainty with the best and worst in mind: Neo-additive capacities," Journal of Economic Theory, 137 (1), 538–567.

Chateauneuf, Alain, José Heleno Faro, Jean-Marc Tallon, and Vassili Vergopoulos (2024): "Alpha-maxmin as an aggregation of two selves," Journal of Mathematical Economics, 103006.

Cusumano, Carlo M and Masaki Miyashita (2021): "Twofold preferences under uncertainty," Tech. rep., Working paper, Yale.

Danan, Eric, Thibault Gajdos, Brian Hill, and Jean-Marc Tallon (2016): "Robust Social Decisions," American Economic Review, 106 (9), 2407–25.

Dubra, Juan, Fabio Maccheroni, and Efe Ok (2004): "Expected utility theory without the completeness axiom," Journal of Economic Theory, 115 (1), 118–133.

Dur, Umut, Robert G Hammond, and Thayer Morrill (2018): "Identifying the harm of manipulable school-choice mechanisms," American Economic Journal: Economic Policy, 10 (1), 187–213.

Dworczak, Piotr and Alessandro Pavan (2022): "Preparing for the worst but hoping for the best: Robust (bayesian) persuasion," Econometrica, 90 (5), 2017–2051.

Echenique, Federico, Masaki Miyashita, Yuta Nakamura, Luciano Pomatto, and Jamie Vinson (2022): "Twofold multiprior preferences and failures of contingent reasoning," Journal of Economic Theory, 202, 105448.

Efe, Ok, Pietro Ortoleva, and Gil Riella (2012): "Incomplete Preferences Under Uncertainty: Indecisiveness in Beliefs versus Tastes," Econometrica, 80 (4), 1791–1808.

Eichberger, Jürgen, Simon Grant, David Kelsey, and Gleb A. Koshevoy (2011): "The α-MEU model: A comment," Journal of Economic Theory, 146 (4), 1684–1698.

Etner, Johanna, Meglena Jeleva, and Jean-Marc Tallon (2012): "Decision theory under ambiguity," Journal of Economic Surveys, 26 (2), 234–270.

Faro, José (2015): "Variational Bewley preferences," Journal of Economic Theory, 157, 699–729.

Fishburn, Peter C. (1970): Utility Theory for Decision Making, John Wiley and Sons, New York.

Frick, Mira, Ryota Iijima, and Yves Le Yaouanq (2022): "Objective rationality foundations for (dynamic) α-MEU," Journal of Economic Theory, 200, 105394.

Fudenberg, Drew and David K. Levine (2006): "A Dual-Self Model of Impulse Control," American Economic Review, 96 (5), 1449–1476.

Galaabaatar, Tsogbadral and Edi Karni (2012): "Expected multi-utility representations," Mathematical Social Sciences, 64 (3), 242–246.

Ghirardato, Paolo, Fabio Maccheroni, and Massimo Marinacci (2004): "Differentiating ambiguity and ambiguity attitude," Journal of Economic Theory, 118 (2), 133–173.

Ghirardato, Paolo and Massimo Marinacci (2002): "Ambiguity Made Precise: A Comparative Foundation," Journal of Economic Theory, 102 (2), 251–289.

Gilboa, Itzhak, Fabio Maccheroni, Massimo Marinacci, and David Schmeidler (2010): "Objective and Subjective Rationality in a Multiple Prior Model," Econometrica, 78 (2), 755–770.

Gilboa, Itzhak and Massimo Marinacci (2016): Ambiguity and the Bayesian Paradigm, Cham: Springer International Publishing, 385–439.

Gilboa, Itzhak and David Schmeidler (1989): "Maxmin expected utility with non-unique prior," Journal of Mathematical Economics, 18 (2), 141 – 153.

Grant, Simon, Patricia Rich, and Jack Stecher (2020): "Worst-and best-case expected utility and ordinal meta-utility," Tech. rep., Working Paper.

Gul, Faruk and Wolfgang Pesendorfer (2015): "Hurwicz expected utility and subjective sources," Journal of Economic Theory, 159, 465–488.

Hartmann, Lorenz (2023): "Strength of preference over complementary pairs axiomatizes alpha-MEU preferences," Journal of Economic Theory, 213, 105719.

Hill, Brian (2016): "Incomplete preferences and confidence," Journal of Mathematical Economics, 65, 83–103.

——— (2023): "Beyond uncertainty aversion," Games and Economic Behavior, 141, 196–222.

Hurwicz, Leonid (1951): "Optimality criteria for decision making under ignorance," Tech. rep., Cowles Commission discussion paper, statistics.

Karni, Edi (2020): "Probabilistic sophistication without completeness," Journal of Mathematical Economics, 89, 8–13.

Klibanoff, Peter, Sujoy Mukerji, Kyoungwon Seo, and Lorenzo Stanca (2022): "Foundations of ambiguity models under symmetry: α-MEU and smooth ambiguity," Journal of Economic Theory, 199, 105202.

Knight, Frank H (1921): "Risk, Uncertainty and Profit," University of Illinois at Urbana-Champaign's Academy for Entrepreneurial Leadership Historical Research Reference in Entrepreneurship.

Kopylov, I (2002): "α-Maxmin expected utility," Tech. rep., mimeo, University of Rochester.

Li, Shengwu (2017): "Obviously strategy-proof mechanisms," American Economic Review, 107(11), 3257–3287.

Marinacci, Massimo (2002): "Probabilistic sophistication and multiple priors," Econometrica, 70 (2), 755–764.

Minardi, Stefania and Andrei Savochkin (2015): "Preferences with grades of indecisiveness," Journal of Economic Theory, 155, 300–331.

Nascimento, Leandro and Gil Riella (2011): "A class of incomplete and ambiguity averse preferences," Journal of Economic Theory, 146 (2), 728–750.

Nielsen, Kirby and Luca Rigotti (2022): "Revealed incomplete preferences," arXiv preprint arXiv:2205.08584.

Pathak, Parag A and Tayfun Sönmez (2008): "Leveling the playing field: Sincere and sophisticated players in the Boston mechanism," American Economic Review, 98 (4), 1636–1652.

Schmeidler, David (1989): "Subjective Probability and Expected Utility without Additivity," Econometrica, 57 (3), 571–587.

Schröder, David (2011): "Investment under ambiguity with the best and worst in mind," Mathematics and Financial Economics, 4, 107–133.

Siniscalchi, Marciano (2009): "Vector expected utility and attitudes toward variation," Econometrica, 77 (3), 801–855.

Taylor, Lauren J, Michael J Nabozny, Nicole M Steffens, Jennifer L Tucholka, Karen J Brasel, Sara K Johnson, Amy Zelenski, Paul J Rathouz, Qianqian Zhao, Kristine L Kwekkeboom, et al. (2017): "A framework to improve surgeon communication in high-stakes surgical decisions: best case/worst case," JAMA surgery, 152 (6), 531–538.

Thaler, Richard H. and H. M. Shefrin (1981): "An Economic Theory of Self-Control," Journal of Political Economy, 89 (2), 392–406.

Trautmann, Stefan T. and Gijs van de Kuilen (2015): Ambiguity Attitudes, John Wiley & Sons, Ltd, chap. 3, 89–116.

Troyan, Peter and Thayer Morrill (2020): "Obvious manipulations," Journal of Economic Theory, 185, 104970.

Xia, Jianming (2020): "Decision Making under Uncertainty: A Game of Two Selves," .